SEVERAL TYPES OF FUZZY COVERINGS

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In this paper, we introduce new types of fuzzy coverings, namely non-inclusive coverings, disjoint coverings and projection coverings and we study their prop-erties and connections with partitions with crisp sets.

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1. Introduction

Lotfi Zadeh [5] introduced the fuzzy sets in 1965. Fuzzy sets are generalizations of sets which are used to describe the truth level with a degree of uncertainty between zero (meaning false) and one (meaning truth). A *covering* of a set is a collection of fuzzy sets such that every element of the set has the degree 1 in at least one of the fuzzy sets, see [1, Definition 1]. Coverings have a high importance in fuzzy control and machine learning, see [2]. The purpose of this article is to continue our previous work [3, 4] on this topic and to broaden the view of coverings establishing new types of coverings and also the relations between coverings.

The article is divided into eight sections, the first one being this Introduction. In the second section we recall some basic definition regarding fuzzy sets, fuzzy coverings and fuzzy partitions of fuzzy sets. In the third section, we introduce the notion of permutations of coverings as two coverings of the same set which have the fuzzy sets reordered, see Definition 3.**1**, which gives an equivalence relation on coverings. In Section 4 we define an inclusion relation between coverings, see Definition 4.**1**, and we prove that this relation is reflexive and transitive, see Proposition 4.**3**.

In Section 5 we introduce the notion *non-inclusive coverings*, i.e. coverings $(X, (A_i)_{i\in I})$ such that for any $i, j \in I$, we have that $A_i(x) \leq A_j(x)$, for all $x \in X$, if and only if $i = j$. In Theorem 5.**4**, we prove that the relation of inclusion previously defined is a relation of partial order on the class of non-inclusive coverings with respect to the equivalence relation given by the permutation of coverings.

In Section 6 we introduce the notion of *disjoint coverings*, i.e. coverings $(X, (A_i)_{i\in I})$ such that for any $x \in X$ and $i \neq j \in I$ we have $A_i(x) \wedge A_j(x) < 1$. Note that non-inclusive and disjoint coverings are both generalization for partitions with crisp sets.

We prove that non-inclusive normal coverings are disjoint, see Proposition 6.**4**, and we give a characterization of them in terms of partitions, see Theorem 6.**5**.

In Section 7 we introduce the projection covering as the family of fuzzy sets $(A_i)_{i\in I}$ defined on a Cartesian product $X \times Y$ such that the projection on each of the coordinates forms a covering. This implies that there exists $f : X \to Y$ and $g : Y \to X$ such that for any $i \in I$, there exist $x \in X$ and $y \in Y$ such that $A_i(x, f(x)) = A_j(g(y), y) = 1$. In Theorem 7.**12**, we prove that if the above functions f and g are unique, then they are bijective and

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$g = f^{-1}$. Also, we discuss the properties of the converse of a projection covering, i.e. the projection covering of $Y \to X$ with $(A_i^{\smile})_{i \in I}$ such that $A_i^{\smile}(y,x) = A_i(x,y)$ for all $i \in I$, $x \in X$ and $y \in Y$. Section 8 is dedicated to conclusions.

2. Preliminaries

In this section, we recall some basic definitions regarding fuzzy sets and coverings. Let X be a nonempty set.

Definition 2.1. *We say that A is a* fuzzy set*, or a* fuzzy subset of X*, if $A : X \to [0,1]$ is a function. $A(x)$ is the membership degree to which x belongs to A.*

We say that A is a crisp set*, if A is a subset of X in the usual sense. Here, we identify A with its characteristic function, i.e.* $A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ *for all $x \in X$.*

Definition 2.2. *Let $A : X \to [0,1]$ be a fuzzy set. We define:*

(1) $A^{\uparrow} : X \to [0,1]$, $A^{\uparrow}(x) = \begin{cases} 1, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}$

(2) $A^{\downarrow} : X \to [0,1]$, $A^{\downarrow}(x) = \begin{cases} 1, & A(x) = 1 \\ 0, & A(x) < 1 \end{cases}$

Definition 2.3. *Let $A, B : X \to [0,1]$ be two fuzzy sets.*
We say that A is included *in B and denote it by $A \subseteq B$ if for all $x \in X$ we have:*

$$A(x) \le B(x).$$

We say that A is strictly included *in B and denote it by $A \subset B$ if A is included in B and there exists an element $x \in X$ such that $A(x) < B(x)$.*

Definition 2.4. *Let $A : X \to [0,1]$ and $B : Y \to [0,1]$ be two fuzzy sets.*

(1) The union *of the fuzzy sets A and B is the fuzzy set:*

$$A \vee B : X \cup Y \to [0,1],\ (A \vee B)(x) = \max(A(x), B(x)).$$

(2) The intersection *of the fuzzy sets A and B is the fuzzy set:*

$$A \wedge B : X \cap Y \to [0,1],\ (A \wedge B)(x) = \min(A(x), B(x)).$$

(3) The Cartesian product *of the fuzzy sets A and B is the fuzzy set:*

$$A \times B : X \times Y \to [0,1],\ (A \times B)(x,y) = \min(A(x), B(y)).$$

Definition 2.5. *A* fuzzy relation *R between the sets X and Y is a fuzzy subset of $X \times Y$, i.e. $R : X \times Y \to [0,1]$.*

