DUAL JET GEOMETRIZATION FOR A CHERNOV-LIKE HAMILTONIAN

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Dedicated to the memory of Prof. Emeritus Dr. Constantin Udrişte

In this paper we geometrize on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ *the Chernov-like Hamiltonian*

$$H(t,p) = \sqrt{h_{11}(t)} \cdot \sqrt[3]{p_1^1 p_2^1 p_3^1 + p_1^1 p_2^1 p_4^1 + p_1^1 p_3^1 p_4^1 + p_2^1 p_3^1 p_4^1},$$

in the sense of construction of Cartan canonical connection, $d-$*torsions and* $d-$*curvatures, Maxwell-like and Einstein-like equations, produced by this Chernov-like Hamiltonian.*

Keywords: dual 1-jet spaces; Cartan canonical connection; $d-$torsions; $d-$curvatures; Maxwell-like equations; Einstein-like equations.

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1. Introduction

In their works, Garasko [5], Pavlov [8] and Chernov [4] emphasize the importance of the four-dimension metrics in the Finsler geometry, which are characterized by the complete equivalence of all non-isotropic directions, promoting in their papers geometric-physical models based on the Minkowski-like, Berwald-Moór or Chernov metrics (see Balan-Neagu's monograph [2]).

The *jet Chernov Finsler metric (Lagrangian)* is given by

$$F(t,y) = \sqrt{h^{11}(t)} \cdot \sqrt[3]{y_1^1 y_1^2 y_1^3 + y_1^1 y_1^2 y_1^4 + y_1^1 y_1^3 y_1^4 + y_1^2 y_1^3 y_1^4}.$$

Its generalized Lagrange geometry on the 1-jet space $J^1(\mathbb{R}, M^4)$ is completely done in [2] and [3] by using the canonical nonlinear connection

$$\Gamma = \left(M_{(1)1}^{(i)} = -\kappa_{11}^1 y_1^i,\ N_{(1)j}^{(i)} = -\frac{\kappa_{11}^1}{2} \delta_j^i \right),$$

where δ_j^i is the Kronecker symbol and, if we have

$$h^{11} = \frac{1}{h_{11}} > 0,$$

then

$$\kappa_{11}^1 = \frac{h^{11}}{2} \frac{dh_{11}}{dt}.$$

By dual analogy, in this paper, we investigate on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ the *Chernov-like Hamiltonian (Cartan metric)*

$$H(t,p) = \sqrt{h_{11}(t)} \cdot \sqrt[3]{p_1^1 p_2^1 p_3^1 + p_1^1 p_2^1 p_4^1 + p_1^1 p_3^1 p_4^1 + p_2^1 p_3^1 p_4^1}, \tag{1}$$

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endowed with the dual nonlinear connection

$$\Gamma^*_{[3]} = \left(M^{(1)}_{(i)1} = \kappa^1_{11} p^1_i,\ N^{(1)}_{(i)j} = \frac{\kappa^1_{11}}{2}\delta_{ij} \right). \tag{2}$$

It is important to note that the general dual jet Hamilton geometry produced by a Hamiltonian and a nonlinear connection is completely developed in the Neagu-Oană's monograph [7]. Consequently, in the sequel, we will construct all geometrical objects (such as Cartan linear connection, $d-$torsions and $d-$curvatures, together with some natural Maxwell-like and Einstein-like equations), that characterize and which are produced by the Chernov-like Hamiltonian (1) and the nonlinear connection (2).

For example, if we use the following notations and formulas

$$S^{111} := S^{111}_{[3]} = p^1_1 p^1_2 p^1_3 + p^1_1 p^1_2 p^1_4 + p^1_1 p^1_3 p^1_4 + p^1_2 p^1_3 p^1_4,$$

$$S^{i11} := S^{i11}_{[3]} = \frac{\partial S^{111}_{[3]}}{\partial p^1_i} = \frac{S^{111}_{[3]} p^1_i - S^{1111}_{[4]}}{\left(p^1_i\right)^2},$$

$$S^{ij1} := S^{ij1}_{[3]} = \frac{\partial S^{i11}_{[3]}}{\partial p^1_j} = \frac{\partial^2 S^{111}_{[3]}}{\partial p^1_i \partial p^1_j} = \begin{cases} S^1_{[1]} - p^1_i - p^1_j, & i \neq j \\ 0, & i = j, \end{cases}$$

