ENERGY LEVELS AND STRUCTURE OF ^{9}B AND ^{9}Be MIRROR NUCLEI IN NUCLEAR MODELS

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In this paper, we have investigated energy levels mirror nuclei of the ^{9}B and ^{9}Be in both shell model and cluster model. In shell model, the nuclei ^{9}B and ^{9}Be can be modeled as one core plus five nucleons. As the same way the mirror nuclei ^{21}Na and ^{21}Ne can be studied in five-body system. In cluster model, we have selected two α clusters and one nucleon in interaction with clusters. Using suitable potential for interaction between particles by applying parametric Nikiforov-Uvarov method in Jacobi coordinate, potential coefficients have been computed. Then we have calculated the energy levels and wave function for mirror nuclei ^{9}B and ^{9}Be and compare with experimental results. The energy levels of mirror nuclei ^{21}Na compared with ^{21}Ne are calculated in shell model.

Key words: cluster model, shell model, modified Eckart potential, PNU method

1. Introduction

Studying the nuclei structure is one of the main challenges in nuclear physics. Some properties of the nucleus like the energy levels present a good picture of their nuclear structures [1]. Several patterns and models such as shell model and cluster model have been offered for nuclear structure up to now. Choosing a suitable potential is the first step of studying the energy levels of nuclei in shell model and cluster model [2]. In this regard, studying the nucleon-nucleon interactions is very useful to find the many important properties of multi-nucleon systems [3]. There are many nucleon-nucleon potentials such as Hulthen potential [4], Titz-Wei potential [5], Woods-Saxon potential [6, 7] and Eckart potential [2, 8]. Studying the wave function is obtained a lot of necessary information for descriptions of quantum system, so solving equations such as Schrödinger equation in nonrelativistic quantum mechanics is very important [9]. The isotope ^{9}Be is an even-odd nucleus in nuclear structure [10]. The mirror nucleus of ^{9}Be is ^{9}B.The isotope ^{21}Na is also an even-odd nucleus, and the mirror nucleus of it, is ^{21}Ne.

The *Be* has 12 known isotopes, but only one of them is stable. Significantly, however ^{8}Be structure involve 2α cluster, its half-life is very short (8.19×10^{-8}ns),

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whereas ^{9}Be with an extra nucleon is stable and strongly deformed [11]. The cluster structure of ^{9}Be is interesting for studying the dynamics of reactions with weakly bound nuclei and also nuclear astrophysics [11].

There are 16 isotopes for *B* that has been discovered with mass number from 6 to 21. But only ^{10}B and ^{11}B naturally can be found. ^{9}B has a structure similar to ^{9}Be, with the odd-neutron replaced by the odd-proton [12].

Filikhin, Suslov and Vlahovic calculated energies of the ^{9}Be low-lying levels in three cluster model, using configuration space Faddeev equations [13]. Nesterov et al. studied the nature of resonance states and wave functions of the ^{9}Be and ^{9}B nuclei within a microscopic three-cluster model, using three-cluster algebraic version of the resonating group method (AV MRG) [14]. Descouvemont et al. in their paper introduced Hamiltonian and phenomenological three-body force for three-body model of ^{9}Be and using the CDCC method for calculating wave function of ^{9}Be [15]. Mengjiao et al. introduced a modified version of the THSR wave function and computed binding energies for the ground state of ^{9}Be [10]. Zhao et al. investigated energy spectrum of ^{9}B using the new THSR wave function in three cluster model of ^{9}B [16].

In nonrelativistic quantum mechanics, the Schrödinger equation with some typical potential used to describe the particles dynamics by different methods. There is the exact solution of this equation just only for a few simple interactions. So, the kinds of various methods have been used for the solution of this equation, exemplar, the super symmetric method [5], Nikiforov-Uvarov method [8] and so on.

In this paper, we use nonrelativistic shell model and cluster model for calculating the energy level and wave function for some mirror nuclei with employing Jacobi coordinates. So we review Jacobi coordinates in section 2. Parametric Nikiforov-Uvarov method which used to solve the Schrödinger equation explains in section 3. We determine the energy levels and wave function in Schrödinger equation by modified Eckart plus repulsive term potentials in section 4, calculating of the mirror nuclei ^{9}B and ^{9}Be and ^{21}Na and ^{21}Ne energy levels is done in section 5 for shell model. The energy levels of the ^{9}B and ^{9}Be in cluster model are calculated in section 6. Discussion and conclusion are given in section 7.