Definition 2.6. *A fuzzy set $A : X \to [0,1]$ is called* normal *if there exists $x \in X$ such that $A(x) = 1$.*

Definition 2.7. *Let $\alpha \in [0,1]$. The $\alpha-$*cut *of the fuzzy set $A : X \to [0,1]$ is denoted A_α and it is defined by:*

$$A_\alpha : X \to [0,1],\ A_\alpha(x) = \begin{cases} 1, & A(x) \ge \alpha \\ 0, & A(x) < \alpha \end{cases}.$$

Remark 2.8. Let A be a fuzzy subset of X. We have that:

(1) If A is a crisp subset, then $A_\alpha = \begin{cases} X, & \alpha = 0 \\ A, & \alpha \in (0,1] \end{cases}$.

(2) $(A_\alpha)_\beta = A_\alpha$ for all $\alpha \in [0,1]$ and $\beta \in (0,1]$.

(3) $\alpha \le \beta$ if and only if $A_\beta \subseteq A_\alpha$, where $\alpha, \beta \in [0,1]$.

Lemma 2.9. If $A, B : X \to [0,1]$ are two fuzzy sets then we have:

$$A \subseteq B \text{ if and only if } A_\alpha \subseteq B_\alpha, \text{ for all } \alpha \in (0,1].$$

Proof. If $A \subseteq B$ then it is clear that $A_\alpha \subseteq B_\alpha$ for any $\alpha \in (0,1]$. In order to prove the converse, assume that $A(x) > B(x)$ and choose $\alpha := A(x)$. Then $A_\alpha(x) = 1 > B_\alpha(x) = 0$, a contradiction. □

Definition 2.10. *(*[1, Definition 1]*) We say that* $(X, (A_i)_{i\in I})$ *is a* fuzzy covering*, or, simply, a* covering *of* X*, if* $A_i : X \to [0,1]$ *are fuzzy sets such that for all* $x \in X$*, there exists* $i \in I$ *with* $A_i(x) = 1$*.*

In this case, we can also say that X *is* covered *by the fuzzy sets* A_i*, where* $i \in I$*.*

Definition 2.11. *Let* $\left(X, (A_i)_{i\in I}\right)$ *and* $\left(I, (B_y)_{y\in Y}\right)$ *be two coverings. Their* composition *is* $\left(X, (A_i)_{i\in I}\right); \left(I, (B_y)_{y\in Y}\right) = \left(X, \bigvee_{i\in I} (A_i(x) \wedge B_y(i))_{y\in Y}\right)$.

Proposition 2.12. *(See* [4, Proposition 2.1] *and* [4, Proposition 3.1]*)*

The composition of two coverings is a covering. Morover, the composition of two normal coverings is a normal covering.

Definition 2.13. *(*[4, Definition 3.1]*) A* normal covering $\left(X, (A_i)_{i\in I}\right)$ *is a covering with the property that for all* $i \in I$*, there exists* $x \in X$ *such that* $A_i(x) = 1$*.*

In other words, a normal covering of X *is a covering with normal fuzzy sets.*

Definition 2.14. *We say that* $(X, (A_i)_{i\in I})$ *is a* partition *if* $(A_i)_{i\in I}$ *is a collection of non-empty, disjoint crisp sets whose union is* X*.*

3. Permutations of coverings

Definition 3.1. *We say that the covering* $\left(X, (B_j)_{j\in J}\right)$ *is a* permutation *of the covering* $\left(X, (A_i)_{i\in I}\right)$ *and we write* $\left(X, (A_i)_{i\in I}\right) \simeq \left(X, (B_j)_{j\in J}\right)$ *if there exists a bijective function* $\rho : I \to J$ *such that*

$$A_i(x) = B_{\rho(i)}(x) \text{ for all } x \in X \text{ and } i \in I.$$

Proposition 3.2. *The relation* $\simeq$ *defined above is an equivalence relation.*

Proof. We have to check that $\simeq$ is:

(1) Reflexive: for any covering $\left(X, (A_i)_{i\in I}\right)$ we have $\left(X, (A_i)_{i\in I}\right) \simeq \left(X, (A_i)_{i\in I}\right)$. Indeed, we can take $\rho : I \to I$ to be the identity function on I.

(2) Symmetric: for any coverings $\left(X, (A_i)_{i\in I}\right)$ and $\left(X, (B_j)_{j\in J}\right)$ we have:

$$\left(X, (A_i)_{i\in I}\right) \simeq \left(X, (B_j)_{j\in J}\right) \text{ if and only if } \left(X, (B_j)_{j\in J}\right) \simeq \left(X, (A_i)_{i\in I}\right).$$

Indeed, if $\rho : I \to J$ satisfies $A_i(x) = B_{\rho(i)}(x)$ for all $x \in X$ and $i \in I$, then $B_j(x) = A_{\rho^{-1}(j)}(x)$ for all $x \in X$ and $j \in J$.