$$S^{jkm} := S^{jkm}_{[3]} = \frac{1}{6}\frac{\partial S^{jk1}_{[3]}}{\partial p^1_m} = \frac{1}{6}\frac{\partial^3 S^{111}_{[3]}}{\partial p^1_j \partial p^1_k \partial p^1_m} = \begin{cases} \dfrac{1}{3!}, & \{j,k,m\} \text{ - distinct indices} \\ 0, & \text{otherwise}, \end{cases}$$

where

$$S^{1111}_{[4]} = p^1_1 p^1_2 p^1_3 p^1_4,$$

$$S^1_{[1]} = p^1_1 + p^1_2 + p^1_3 + p^1_4,$$

then the *fundamental metrical d-tensor* produced by H is given by the formula

$$g^{ij}(t,p) = \frac{h^{11}(t)}{2}\frac{\partial^2 H^2}{\partial p^1_i \partial p^1_j} = \frac{\left(S^{111}\right)^{-1/3}}{3}\left[S^{ij1} - \frac{1}{3S^{111}} S^{i11} S^{j11}\right]. \tag{3}$$

Moreover, because the matrix $\left(S^{ij1}\right)$ is non-degenerate since we have

$$0 \neq \det\left(S^{ij1}\right)_{i,j=\overline{1,4}} = 4\left[4S^{1111}_{[4]} - S^1_{[1]} S^{111}_{[3]}\right] := \mathbf{T}^{1111},$$

we deduce that its inverse matrix (S_{jk1}) is given by the entries

$$S_{jk1} := S_{[3]jk1} = \begin{cases} \dfrac{-2}{\mathbf{T}^{1111}}\left(p^1_j + p^1_k\right)\left[p^1_j p^1_k + \dfrac{S^{1111}_{[4]}}{p^1_j p^1_k}\right], & j \neq k \\ \dfrac{1}{\mathbf{T}^{1111}} \cdot \dfrac{1}{p^1_j}\left[\prod_{l=1}^4 \left(p^1_j + p^1_l\right)\right], & j = k. \end{cases}$$

It follows that the matrix $g = (g^{ij})$ admits the inverse matrix $g^{-1} = (g_{jk})$, whose entries are

$$g_{jk} = 3\left(S^{111}\right)^{1/3}\left[S_{jk1} + \frac{S^1_j S^1_k}{3\left(S^{111} - \mathbf{S}^{111}\right)}\right], \tag{4}$$

where $S^1_j = S_{jp1} S^{p11}$ and $3\mathbf{S}^{111} = S_{pq1} S^{p11} S^{q11}$.

Note that, for $i \neq j$, the following equalities hold true as well:

$$\sum_{k=1}^4 \frac{S^{1111}_{[4]}}{p^1_k} = S^{111}_{[3]}, \quad p^1_k \cdot S^{k11}_{[3]} = 3S^{111}_{[3]}, \tag{5}$$

$$S_{[3]}^{i11} \cdot S_{[3]}^{j11} = S_{[3]}^{111}\left(S_{[1]}^{1} - p_i^1 - p_j^1\right) + \frac{\left(S_{[4]}^{1111}\right)^2}{\left(p_i^1\right)^2\left(p_j^1\right)^2}. \tag{6}$$

Further, some laborious computations lead to:

$$\begin{aligned} S_j^1 &:= S_{[3]j}^1 = S_{[3]jr1}S_{[3]}^{r11} = \frac{1}{2}p_j^1, \\ \mathbf{S}^{111} &:= \mathbf{S}_{[3]}^{111} = S_{[3]rq1}S_{[3]}^{r11}S_{[3]}^{q11} = \frac{1}{2}S_{[3]}^{111}. \end{aligned} \tag{7}$$

Using the formulas (3) – (7), by direct computations, we conclude that we have

$$g^{ij} := g_{[3]}^{ij} = \frac{1}{3}\left(S_{[3]}^{111}\right)^{-1/3} S_{[3]}^{ij1} - \frac{1}{9}\left(S_{[3]}^{111}\right)^{-4/3} S_{[3]}^{i11} \cdot S_{[3]}^{j11}. \tag{8}$$

and

$$g_{jk} := g_{[3]jk} = 3\left(S_{[3]}^{111}\right)^{1/3}\left[S_{jk1} + \frac{1}{6S_{[3]}^{111}}p_j^1 p_k^1\right]. \tag{9}$$