2. Review of Jacobi coordinates

In the theory of N-body system, Jacobi coordinates are used to simplify the mathematical formulation. For a system containing *A* particles, an $N = A - 1$ Jacobi vector $\boldsymbol{\zeta}$ and 3*A* Jacobi components are determined [17]. In these calculations, the mass difference between nucleons and protons is considered negligible. For such system, the *N-1* Jacobi vector can be defined as follows [17]:

$$\zeta_i = \sqrt{\frac{i}{i+1}}\left(\boldsymbol{r}_{i+1} - \frac{1}{i}\sum_{j=1}^{i}\boldsymbol{r}_j\right), i = 1,2,...,N-1 \tag{1}$$

The ζ_i vector is any point relative to the previous mass center. The center of mass vector for *A* particles is defined as:

$$\boldsymbol{R} = \frac{1}{A}\left(\boldsymbol{r}_1 + \boldsymbol{r}_2 + ... + \boldsymbol{r}_A\right) = \frac{1}{N+1}\sum_{i=1}^{N+1}\boldsymbol{r}_i \tag{2}$$

The volume element in this coordinate is as follows:

$$\prod\nolimits_{i=1}^{N} dr_i = N^{\frac{3}{2}} dR \prod\nolimits_{j=1}^{N-1} d\zeta_j = dx \tag{3}$$

Where the hyper-radius *x* is defined as:

$$x^2 = \sum_{i=1}^{N-1}\left(\zeta_i^2\right) \tag{4}$$

For example, a three-particle system after eliminating the center-of-mass motion becomes a six-dimensional one (D=9-3=6). From Eq. (1) and (2), the internal three-identical-particle motion is described by means of the Jacobi relative coordinates ζ_1, ζ_2 and $R=R_3$ as [18]:

$$\boldsymbol{\zeta_1} = \frac{\boldsymbol{r_1} - \boldsymbol{r_2}}{\sqrt{2}}, \quad \boldsymbol{\zeta_2} = \frac{\boldsymbol{r_1} + \boldsymbol{r_2} - 2\boldsymbol{r_3}}{\sqrt{6}}, \quad \boldsymbol{R_3} = \frac{\boldsymbol{r_1} + \boldsymbol{r_2} + \boldsymbol{r_3}}{3} \tag{5}$$

The hyper-spherical coordinates are defined in terms of the absolute values of ζ_1, ζ_2 and from Eq. (5) it follows

$$x = \sqrt{\zeta_1^2 + \zeta_2^2}, \quad t = \arctan\left(\frac{\zeta_1}{\zeta_2}\right) \tag{6}$$

If the potential between the particles depends solely on the powers of their relative distance, they can be written in terms of the hyperradius. In this case, these potentials are called hypercentral potentials [19].

3. Parametric Nikiforov-Uvarov method

The exact solution of Schrödinger, Klein-Gordon and Dirac wave equation can be possible only for a few simple interactions. The PNU method can be used for a certain potential [20]. In this method the differential equation can be written as follows [6, 20, and 21]:

$$\left[\frac{d^2}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{(s(1-\alpha_3 s))^2}\right]\Psi = 0 \tag{7}$$

According to the Nikiforov-Uvarov method [22, 23], the energy eigen-value is obtained by the following equation:

$$\alpha_2 n - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0 \quad (8)$$

The corresponding wave function can be obtained using bellow equation:

$$\Psi_{n,l}(s) = N_{n,l} s^{\alpha_{12}} (1-\alpha_3 s)^{\alpha_{13}} P_n^{(\alpha_{10},\alpha_{11})}(1-2\alpha_3 s) \quad (9)$$

Where, $N_{n,l}$ is the normalization constant and $P_n^{(\mu,\upsilon)}(x)$, $\mu > -1$, $\nu > -1$ and x ϵ [-1, 1] are the Jacobi polynomials with [24]:

$$P_n^{(\mu,\upsilon)}(x) = \frac{(\mu+1)_n}{n!}\, {}_2F_1\left(-n, 1+\mu+\upsilon+n; \mu+1; \frac{1}{2}(1-x)\right) \quad (10)$$

The α_i constants are given by [25, 26]:

$$\alpha_4 = \frac{1}{2}\left(1-\alpha_1\right) \qquad \alpha_5 = \frac{1}{2}\left(\alpha_2 - 2\alpha_3\right)$$

$$\alpha_6 = \alpha_5^2 + \xi_1 \qquad \alpha_7 = 2\alpha_4\alpha_5 - \xi_2$$

$$\alpha_8 = \alpha_4^2 + \xi_3 \qquad \alpha_9 = \alpha_3\left(\alpha_7 + \alpha_3\alpha_8\right) + \alpha_6$$