(3) Transitive: for any coverings $\left(X, (A_i)_{i\in I}\right)$, $\left(X, (B_j)_{j\in J}\right)$ and $\left(X, (C_k)_{k\in K}\right)$ such that $\left(X, (A_i)_{i\in I}\right) \simeq \left(X, (B_j)_{j\in J}\right)$ and $\left(X, (B_j)_{j\in J}\right) \simeq \left(X, (C_k)_{k\in K}\right)$ we have:

$$\left(X, (A_i)_{i\in I}\right) \simeq \left(X, (C_k)_{k\in K}\right).$$

Indeed, if $A_i(x) = B_{\rho(i)}(x)$ for all $x \in X$ and $i \in I$ and $B_j(x) = C_{\tau(j)}(x)$ for all $x \in X$ and $j \in J$, then $A_i(x) = C_{(\tau\circ\rho)(i)}(x)$ for all $x \in X$ and $i \in I$.

□

Proposition 3.3. *Let* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(X,(B_j)_{j\in J}\right)$ *be two coverings. The following are equivalent:*

(1) $\left(X,(A_i)_{i\in I}\right) \simeq \left(X,(B_j)_{j\in J}\right)$.

(2) $\left(X,((A_i)_\alpha)_{i\in I}\right) \simeq \left(X,((B_j)_\alpha)_{j\in J}\right)$ *for all* $\alpha \in (0,1]$.

Proof. (1) ⇒ (2) It is clear.

(2) ⇒ (1) Follows from Lemma 2.**9**. □

4. Inclusions of coverings

Definition 4.1. *Let* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(X,(B_j)_{j\in J}\right)$ *be two coverings. We say that the covering* $\left(X,(A_i)_{i\in I}\right)$ *is* included *in the covering* $\left(X,(B_j)_{j\in J}\right)$ *and we denote it by*

$$\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right),$$

if there exists a function $\rho: I \to J$ *such that for all* $x \in X$ *and* $i \in I$ *we have:*

$$A_i(x) \leq B_{\rho(i)}(x).$$

ρ *is called the* inclusion function *associated to the covering inclusion.*

Proposition 4.2. *If* $\left(X,(A_i)_{i\in I}\right)$ *is a fuzzy covering then we have the inclusions:*

(1) $\left(X,(\{x\})_{x\in X}\right) \subseteq \left(X,(A_i)_{i\in I}\right) \subseteq (X,\{X\})$.

(2) $\left(X,\left(A_i^{\downarrow}\right)_{i\in I}\right) \subseteq \left(X,(A_i)_{i\in I}\right) \subseteq \left(X,\left(A_i^{\uparrow}\right)_{i\in I}\right)$.

(3) $\left(X,\left(A_i^{\downarrow}\right)_{i\in I}\right) \subseteq \left(X,((A_i)_\alpha)_{i\in I}\right) \subseteq \left(X,\left(A_i^{\uparrow}\right)_{i\in I}\right)$ *for all* $\alpha \in (0,1]$.

Proof. (1) The first inclusion follows from the fact that for any $x \in X$, there exists $i \in I$ with $A_i(x) = 1$. The second inclusion is clear.

(2) and (3) Taking $\rho = 1_I$, the inclusions follow from the inequalities:

$$A_i^{\downarrow}(x) \leq A_i(x) \leq A_i^{\uparrow}(x), \text{ for all } x \in X,\ i \in I.$$
$$A_i^{\downarrow}(x) \leq (A_\alpha)_i(x) \leq A_i^{\uparrow}(x), \text{ for all } x \in X,\ i \in I.$$

□

Proposition 4.3. *The inclusion relation* $\subseteq$ *between two coverings of the set* X *is:*

(1) *Reflexive: for any covering* $\left(X,(A_i)_{i\in I}\right)$ *we have:* $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(A_i)_{i\in I}\right)$.

(2) *Transitive: for any coverings* $\left(X,(A_i)_{i\in I}\right)$, $\left(X,(B_j)_{j\in J}\right)$ *and* $\left(X,(C_k)_{k\in K}\right)$ *such that*

$$\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right) \text{ and } \left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(C_k)_{k\in K}\right),$$

we have that $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(C_k)_{k\in K}\right)$.

Proof. (1) It is clear, taking $\rho = 1_I$.

(2) Since $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right)$, it follows that there exists $\rho: I \to J$ such that $A_i(x) \leq B_{\rho(i)}(x)$ for all $x \in X$ and $i \in I$.

Since $\left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(C_k)_{k\in K}\right)$, it follows that there exists $\theta: J \to K$ such that $B_j(x) \leq C_{\theta(j)}(x)$ for all $x \in X$ and $j \in J$.