2. **Cartan canonical connection.** $d-$**Torsions and** $d-$**curvatures**

The nonlinear connection (2) allows us to construct the dual *local adapted bases* of $d-$vector fields

$$\left\{\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - \kappa_{11}^1 p_r^1 \frac{\partial}{\partial p_r^1} \;;\; \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \frac{\kappa_{11}^1}{2}\frac{\partial}{\partial p_i^1} \;;\; \frac{\partial}{\partial p_i^1}\right\} \subset \mathfrak{X}(E^*) \tag{10}$$

and of $d-$covector fields

$$\left\{dt \;;\; dx^i \;;\; \delta p_i^1 = dp_i^1 + \kappa_{11}^1 p_i^1 dt + \frac{\kappa_{11}^1}{2}dx^i\right\} \subset \mathfrak{X}^*(E^*), \tag{11}$$

where $E^* = J^{1*}(\mathbb{R}, M^4)$, whose elements transform as classical tensors. Consequently, all subsequent geometrical objects on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ (as Cartan canonical connection, torsion, curvature etc.) will be described in local adapted components.

Remark 2.1. *A similar local adapted approach is used also by Miron et al. in* [6] *on cotangent bundles or by Udrişte et al. in* [9] *in the multitime Hamiltonian context.*

A general result from [7] and some direct computations imply the following important geometrical result:

Theorem 2.1. *The Cartan canonical* $\Gamma_{[3]}^*$*-linear connection produced by the Chernov-like Hamiltonian (1) has the following adapted local components:*

$$C\Gamma_{[3]}^* = \left(\kappa_{11}^1,\ A_{j1}^k,\ H_{jk}^i,\ C_{j(1)}^{i(k)}\right),$$

where

$$A_{j1}^k = -\frac{\kappa_{11}^1}{2}\left(\delta_j^k - 6S_{[3]jm1}p_r^1 S_{[3]}^{rkm} - \frac{1}{6}\frac{1}{S_{[3]}^{111}}S_{[3]}^{k11}p_j^1 + \right.$$
$$\left. + \frac{1}{S_{[3]}^{111}}S_{[3]}^{k11}S_{[3]jm1}S_{[3]}^{rml}p_r^1 p_l^1\right),$$

$$C_{i(1)}^{j(k)} = -3S_{[3]im1}S_{[3]}^{jkm} + \frac{1}{6}\frac{1}{S_{[3]}^{111}}\left[S_{[3]}^{jk1}\frac{p_i^1}{2} + \delta_i^j S_{[3]}^{k11} + \delta_i^k S_{[3]}^{j11}\right] -$$
$$-\frac{1}{9}\frac{1}{\left(S_{[3]}^{111}\right)^2}S_{[3]}^{j11}S_{[3]}^{k11}p_i^1 - \frac{1}{2}\frac{1}{S_{[3]}^{111}}S_{[3]}^{jkm}p_i^1 p_m^1 -$$

$$-\frac{1}{27}\frac{1}{\left(S_{[3]}^{111}\right)^{3}}S_{[3]}^{j11}S_{[3]}^{k11}S_{[3]}^{m11}p_i^1p_m^1+$$

$$+\frac{1}{36}\frac{1}{\left(S_{[3]}^{111}\right)^{2}}\left[S_{[3]}^{m11}S_{[3]}^{jk1}+S_{[3]}^{j11}S_{[3]}^{km1}+S_{[3]}^{k11}S_{[3]}^{jm1}\right]p_i^1p_m^1,$$

$$H_{jk}^i=\frac{\kappa_{11}^1}{4}\left(g_{[3]jm}\frac{\partial g_{[3]}^{im}}{\partial p_k^1}+g_{[3]km}\frac{\partial g_{[3]}^{im}}{\partial p_j^1}-g_{[3]}^{ir}g_{[3]jm}\frac{\partial g_{[3]}^{ms}}{\partial p_r^1}g_{[3]sk}\right)=$$

$$=-\frac{\kappa_{11}^1}{2}\left(C_{j(1)}^{i(k)}+C_{k(1)}^{i(j)}-g_{[3]}^{ir}C_{k(1)}^{m(r)}g_{[3]mj}\right).$$