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8} - 1 \qquad \alpha_{11} = 1 - \alpha_1 - 2\alpha_4 + \frac{2}{\alpha_3}\sqrt{\alpha_9}, \quad \alpha_3 \neq 0$$

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8} \qquad \alpha_{13} = -\alpha_4 + \frac{1}{\alpha_3}\left(\sqrt{\alpha_9} - \alpha_5\right), \quad \alpha_3 \neq 0 \quad (11)$$

In some problems α_3=0, so the α_{11}, α_{12} becomes as:

$$\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \qquad \alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \quad (12)$$

The Eq. (9) becomes as:

$$\Psi(s) = s^{\alpha_{12}} e^{\alpha_{13} s} L_n^{\alpha_{10}-1}(\alpha_{11} s) \quad (13)$$

Where $L_n^{\alpha}(x)$ are the Laguerre polynomials [27].

4. Eigenvalue and wave function in Schrödinger equation

To study the energy spectrum for A-body system, we use the time independent Schrödinger equation [28]. That is as follow

$$\frac{d^2R}{dx^2} + \frac{D-1}{x}\frac{dR}{dx} + \frac{2\mu}{h^2}(E_{n,l} - V\left(x\right) - \frac{h^2}{2\mu}\frac{l(l+D-2)}{x^2})R = 0 \quad (14)$$

Where $D=3N-3$ and μ is the reduced mass. $V(x)$ is the interaction between particles. Eckart potential is one of the most important potential which has been studied by many researchers in physics and chemical physics [2]. So we use modified Eckart potential [2, 8] plus repulsive term potential as interaction between particles. That is defined as

$$V(x) = 4V_0 \frac{e^{-2\alpha x}}{(1-e^{-2\alpha x})^2} + \frac{\alpha V_1}{x} e^{-2\alpha x} \tag{15}$$

In the above equation, V_0 and V_1 are the actual parameter describing the potential well depth and the parameter α representing the potential range.

By substituting the Eq. (15) in Eq. (14), the radial Schrödinger equation is obtained as:

$$\frac{d^2R(x)}{dx^2} + \frac{D-1}{x}\frac{dR(x)}{dx} + \frac{2\mu}{h^2}(E - 4V_0 \frac{e^{-2\alpha x}}{(1-e^{-2\alpha x})^2} - \frac{\alpha V_1}{x} e^{-2\alpha x} - \frac{h^2}{2\mu}\frac{l(l+D-2)}{x^2})R(x) = 0 \tag{16}$$

It is seen from Eq. (16) that the equation has the exponential and inverse square terms, which cannot be solved analytically. By applying suitable approximation, this problem can be solved. We can take Pekeris approximation for our aim. This approximation is based on the expansion of the centrifugal term [29]. That is valid for $\alpha \leq 1$ [25].

$$\frac{1}{x^2} \approx \frac{4\alpha^2}{(1-e^{-2\alpha x})^2} \tag{17}$$

Now the Eq. (16) with $s = e^{-2\alpha x}$ can be written bellow:

$$\frac{d^2R}{ds^2} + \frac{(2-D)-s}{s(1-s)}\frac{dR}{ds} + \frac{1}{s^2(1-s)^2}(-\xi_1 s^2 + \xi_2 s - \xi_3)R = 0 \tag{18}$$

Where the parameters ξ_1, ξ_2 and ξ_3 are as follows:

$$\begin{aligned} \xi_1 &= -\frac{\mu}{2\alpha^2\hbar^2}E - \frac{\mu}{\hbar^2}v_1 \\ \xi_2 &= -\frac{\mu}{\alpha^2\hbar^2}E - \frac{2\mu}{\alpha^2\hbar^2}v_0\frac{\mu}{\hbar^2}v_1 \\ \xi_3 &= -\frac{\mu}{2\alpha^2\hbar^2}E + l\left(l+D-2\right) \end{aligned} \tag{19}$$