Therefore, for all $x \in X$ and all $i \in I$ we have $A_i(x) \leq B_{\rho(i)}(x) \leq C_{(\theta\circ\rho)(i)}(x)$, hence $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(C_k)_{k\in K}\right)$. □

Proposition 4.4. *Any two fuzzy coverings* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(X,(B_j)_{j\in J}\right)$ *of the set* X *can be in one of the mutual disjoint cases:*

(1) $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right)$ *and* $\left(X,(B_j)_{j\in J}\right) \nsubseteq \left(X,(A_i)_{i\in I}\right)$.

(2) $\left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(A_i)_{i\in I}\right)$ *and* $\left(X,(A_i)_{i\in I}\right) \nsubseteq \left(X,(B_j)_{j\in J}\right)$.

(3) $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right)$ *and* $\left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(A_i)_{i\in I}\right)$ *and* $\left(X,(A_i)_{i\in I}\right) \not\simeq \left(X,(B_j)_{j\in J}\right)$.

(4) $\left(X,(A_i)_{i\in I}\right) \simeq \left(X,(B_j)_{j\in J}\right)$.

(5) Neither of them is included in the other one.

Proof. We can easily produce examples for (1), (2), (4) and (5).

For (3) we can choose two fuzzy coverings $\left(X,(A_i)_{i\in I}\right)$ and $\left(X,(B_j)_{j\in J}\right)$ such that $|I| \neq |J|$ and $A_{i_0} = B_{j_0} = X$ for some $i_0 \in I$ and $j_0 \in J$. Then the inclusion function of $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right)$ is $\rho : I \to J$, $\rho(i) = j_0$ for all $i \in I$, respectively the inclusion function of $\left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(A_i)_{i\in I}\right)$ is $\theta : J \to I$, $\theta(j) = i_0$ for all $j \in J$. □

5. **Non-inclusive coverings**

Definition 5.1. *We say that* $\left(X,(A_i)_{i\in I}\right)$ *is a* non-inclusive covering *if for any* $i, j \in I$*:*

$$A_i(x) \leq A_j(x), \text{for all } x \in X \text{ if and only if } i = j.$$

A covering which is not non-inclusive it is called inclusive.

Proposition 5.2. *A partition with crisp sets is a non-inclusive covering.*

Proof. Let $\left(X,(A_i)_{i\in I}\right)$ be a partition of X, i.e. A_i are nonempty crisp subsets of X. It is enough to prove that the only function $\rho : I \to I$ with the property

$$A_i(x) \leq A_{\rho(i)}(x) \text{ for all } x \in X \text{ and } i \in I, \tag{5.1}$$

is $\rho = 1_I$. Indeed, assume there exists ρ as above with $\rho(i_0) \neq i_0$ for some $i_0 \in I$. Let $x_0 \in X$ such that $A_{i_0}(x_0) = 1$. From (5.1) it follows that $A_{\rho(i_0)}(x_0) = 1$, hence A_{i_0} and $A_{\rho(i_0)}$ are not disjoint, which is a contradiction. □

Lemma 5.3. *Let* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(X,(B_j)_{j\in J}\right)$ *be two non-inclusive coverings such that* $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right)$ *and* $\left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(A_i)_{i\in I}\right)$*. Then:*

$$\left(X,(A_i)_{i\in I}\right) \simeq \left(X,(B_j)_{j\in J}\right).$$

Proof. If $\left(X,(A_i)_{i\in I}\right) \subseteq \left(X,(B_j)_{j\in J}\right)$ then there exists a function $\rho : I \to J$ such that for any $x \in X$ and $i \in I$ we have $A_i(x) \leq B_{\rho(i)}(x)$. Similarly if $\left(X,(B_j)_{j\in J}\right) \subseteq \left(X,(A_i)_{i\in I}\right)$ we have $\theta : J \to I$ such that for any $x \in X$ and $j \in J$ we have $B_j(x) \leq A_{\theta(j)}(x)$.

It follows that for any $x \in X$ and $i \in I$, we have

$$A_i(x) \leq B_{\rho(i)}(x) \leq A_{(\rho\circ\theta)(i)}(x). \tag{5.2}$$

Since $(X,(A_i)_{i\in I})$ is non-inclusive, from (5.2) it follows that $\rho \circ \theta = 1_I$ and, in particular, that $A_i(x) = B_{\rho(i)}(x)$.

Similarly, for any $x \in X$ and $j \in J$ we have $B_j(x) = A_{\theta(j)}(x)$ and $\theta \circ \rho = 1_J$. It follows that ρ and θ are bijective functions and therefore $\left(X,(A_i)_{i\in I}\right) \simeq \left(X,(B_j)_{j\in J}\right)$. □

Theorem 5.4. *Let $(X,(A_i)_{i\in I})$ and $(Y,(B_j)_{j\in J})$ be two non-inclusive coverings. We denote $\overline{(X,(A_i)_{i\in I})}$, respectively $\overline{(Y,(B_j)_{j\in J})}$, their classes with respect to $\simeq$.*

We say that $\overline{(X,(A_i)_{i\in I})}$ is included in $\overline{(Y,(B_j)_{j\in J})}$, if $(X,(A_i)_{i\in I})$ is included in $(Y,(B_j)_{j\in J})$ and we write $\overline{(X,(A_i)_{i\in I})} \subseteq \overline{(Y,(B_j)_{j\in J})}$.