Proof. The Chernov derivative operators (10) and (11) and the formulas (8) and (9), applied to the general formulas of the adapted components of the Cartan canonical connection from [7], lead us to the required results:

$$A_{j1}^k=\frac{g_{[3]}^{km}}{2}\frac{\delta g_{[3]mj}}{\delta t},$$

$$H_{jk}^i=\frac{g_{[3]}^{im}}{2}\left(\frac{\delta g_{[3]jm}}{\delta x^k}+\frac{\delta g_{[3]km}}{\delta x^j}-\frac{\delta g_{[3]jk}}{\delta x^m}\right),$$

$$C_{i(1)}^{j(k)}=-\frac{g_{[3]im}}{2}\left(\frac{\partial g_{[3]}^{jm}}{\partial p_k^1}+\frac{\partial g_{[3]}^{km}}{\partial p_j^1}-\frac{\partial g_{[3]}^{jk}}{\partial p_m^1}\right)=-\frac{g_{[3]im}}{2}\frac{\partial g_{[3]}^{jk}}{\partial p_m^1},$$

where, by computations, we have

$$\begin{aligned}\frac{\partial g_{[3]}^{jk}}{\partial p_m^1} &= 2\left(S_{[3]}^{111}\right)^{-1/3}S_{[3]}^{jkm}-\\ &-\frac{1}{9}\left(S_{[3]}^{111}\right)^{-4/3}\left\{S_{[3]}^{jk1}S_{[3]}^{m11}+S_{[3]}^{km1}S_{[3]}^{j11}+S_{[3]}^{mj1}S_{[3]}^{k11}\right\}+\\ &+\frac{4}{27}\left(S_{[3]}^{111}\right)^{-7/3}S_{[3]}^{j11}S_{[3]}^{k11}S_{[3]}^{m11},\end{aligned}$$

$$\begin{aligned}\frac{\partial g_{[3]jk}}{\partial p_m^1} &= -18\left(S_{[3]}^{111}\right)^{1/3}S_{[3]}^{mrs}S_{[3]jr1}S_{[3]ks1}+\\ &+\left(S_{[3]}^{111}\right)^{-2/3}S_{[3]}^{m11}S_{[3]jk1}-\frac{1}{3}\left(S_{[3]}^{111}\right)^{-5/3}S_{[3]}^{m11}p_j^1p_k^1+\\ &+\frac{1}{2}\left(S_{[3]}^{111}\right)^{-2/3}\left(\delta_j^mp_k^1+\delta_k^mp_j^1\right).\end{aligned}$$

For other details, consult the works [1] and [3]. □

Remark 2.2. *It is important to note that the following properties of the d−tensor* $C_{i(1)}^{j(k)}$ *are verified:*

$$C_{i(1)}^{j(k)}=C_{i(1)}^{k(j)},\quad C_{i(1)}^{j(m)}p_m^1=0.$$

Theorem 2.2. *The Cartan canonical connection* $C\Gamma_{[3]}^*$ *of the Chernov-like Hamiltonian (1) has* ***three*** *effective local torsion d−tensors:*

$$P_{(k)i(1)}^{(1)\ (j)}=H_{ki}^j,\quad P_{k(1)}^{i(j)}=C_{k(1)}^{i(j)},\quad R_{(k)1j}^{(1)}=-\frac{1}{2}\left(\frac{d\kappa_{11}^1}{dt}+\kappa_{11}^1\kappa_{11}^1\right)\delta_{kj}.$$

Proof. On the dual 1-jet space $J^{1*}(\mathbb{R},M^4)$ our Cartan canonical connection $C\Gamma^*_{[3]}$ has only three nonzero $d-$torsions (see [7, p. 44]):

$$P^{(1)\ (j)}_{(k)i(1)}=\frac{\partial N^{(1)}_{(k)i}}{\partial p^1_j}+H^j_{ki},\quad R^{(1)}_{(k)1j}=\frac{\delta M^{(1)}_{(k)1}}{\delta x^j}-\frac{\delta N^{(1)}_{(k)j}}{\delta t},\quad P^{k(j)}_{i(1)}=C^{k(j)}_{i(1)}.$$