By comparing Eq. (18) with Eq. (7) and (11), the coefficients α_i can be obtained as follows:

$$\alpha_1 = 2 - D, \qquad \alpha_2 = \alpha_3 = 1, \qquad \alpha_4 = \frac{1}{2}(D-1), \qquad \alpha_5 = \frac{-1}{2},$$
$$\alpha_6 = \frac{1}{4} + \xi_1, \qquad \alpha_7 = -\frac{1}{2}(D-1) - \xi_2, \qquad \alpha_8 = \frac{1}{4}(D-1)^2 + \xi_3,$$
$$\alpha_9 = \xi_1 - \xi_2 + \xi_3 - \frac{1}{2}(D-1) + \frac{1}{4}(D-1)^2 + \frac{1}{4}, \qquad \alpha_{10} = 2\sqrt{\frac{1}{4}(D-1)^2 + \xi_3},$$
$$\alpha_{11} = 2\sqrt{\xi_1 - \xi_2 + \xi_3 - \frac{1}{2}(D-1) + \frac{1}{4}(D-1)^2 + \frac{1}{4}},$$
$$\alpha_{12} = \frac{1}{2}(D-1) + \sqrt{\frac{1}{4}(D-1)^2 + \xi_3},$$
$$\alpha_{13} = -\frac{1}{2}(D-1) + \frac{1}{2} + \sqrt{\xi_1 - \xi_2 + \xi_3 - \frac{1}{2}(D-1) + \frac{1}{4}(D-1)^2 + \frac{1}{4}} \tag{20}$$

Using the Eq. (8), we can write the energy Eigen-values formula as:

$$2n + \frac{1}{2} + n(n-1) - \frac{1}{2}(D-1)(2-D) + \lambda + 2l(l+D-2) + (2n+1)(\sqrt{\alpha_9} + \sqrt{\alpha_8}) + 2\sqrt{\alpha_9}\sqrt{\alpha_8} = 0 \tag{21}$$

Where $\lambda = \frac{2\mu}{h^2\alpha^2}V_0 + \frac{\mu}{h^2}V_1$

The wave function is obtained as follow

$$R_{n,l}(s) = N_{n,l}\, s^{\frac{1}{2}(D-1)+\sqrt{\alpha 8}}\,(1-s)^{-\frac{1}{2}(D-1)+\frac{1}{2}+\sqrt{\alpha 9}}\, P_n^{(2\sqrt{\alpha_8},2\sqrt{\alpha_9})}(1-2s) \tag{22}$$

Where, *N* is the normalization constant.

5. Calculation energy level of mirror nuclei in shell model

The mirror nuclei are expected to have approximately identical structure at nearly same excitation energy [30]. In nuclear structure a shell that contains the maximum number of nucleons permitted by the exclusion principle is called closed shell. The nuclei ^9B and ^9Be can be modeled as a core, with five nucleons in $1p_{3/2}$ level. The ground state spin and parity of ^9B and ^9Be are $J^{\pi}=3/2^-$ [31]. These isotopes can be considered as many-body system in shell model. The energy values of ^9B and ^9Be for the ground state and $1f_{5/2}$ level are calculated by choosing suitable coefficients for the proposed potential through fitting with the experimental results [32]. These results are compared with the experimental data for the ground state and the excited state in Table 1.

Table 1

The ground state and the excited state energy of ^9B and ^9Be in shell model

isotope	*Parameters of potential*			J^π	*E(Mev)*	
	$\alpha(fm^{-1})$	$V_0(Mev)$	$V_1(Mev.fm)$		*Our work*	*Exp.[32]*
^9B	0.035	53.409	-22.281	$3/2^-$	-56.313	-56.313
				$5/2^-$	-53.689	-53.968
^9Be	0.035	55.154	-25.585	$3/2^-$	-58.164	-58.164
				$5/2^-$	-55.517	-55.735

The ^{21}Na and ^{21}Ne can be modeled as a core with five nucleons, too. The ground state spin and parity of ^{21}Na and ^{21}Ne are $J^\pi = 3/2^+$ [33]. The energy values of ^{21}Na and ^{21}Ne can be computed, by determination suitable parameters of proposed potential.

Table 2

The ground state and the excited state energy of ^{21}Na and ^{21}Ne in shell model

isotope	*Parameters of potential*			J^π	*E(Mev)*	
	$\alpha(fm^{-1})$	$V_0(Mev)$	$V_1(Mev.fm)$		*our*	*Exp.[33,34]*
^{21}Na	0.024	157.145	-30.385	$3/2^+$	-163.046	-163.046
				$7/2^+$	-161.282	-161.33
^{21}Ne	0.024	161.385	-35.755	$3/2^+$	-167.405	-167.405
				$7/2^+$	-165.636	-165.66

Here, we have modeled our nuclei as a core with five nucleons. Using Jacobi coordinates; the ground state and the excited state energy are calculated. As it is shown in the above tables, there is a good agreement between our results and experimental results.