Then $\subseteq$ is an order relation on $\mathcal{N}/\simeq$, where $\mathcal{N}$ is the family of all non-inclusive coverings.

Proof. It is easy to see that $\subseteq$ is well defined for classes of non-inclusive coverings. Hence, the conclusion follows from Proposition 4.**3** and Lemma 5.**3**. □

6. **Disjoint coverings**

Definition 6.1. *We say that $\left(X,(A_i)_{i\in I}\right)$ is a* disjoint covering *if for any $x \in X$ and $i \neq j \in I$ we have:*

$$A_i(x) \wedge A_j(x) < 1.$$

Remark 6.2. A partition with crisp sets is a disjoint covering. Indeed, if $(A_i)_{i\in I}$ is a partition of X, then $A_i(x) \wedge A_j(x) = 0 < 1$ for any $i \neq j$.

Proposition 6.3. *If $\left(X,(A_i)_{i\in I}\right)$ is a disjoint covering then for any $x \in X$ there exists a unique $i \in I$ such that $A_i(x) = 1$.*

Proof. Assume there exist $i, j \in I$ with $i \neq j$ such that $A_i(x) = A_j(x) = 1$. We get $A_i(x) \wedge A_j(x) = 1$. Contradiction. □

Proposition 6.4. *Let $\left(X,(A_i)_{i\in I}\right)$ be a disjoint normal covering. Then $\left(X,(A_i)_{i\in I}\right)$ is non-inclusive.*

Proof. Since $\left(X,(A_i)_{i\in I}\right)$ is a normal covering then for any $i \in I$ there exists $x \in I$ such that $A_i(x) = 1$.

Assume, by contradiction, that there exist $i \neq j \in I$ such that for any $x \in X$ we have $A_i(x) \leq A_j(x)$. Let $x \in X$ such that $A_i(x) = 1$. It follows that $A_j(x) = 1$ and hence $A_i(x) \wedge A_j(x) = 1$. Contradiction. □

Theorem 6.5. *Let $\left(X,(A_i)_{i\in I}\right)$ be a covering. The following are equivalent:*

(1) $\left(X,(A_i)_{i\in I}\right)$ *is a disjoint normal covering.*
(2) $\left(X,\left(A_i^{\downarrow}\right)_{i\in I}\right)$ *is a partition.*

Proof. (1) ⇒ (2) Let $\left(X,(A_i)_{i\in I}\right)$ be a normal disjoint covering.

(i) For any $x \in X$ we have $\max_{i\in I} A_i(x) = 1$. Applying $\downarrow$ we get $\bigcup_{i\in I} A_i^{\downarrow}(x) = X$.
(ii) For any $i, j \in I$, $i \neq j$ and $x \in X$ we have $A_i(x) \wedge A_j(x) < 1$. Applying $\downarrow$ we get $A_i^{\downarrow} \cap A_j^{\downarrow} = \emptyset$;
(iii) For any $i \in I$ there exists $x \in X$ such that $A_i(x) = 1$. Applying $\downarrow$ we get $A_i^{\downarrow} \neq \emptyset$.

From (i),(ii) and (iii) it follows that $\left(X,\left(A_i^{\downarrow}\right)_{i\in I}\right)$ is a partition.

(2) ⇒ (1) Let $\left(X,(A_i)_{i\in I}\right)$ be a partition. Similarly we have:

(i) $\bigcup_{i\in I} A_i^{\downarrow}(x) = X$ implies that $\max_{i\in I} A_i(x) = 1$ for all $x \in X$;
(ii) For any $i \neq j \in I$ we have $A_i^{\downarrow} \cap A_j^{\downarrow} = \emptyset$, which implies that for any $x \in X$ we have $A_i(x) \wedge A_j(x) < 1$;
(iii) For any $i \in I$ we have $A_i^{\downarrow} \neq \emptyset$, hence for any $i \in I$ there exists $x \in X$ such that $A_i(x) = 1$.

From (i),(ii) and (iii) it follows that $\big(X,(A_i)_{i\in I}\big)$ is a disjoint normal covering. □

Theorem 6.6. *Let* $\big(X,(A_i)_{i\in I}\big)$ *and* $\Big(X,(B_j)_{j\in J}\Big)$ *be two disjoint coverings such that* $\big(X,(A_i)_{i\in I}\big)\subseteq\Big(X,(B_j)_{j\in J}\Big)$ *with the associated inclusion function* $\rho: I\to J$. *Then:*

(1) $\Big(B_j^{\downarrow},\big(A_i|_{B_j}\big)_{i\in\rho^{-1}(j)}\Big)$ *is a disjoint covering for any* $j\in J$.

(2) $\Big(X,(A_i)_{\rho(i)\in K},(B_j)_{j\in J\setminus K}\Big)$ *is a disjoint covering for any non-empty subset* $K\subset J$.