□

Theorem 2.3. *The Cartan canonical connection $C\Gamma^*_{[3]}$ of the Chernov-like Hamiltonian (1) has* ***five*** *effective local curvature $d-$tensors (see* [7, p. 44]*):*

$$R^l_{i1k}=\frac{\delta A^l_{i1}}{\delta x^k}-\frac{\delta H^l_{ik}}{\delta t}+A^m_{i1}H^l_{mk}-H^m_{ik}A^l_{m1}+C^{l(m)}_{i(1)}R^{(1)}_{(m)1k},$$
$$R^l_{ijk}=\frac{\delta H^l_{ij}}{\delta x^k}-\frac{\delta H^l_{ik}}{\delta x^j}+H^m_{ij}H^l_{mk}-H^m_{ik}H^l_{mj}+C^{l(m)}_{i(1)}R^{(1)}_{(m)jk},$$
$$P^{l\ (k)}_{i1(1)}=\frac{\partial A^l_{i1}}{\partial p^1_k}-C^{l(k)}_{i(1)/1}+C^{l(m)}_{i(1)}P^{(1)\ (k)}_{(m)1(1)},$$
$$P^{l\ (k)}_{ij(1)}=\frac{\partial H^l_{ij}}{\partial p^1_k}-C^{l(k)}_{i(1)|j}+C^{l(m)}_{i(1)}P^{(1)\ (k)}_{(m)j(1)},$$
$$S^{l(j)(k)}_{i(1)(1)}=\frac{\partial C^{l(j)}_{i(1)}}{\partial p^1_k}-\frac{\partial C^{l(k)}_{i(1)}}{\partial p^1_j}+C^{m(j)}_{i(1)}C^{l(k)}_{m(1)}-C^{m(k)}_{i(1)}C^{l(j)}_{m(1)},$$

where

$$C^{l(k)}_{i(1)/1}=\frac{\delta C^{l(k)}_{i(1)}}{\delta t}-\kappa^1_{11}C^{l(k)}_{i(1)},$$
$$C^{l(k)}_{i(1)|j}=\frac{\delta C^{l(k)}_{i(1)}}{\delta x^j}+C^{m(k)}_{i(1)}H^l_{mj}-C^{l(k)}_{m(1)}H^m_{ij}-C^{l(m)}_{i(1)}H^k_{mj}.$$

3. From Chernov-like Hamiltonian to Maxwell-like equations

A given Hamiltonian function H on the dual 1-jet space $J^{1*}(\mathbb{R},M^4)$ determines a geometrical theory dependent of momenta for electromagnetism (see [7]). In the context of Chernov-like Hamilton geometry, we work with an *electromagnetic distinguished* 2*-form*

$$\mathbb{F}^*_{[3]}=F^{(i)}_{(1)j}\delta p^1_i\wedge dx^j,$$

whose local components

$$F^{(i)}_{(1)j}=\frac{h^{11}}{2}\left[g^{jm}_{[3]}N^{(1)}_{(m)i}-g^{im}_{[3]}N^{(1)}_{(m)j}+\left(g^{ir}_{[3]}H^m_{jr}-g^{jr}_{[3]}H^m_{ir}\right)p^1_m\right]$$

are characterized by the following *geometrical Maxwell-like equations* [7, p. 46]

$$F^{(i)}_{(1)j/1}=\frac{1}{2}\mathcal{A}_{\{i,j\}}\left\{\Delta^{(i)}_{(1)1|j}+\mathcal{V}^{(i)(m)}_{(1)(1)}R^{(1)}_{(m)1k}+R^i_{r1k}p^{(r)}_{(1)}\right\},$$
$$\sum_{\{i,j,k\}}F^{(i)}_{(1)j\ |\ k}=-\frac{1}{2}\sum_{\{i,j,k\}}\left[\mathcal{V}^{(i)(m)}_{(1)(1)}R^{(1)}_{(m)jk}+R^i_{rjk}p^{(r)}_{(1)}\right],$$
$$F^{(1)}_{(i)j}\,\Big|^{(k)}_{(1)}=\frac{1}{2}\mathcal{A}_{\{i,j\}}\left\{\mathcal{V}^{(i)(k)}_{(1)(1)|j}-P^{i\ (k)}_{rj(1)}p^{(r)}_{(1)}-\Delta^{(i)}_{(1)r}C^{r(k)}_{j(1)}-\mathcal{V}^{(i)(m)}_{(1)(1)}P^{(1)\ (k)}_{(m)j(1)}\right\},$$