6. Calculation energy level of ^9B and ^9Be in cluster model

From the beginning of nuclear science, the clustering phenomena has been considered. The cluster interpretation is suitable to describe nuclear states and has been successful in reproducing the energy spectra and other nuclear properties [35]. Deformation is played a significant role in the light nuclei [33]. Because the nucleus ^9Be is strongly deformed, its structure can be identified in cluster model

[11]. The two-body picture of cluster configuration of ^{9}Be described as $^{8}Be + n$ or $^{5}He + \alpha$ [36, 37]. But the three-body structure of ^{9}Be is most possible [11]. The three-body models of ^{9}Be and ^{9}B relies on two α and nucleon interactions [11]. The structure of neutron (proton)-rich nucleus can be described with the correlation between α-clusters and valence nucleons [10, 16]. So, the nuclei ^{9}B and ^{9}Be are a nucleus with $\alpha+\alpha+n$ structure [14]. We choose the modified Eckart potential [2, 8] plus repulsive term potential for interaction between particles. Therefore, phenomenological three-body force has been introduced as Eq. (14).

By choosing the appropriate coefficients for the proposed potential with the experimental results, the results are presented in the table below.

Table 3

The ground state and the excited state energy of ^{9}B and ^{9}Be in cluster model

isotope	*Parameters of potential*			J^{π}	*E(Mev)*		
	$\alpha(fm^{-1})$	$V_0(Mev)$	$V_1(Mev.fm)$		*Our work*	*Exp.[32]*	*Other [10,16]*
^{9}B	0.08	49.050	--20.281	$3/2^-$	-56.313	-56.313	-55.2
				$5/2^-$	-53.784	-53.968	-52.9
^{9}Be	0.08	50.759	-22.080	$3/2^-$	-58.164	-58.164	-56.4
				$5/2^-$	-55.621	-55.735	-53.8

Here, we have modeled ^{9}B and ^{9}Be as two α clusters with extra nucleon. By applying Jacobi coordinates in three-body system; the ground state and the excited state energy of ^{9}Be and ^{9}B have been calculated. As it is shown in the above table, there is a good agreement between our results and experimental results.

7. Discussion and Conclusion

In this study, we calculated the energy levels and the wave function of N-particle system in a non-relativistic system by choosing the modified Eckart potential plus repulsive potential for shell model and cluster model. By Jacobi coordinates and parametric *Nikiforov-Uvarov method* we have solved the Schrödinger equation and calculated the energy of the ground state and the excited state for the mirror nuclei of ^{9}B and ^{9}Be and ^{21}Na and ^{21}Ne in shell model. ^{9}B and ^{9}Be can be descripted based on a core with five nucleon and also combination as two α clusters plus extra nucleon. For investigation the ^{21}Na and ^{21}Ne nuclei, it can be modeled as a core with five nucleons. In tables 1 and 3, the ground state and the excited state energy of ^9B and ^9Be have been displayed in shell model and cluster model, respectively. The ground state and the excited state energy of ^{21}Na and ^{21}Ne in shell model have been presented in table 2. As it is seen, there is good agreement between our results and experimental result especially in cluster model. The wave function of the ground state of ^9B and ^9Be are plotted in Fig.1.

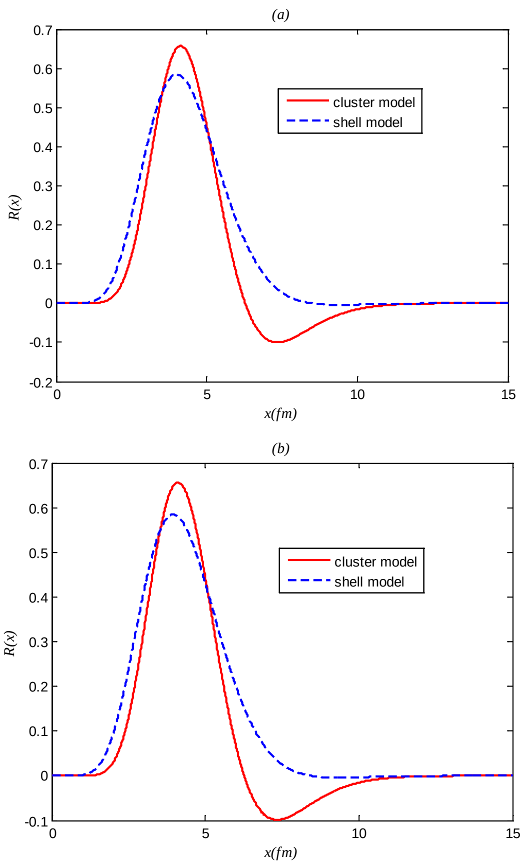

Fig1. a) The wave function of a) ^9Be and b) ^9B

The wave function diagram of ^9B and ^9Be in Fig1 demonstrate acceptable consequence about both models special cluster model.

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