Proof. (1) We notice that $B_j^{\downarrow}$ is a set for any $j\in J$ and it can be covered by the B_j-fuzzy sets $A_i|_{B_j}$.

Let $x\in B_j^{\downarrow}$. We prove that there exists $i\in\rho^{-1}(j)$ such that $A_i(x)=1$. If this is not the case, then there exists $i'\in I\setminus\rho^{-1}(j)$ such that $A_{i'}(x)=1$. Since $\big(X,(A_i)_{i\in I}\big)\subseteq\Big(X,(B_j)_{j\in J}\Big)$ it follows that $1=A_{i'}(x)\le B_{\rho(i')}(x)$. Therefore, $B_j(x)\wedge B_{\rho(i')}(x)=1$ and $j\ne\rho(i')$. This contradicts the fact that $\Big(X,(B_j)_{j\in J}\Big)$ is a disjoint covering.

(2) The proof is similar. □

7. Projection coverings

Definition 7.1. *A* projection covering *of* $X\times Y$ *is a pair* $\big(X\times Y,(A_i)_{i\in I}\big)$ *such that* $\Big(X,(A_i(-,y))_{(y,i)\in Y\times I}\Big)$ *and* $\Big(Y,(A_i(x,-))_{(x,i)\in X\times I}\Big)$ *are coverings.*

Proposition 7.2. *Let* $A_i: X\times Y\to[0,1]$ *be fuzzy sets. The following are equivalent:*

(1) $\big(X\times Y,(A_i)_{i\in I}\big)$ *is a projection covering.*

(2) There exist two functions $f: X\to Y$ *and* $g: Y\to X$ *such that for all* $x\in X$ *there exists* $i\in I$ *with* $A_i(x,f(x))=1$ *and similarly for all* $y\in Y$ *there exists* $j\in I$ *with* $A_j(g(y),y)=1$.

(3) $\Big(X,\big(\bigvee_{i\in I}(A_i(-,y))\big)_{y\in Y}\Big)$ *and* $\Big(Y,\big(\bigvee_{i\in I}(A_i(x,-))\big)_{x\in X}\Big)$ *are normal coverings.*

The function $f: X\to Y$ *is called a* left subfunction *of the projection covering* $\big(X\times Y,(A_i)_{i\in I}\big)$ *and the function* $g: Y\to X$ *is called a* right subfunction *of the projection covering.*

Proof. (1) ⇒ (2) If $\big(X\times Y,(A_i)_{i\in I}\big)$ is a projection covering then we know that:

(i) $\Big(X,(A_i(-,y))_{(y,i)\in Y\times I}\Big)$ is a covering which means that for all $x\in X$ there exist $y\in Y$ and $i\in I$ such that $A_i(x,y)=1$. Then we can choose $f(x)=y$.

(ii) $\Big(Y,(A_i(x,-))_{(x,i)\in X\times I}\Big)$ is covering which means that for all $y\in Y$ there exist $x\in X$ and $i\in I$ such that $A_i(x,y)=1$. Then we can choose $g(y)=x$.

Hence, the functions f and g are well defined with the required properties.

(2) ⇒ (1) For all $x\in X$ there exist $y=f(x)\in Y$ and $i\in I$ such that $A_i(x,f(x))=1$ which means that $\Big(X,(A_i(-,y))_{(y,i)\in Y\times I}\Big)$ is a covering. Similary $\Big(Y,(A_i(x,-))_{(x,i)\in X\times I}\Big)$ is a covering.

(1) ⇒ (3) Assume that there exists an $x\in X$ such that for all $y\in Y$ we have $\vee_{i\in I}A_i(x,y)<1$. Then it means that $\Big(X,(A_i(-,y))_{(y,i)\in Y\times I}\Big)$ isn't a covering. Contradiction.

(3) ⇒ (2) For all $x\in X$ there exists $y\in Y$ and $i\in I$ such that $A_i(x,y)=1$. We can define $f: X\to Y$ such that $f(x)=y$. Similarly for y. □

Remark 7.3. If X and Y are finite sets and $\left(X \times Y, (A_i)_{i\in I}\right)$ is a projection covering then the number of left subfunctions is $\prod_{x\in X}\left(\Sigma_{y\in Y}\left(\max_{i\in I}\left(A_i^{\downarrow}(x,y)\right)\right)\right)$ and the number of right subfunctions is $\prod_{y\in Y}\left(\Sigma_{x\in X}\left(\max_{i\in I}\left(A_i^{\downarrow}(x,y)\right)\right)\right)$.

Indeed, the result follows from the fact that for each $x \in X$, a left subfunction can assign any $y \in Y$ such that there exists $i \in I$ with $A_i(x, y) = 1$.