where $\mathcal{A}_{\{i,j\}}$ denotes an alternate sum, we have $p^{(r)}_{(1)}=h_{11}g^{rs}_{[3]}p^1_s$, $\sum_{\{i,j,k\}}$ means a cyclic sum, and the following formulas are good:

$$\Delta^{(i)}_{(1)1}=-h_{11}g^{ik}_{[3]}A^m_{k1}p^1_m,\quad \Delta^{(i)}_{(1)j}=-h_{11}g^{ik}_{[3]}\left[\frac{\kappa^1_{11}}{2}\delta_{kj}+H^m_{kj}p^1_m\right],$$

$$\mathcal{V}^{(i)(j)}_{(1)(1)} = h_{11}\left[g^{ij}_{[3]} - g^{ik}_{[3]}C^{r(j)}_{k(1)}p^1_r\right], \quad \Delta^{(i)}_{(1)1|j} = \frac{\delta\Delta^{(i)}_{(1)1}}{\delta x^j} + \Delta^{(m)}_{(1)1}H^i_{mj},$$

$$A^k_{i1|j} = \frac{\delta A^k_{i1}}{\delta x^j} + A^m_{i1}H^k_{mj} - A^k_{m1}H^m_{ij},$$

$$F^{(1)}_{(i)j/1} = \frac{\delta F^{(1)}_{(i)j}}{\delta t} + F^{(1)}_{(i)j}\kappa^1_{11} - F^{(1)}_{(m)j}A^m_{i1} - F^{(1)}_{(i)m}A^m_{j1},$$

$$F^{(1)}_{(i)j\,|\,k} = \frac{\delta F^{(1)}_{(i)j}}{\delta x^k} - F^{(1)}_{(m)j}H^m_{ik} - F^{(1)}_{(i)m}H^m_{jk},$$

$$F^{(1)}_{(i)j}|_k = \frac{\partial F^{(1)}_{(i)j}}{\partial p^1_k} - F^{(1)}_{(m)j}C^{m(1)}_{i(k)} - F^{(1)}_{(i)m}C^{m(1)}_{j(k)},$$

$$\mathcal{V}^{(i)(k)}_{(1)(1)|j} = \frac{\delta \mathcal{V}^{(i)(k)}_{(1)(1)}}{\delta x^j} + \mathcal{V}^{(m)(k)}_{(1)(1)}H^i_{mj} + \mathcal{V}^{(i)(m)}_{(1)(1)}H^k_{mj}$$

4. From Chernov-like Hamiltonian to Einstein-like equations

From a physical point of view, on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$, the Chernov-like Hamiltonian metric (1) defines the metrical $d-$tensor (gravitational potential)

$$\mathbb{G}^*_{[3]} = h_{11}dt\otimes dt + g_{[3]ij}dx^i\otimes dx^j + h_{11}g^{ij}_{[3]}\delta p^1_i\otimes\delta p^1_j, \tag{12}$$

where $g^{ij}_{[3]}$ and $g_{[3]jk}$ are expressed by (8) and (9).

Remark 4.1. *In the above physical context, the nonlinear connection $\Gamma^*_{[3]}$ (which produces the distinguished 1-forms δp^1_i) prescribes a kind of "interaction" between (t)-, (x)- and (p)-fields.*

We postulate that the gravitational potential $\mathbb{G}_{[3]}$ is governed by the *geometrical Einstein equations* ([7, p. 48])

$$\text{Ric}\left(C\Gamma^*_{[3]}\right) - \frac{\text{Sc}\left(C\Gamma^*_{[3]}\right)}{2}\mathbb{G}^*_{[3]} = \mathcal{K}\mathcal{T}, \tag{13}$$

where $\text{Ric}\left(C\Gamma^*_{[3]}\right)$ is the *Ricci $d-$tensor* associated to the Cartan canonical connection $C\Gamma^*_{[3]}$, $\text{Sc}\left(C\Gamma^*_{[3]}\right)$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathcal{T}$ is the intrinsic *stress-energy $d-$tensor of matter.*