Definition 7.4. *The* composition *of the projection coverings* $\left(X \times Y, (A_i)_{i\in I}\right)$ *and* $\left(Y \times Z, (B_j)_{j\in J}\right)$ *is:*

$$\left(X \times Y, (A_i)_{i\in I}\right); \left(Y \times Z, (B_j)_{j\in J}\right) = \left(X \times Z, \left(\bigvee_{y\in Y} (A_i(x,y) \wedge B_j(y,z))\right)_{(i,j)\in I\times J}\right).$$

Proposition 7.5. *The composition of two projection coverings is a projection covering.*

Proof. Since $\left(X \times Y, (A_i)_{i\in I}\right)$ and $\left(Y \times Z, (B_j)_{j\in J}\right)$ are projection coverings for all $x \in X$ there exists $y \in Y$ and $i \in I$ such that $A_i(x, y) = 1$ and also for this $y \in Y$ there exists $z \in Z$ and $j \in J$ such that $B_j(y, z) = 1$.

Then $\left(X, \left(\bigvee_{y\in Y} (A_i(-,y) \wedge B_j(y,z))\right)_{(i,j,z)\in I\times J\times Z}\right)$ is a covering.

Similarly $\left(Z, \left(\bigvee_{y\in Y} (B_j(x,y) \wedge B_j(y,-))\right)_{(i,j,x)\in I\times J\times X}\right)$ is a covering. □

Proposition 7.6. *Let* $\left(X \times Y, (A_i)_{i\in I}\right)$ *and* $\left(Y \times Z, (B_j)_{j\in J}\right)$ *be projection coverings.*

(1) If f_1 *and* f_2 *are left subfunctions of* $\left(X \times Y, (A_i)_{i\in I}\right)$ *and* $\left(Y \times Z, (B_j)_{j\in J}\right)$ *then* $f_2 \circ f_1$ *is a left subfunction of* $\left(X \times Y, (A_i)_{i\in I}\right); \left(Y \times Z, (B_j)_{j\in J}\right)$.

(2) If g_1 *and* g_2 *are right subfunctions of* $\left(X \times Y, (A_i)_{i\in I}\right)$ *and* $\left(Y \times Z, (B_j)_{j\in J}\right)$ *then* $g_1 \circ g_2$ *is a right subfunction of* $\left(X \times Y, (A_i)_{i\in I}\right); \left(Y \times Z, (B_j)_{j\in J}\right)$.

(3) If f *is a bijective left subfunction of* $\left(X \times Y, (A_i)_{i\in I}\right)$ *then* f^{-1} *is a right subfunction of* $\left(X \times Y, (A_i)_{i\in I}\right)$.

Proof. (1) If f_1 is a left subfunction of $\left(X \times Y, (A_i)_{i\in I}\right)$ then for all $x \in X$ there exists $i \in I$ such that $A_i(x, f(x)) = 1$. Similarly if f_2 is a left covering of $\left(Y \times Z, (B_j)_{j\in J}\right)$ the for all $y \in I$ there exists $j \in J$ such that $B_j(y, g(y)) = 1$.

Let $x \in X$ such that $f_1(x) = y$. We have that:

$$A_i(x, f_1(x)) \wedge B_j(f_1(x), f_2(f_1(x))) = 1.$$

(2) The proof is similar with the proof of (1).

(3) If f is bijective, that is $f(x) = y$ is equivalent with $f^{-1}(y) = x$, then

$$1 = A_i(x, f(x)) = A_i(f^{-1}(y), y) \text{ for all } y \in Y.$$

□

Example 7.7. Let $X = \{x, y, z\}$, $Y = \{a, b\}$ be two sets and $R = \{(x, a), (y, a), (y, b), (z, b)\}$ be a relation. Then $(X \times Y, R)$ is a projection covering. The projection covering has two left subfunctions:

	x	y	z
f_1	a	a	b
f_2	a	b	b

and four right subfunctions:

	a	b
g_1	x	y
g_2	y	y
g_3	x	z
g_4	y	z

Note that all the left subfunctions are surjective, but not all the right subfunctions are injective.

Definition 7.8. *The* converse *of the projection covering* $\big(X \times Y, (A_i)_{i\in I}\big)$ *is*

$$\big(X \times Y, (A_i)_{i\in I}\big)^{\smile} := \big(Y \times X, (A_i^{\smile})_{i\in I}\big),$$

where $A_i^{\smile}(y,x) := A_i(x,y)$ *for all* $x \in X$ *and* $y \in Y$*.*

Proposition 7.9. *The converse of a projection covering is a projection covering.*

Proof. If $\big(X \times Y, (A_i)_{i\in I}\big)$ is a projection covering then $\Big(X, (A_i(-,y))_{(i,y)\in I\times Y}\Big)$ is a covering which means that $\Big(X, (A_i^{\smile}(y,-))_{(i,y)\in Y\times I}\Big)$ is a covering.

Similarly, $\Big(Y, (A_i^{\smile}(-,x))_{(i,y)\in Y\times I}\Big)$ is a covering. □

Proposition 7.10. *(1) If f is a left subfunction of $\big(X \times Y, (A_i)_{i\in I}\big)$ then f is a right subfunction of $\big(X \times Y, (A_i)_{i\in I}\big)^{\smile}$.*

(2) If g is a right subfunction of $\big(X \times Y, (A_i)_{i\in I}\big)$, g is a left subfunction of $\big(X \times Y, (A_i)_{i\in I}\big)^{\smile}$.

(3) If $\big(X \times Y, (A_i)_{i\in I}\big)$ is a projection covering then $\Big(G_f, \big(A_i|_{G_f}\big)_{i\in I}\Big)$ and $\Big(G_g, \big(A_i^{\smile}|_{G_g}\big)_{i\in I}\Big)$ are coverings, where $f : X \to Y$ and $g : Y \to X$ are (left, respectively right) subfunctions and G_f denotes the graph of the function f.

Proof. (1) and (2) The results follow from the fact that $A_i(x, f(x)) = A_i^{\smile}(f(x), x)$ for all $x \in X$. (3) Since $f : X \to Y$ is a left subfunction, it follows that for all $x \in X$ there exists $i \in I$ such that $A_i(x, f(x)) = 1$ which means that for all $(x, y) \in G_f$ there exists $i \in I$ such that $A_i(x, y) = 1$. Hence $\Big(G_f, \big(A_i|_{G_f}\big)_{i\in I}\Big)$ is a covering. □

Proposition 7.11. *If $\big(X \times Y, (A_i)_{i\in I}\big)$ is a projection covering, the following are equivalent:*

(1) *The projection covering has a single left subfunction;*
(2) *All the right subfunctions are injective.*

Proof. (1) ⇒ (2) Assume that there exist two different $y_1, y_2 \in Y$ and $i, j \in I$ such that for an element $x_0 \in X$ we have $A_i(x_0, y_1) = A_j(x_0, y_2) = 1$. Then we can assume that the unique left subfunction of the projection covering $f : X \to Y$ has $f(x_0) = y_1$. But the function $g : X \to Y$, where $g(x) = f(x)$ for all $x \neq x_0$ and $g(x_0) = y_2$ is a different left subfunction. We get a contradiction.

This means that for all $x \in X$ there exists a unique $y_0 \in Y$ such that $A_i(x, y_0) = 1$, where $i \in I$ and $A_i(x, y) < 1$ for all $y \neq y_0$ and for all $i \in I$. We define $h : Y \to X$, where $h(y) = x$ such that $A_i(x, y) = 1$. We can notice that:

- The function is correctly defined.
- The function is injective.
- The function is a right subfunction.

(2) $\Rightarrow$ (1) Let $X_y = \{x \in X | A_i(x, y) = 1, i \in I\}$ for all $y \in Y$. Assume there exist two different $y_1, y_2 \in Y$ such that $X_{y_1} \cap X_{y_2} \neq \emptyset$. We can define right subfunction $g : Y \to X$ such that $g(y_1) = g(y_2)$, which is a contradiction. So for all $y \in Y$ there exist a unique $x \in X$ such that $A_i(x, y) = 1$. Then the we notice that $\big(X, (Y_x)_{x \in X}\big)$ is a partition and that the only right function we can define $f : X \to Y$, where $f(x) = y$ such that $x \in Y_x$ is surjective. □

Theorem 7.12. *Let $\big(X \times Y, (A_i)_{i \in I}\big)$ be a projection covering.*

(1) If the projection covering has a single left subfunction $f : X \to Y$ then f is surjective.
(2) If the projection covering has a single left subfunction $f : X \to Y$ and a single right subfunction $g : Y \to X$ then f and g are bijective and $f^{-1} = g$.

Proof. (1) Since $\Big(Y, (A_i(x, -))_{(x,i) \in X \times I}\Big)$ is a covering, it follows that for all $y \in Y$ there exists $x \in X$ and $i \in I$ such that $A_i(x, y) = 1$. We define a function $f : X \to Y$, with the property that $A_i(x, f(x)) = 1$ for all $x \in X$ and we note that:

- The function is correctly defined.
- The function is surjective by definition.
- The function f is a left subfunction of $\big(X \times Y, (A_i)_{i \in I}\big)$.

(2) From Proposition 7.**11** it follows that g is injective. From Proposition 7.**10** we know that the unique right subfunction in $\Big(X \times Y, (A_i)_{(i \in I}\Big)$ is, in the same time, the unique left subfunction in $\big(Y \times X, (A_i^{\smile})_{i \in I}\big)$ and applying (1) again we get that g is surjective, which means g is bijective. Similarly f is bijective. We notice that f^{-1} is a right subfunction of the projection covering and since g is the unique right subfunction it means that $f^{-1} = g$. □

8. **Conclusion**

In this article, we introduced and studied new types of fuzzy coverings: *non-inclusive coverings*, i.e. coverings $(X, (A_i)_{i \in I})$ such that for any $i, j \in I$ and $x \in X$ we have that $A_i(x) \leq A_j(x)$ if and only if $i = j$, *disjoint coverings*, i.e. coverings $(X, (A_i)_{i \in I})$ such that for any $x \in X$ and $i \neq j \in I$ we have $A_i(x) \wedge A_j(x) < 1$, and *projection coverings*, i.e. family $(A_i)_{i \in I}$ of subsets of $X \times Y$ such that the projection on each of the coordinates forms a covering. Further research include a category theory approach of these notions.

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