In order to describe the local geometrical Einstein equations for the Chernov-like Hamiltonian (1), we use the adapted basis of vector fields (10). Consequently, by direct computations, we infer [7, pp. 48-49]:

Theorem 4.1. *The Ricci $d-$tensor of the Cartan canonical connection $C\Gamma^*_{[3]}$ produced by the Chernov-like Hamiltonian (1) has the following nonzero local Ricci $d-$tensor components:*

$$\begin{array}{ll} R_{i1} := R^r_{i1r}, & R_{ij} := R^r_{ijr}, \\ R^{(i)}_{(1)1} := -P^{(i)}_{(1)1} = -P^{i\ (r)}_{r1(1)}, & R_{i(1)}^{\ (j)} := -P_{i(1)}^{\ (j)} = P^{r\ (j)}_{ir(1)}, \\ R^{(i)(j)}_{(1)(1)} := -\mathbb{S}^{(i)(j)}_{(1)(1)}, & R^{(i)}_{(1)j} := -P^{(i)}_{(1)j} = -P^{i\ (r)}_{rj(1)}. \end{array}$$

where $\mathbb{S}^{(i)(j)}_{(1)(1)} = S^{i(j)(r)}_{r(1)(1)}$ is the vertical Ricci $d-$tensor field.

Proposition 4.1. *The scalar curvature of the Cartan canonical connection $C\Gamma^*_{[3]}$ produced by the Chernov-like Hamiltonian (1) is given by*

$$\mathrm{Sc}\left(C\Gamma^*_{[3]}\right) = R - S,$$

where $R = g^{ij}_{[3]} R_{ij}$ *and* $S = h^{11} g_{[3]pq} \mathbb{S}^{(p)(q)}_{(1)(1)}$.

Describing the global geometrical Einstein equations (13) in the adapted basis of vector fields (10), the following important geometrical and physical result is derived (cf. [7]):

Theorem 4.2. *The local* ***geometrical Einstein equations*** *that govern the gravitational potential (12) produced by the Chernov-like Hamiltonian (1) are expressed by:*

$$\begin{cases} \dfrac{R-S}{2} h_{11} = -\mathcal{K}\mathcal{T}_{11} \\ R_{ij} - \dfrac{R-S}{2} g_{[3]ij} = \mathcal{K}\mathcal{T}_{ij} \\ \mathbb{S}^{(i)(j)}_{(1)(1)} + \dfrac{R-S}{2} h_{11} g^{ij}_{[3]} = -\mathcal{K}\mathcal{T}^{(i)(j)}_{(1)(1)}, \end{cases} \tag{14}$$

$$\begin{cases} 0 = \mathcal{T}_{1i}, \quad R_{i1} = \mathcal{K}\mathcal{T}_{i1}, \quad -P^{(i)}_{(1)1} = \mathcal{K}\mathcal{T}^{(i)}_{(1)1}, \\ 0 = \mathcal{T}_{1(1)}^{\;\;(i)}, \quad -P_{i(1)}^{\;\;(j)} = \mathcal{K}\mathcal{T}_{i(1)}^{\;\;(j)}, \quad -P^{(i)}_{(1)j} = \mathcal{K}\mathcal{T}^{(i)}_{(1)j}. \end{cases} \tag{15}$$

5. **Conclusion**

Taking into account that the 1-jet spaces and their duals are natural houses for classical and quantum field theories (in their Finsler-Lagrange and Cartan-Hamilton approaches) and correlating with the fact that the equivalence of all non-isotropic directions is also natural one in Finsler and Cartan geometries, we consider that the Chernov-like Hamiltonian (which is a Cartan metric) used in this paper is a good candidate to geometrically model the anisotropic physical phenomena dependent on position and momenta. Consequently, we think that the electromagnetic-like or gravitational-like geometrical theories characterized by some natural Maxwell-like or Einstein-like equations (which are naturally attached to our Chernov-like Hamiltonian) could be useful for the studies of physical phenomena characterized by the dependence on anisotropic directions induced by momenta. Obviously, possible physical interpretations of all geometrical objects constructed in this paper represent an open problem which we hope it will finally find a solution in the future.

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