ITERATIVE APPROXIMATION OF FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS ON HADAMARD MANIFOLDS

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In this paper, we propose a new iterative algorithm for finding the common solution of the fixed points of nonexpansive mapping and the solution of the pseudomonotone variational inequality on Hadamard manifolds, and we proved the strong convergence theorem of the generated algorithm, which mainly extended and improved some recent related results.

Keywords: Hadamard manifold, fixed points, variational inequality problem, Convergence.

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1. Introduction

The fixed point problem of a nonlinear mapping T is to find x such that

$$x = Tx.$$

Many practical problems can be converted into the fixed point problem, such as optimization problems, variational inequality problems, equilibrium problems and split feasibility problems and so on. Fixed point problems and related problems have been extensively studied by many researchers, see, e.g., [1, 3, 4, 5, 7, 9, 11, 12, 14, 20, 21, 23, 27] and [30]-[57].

Let K be a nonempty closed convex subset of real Hilbert space H, and $T : K \to H$ be a mapping. The variational inequality problem (VIP) is to find a point $x^* \in K$ such that

$$\langle Tx^*, y - x^*\rangle \geq 0, \forall y \in K. \quad (1)$$

The variational inequality problem of nonlinear mapping was firstly introduced by Stampacchia [25], this theory has extensive and significant applications in so many fields, such as optimization problems, equilibrium problems, split feasibility problems and so on. The variational inequality problem has been widely studied and numerous iterative algorithms for solving VIP have been proposed and analyzed, such as projection algorithm [10], extragradient algorithm [17], subgradient extragradient algorithm [6] and so on. However, the post researches have been mainly concentrated on the linear space, see [13, 10, 15, 33, 34].

In 2004, Xu [37] presented a viscosity algorithm and obtained the strong theorems in Hilbert spaces and Banach spaces.

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Let M be an Hadamard manifold, TM the tangent bundle of M, K a nonempty closed geodesic convex subset of M. the exp is a exponential mapping. In 2003, Nemeth [22] introduced the variational inequality problem on Hadamard manifold, which is as follows:

$$find\ x^* \in K\ such\ that\ \langle Tx^*, \exp_{x^*}^{-1} y\rangle \geq 0,\ \forall y \in K, \tag{2}$$

where $T : K \to TM$ is a vector field, that is $Tx^* \in T_xM$ for each $x \in K$, and $\exp^{-1}$ is the inverse of exponential mapping.

It is easily seen that variational inequality problem(2) on Hadamard manifold is an extension of variational inequality problem (1). If $M = R^n$, the variational inequality problem (2) will be ascribed to the variational inequality problem (1).

Limited by the nonlinearity of manifolds, the research progress of VIP (2) is slow. However, some algorithms were proposed and analyzed. In 2009, Li [18] studied VIP (2) on Riemannian manifold. Recently, Tang [27, 28, 29] introduced the Korpelevich algorithm, proximal point algorithm and projection algorithm and studied the VIP (2) on Hadamard manifold. Very recently, Chen [8] proposed two modified extragradient algorithm with pseudomonotone vector field for solving the VIP (2) on Hadamard manifold. Konrawut [16] introduced the new Tseng's extragradient methods with pseudomonotone vector field for solving the VIP (2) on Hadamard manifold.

Recently, Li [19] studied the fixed point problem for Halpern iterative algorithm (3) and obtain the strong convergence on Hadamard manifolds, this results extended the results of (3) from the classical linear spaces to the setting of manifolds,

$$x_{n+1} = \exp_u(1-\alpha_n)\exp_u^{-1} Tx_n, \quad n \geq 0, \tag{3}$$

where $u, x_0 \in K$ and the sequence $\{\alpha_n\} \subset (0, 1)$.

The Halpern algorithm (3) is equivalent to

$$x_{n+1} = \gamma_n(1-\alpha_n), \quad n \geq 0, \tag{4}$$

where $\gamma_n : [0, 1] \to M$ is the geodesic joining u to Tx_n (i.e. $\gamma(0) = u$ and $\gamma(1) = T(x_n)$).

Motivated by the above works of Li [19], Xu [37] and Konrawut [16], in this paper, we consider the problem of finding

$$x^* \in Fix(S) \cap VIP(T, K) \tag{5}$$

in the setting of Hadamard manifold, where S is nonexpansive mapping, T is pseudomonotone vector field, $Fix(S)$ denotes the set of fixed point of the nonexpansive mapping S, $VIP(T, K)$ denotes the set of solutions of the VIP (2). Especially, The solution set of the problem (5) is denoted by $\mathbf{S} \triangleq Fix(S) \cap VIP(T, K)$. we present a new iterative algorithm and prove that the sequence generated by the algorithm converges strongly to a common element of problem (5) on Hadamard manifolds.

2. **Preliminaries**

Let M be a connected $m-$dimensional manifold and $p \in M$. T_pM denotes the tangent space of M at p. To become Riemannian manifold, we always assume M can be endowed with the Riemannian metric $\langle , \rangle$ and the corresponding norm $\|\cdot\|$.

Given a piecewise smooth curve $c : [a, b] \to M$ joining p to q, we define the length of c by

$$L(c) = \int_a^b \|c'(t)\| dt.$$

Then, the Riemannian distance $d(p, q)$ is the minimal length over all such curves joining p to q.

Let ∇ be a Levi-Civita connection associated with the Riemannian manifold M. If ϕ is a smooth curve, a smooth vector field F along ϕ is called parallel if $\nabla_{\phi'} F = 0$. If ϕ' is parallel, then ϕ is a geodesic, and $\|\phi'\|$ is a constant. Based on the definition of Riemannian distance $d(p, q)$, it is easily seen that a geodesic joining p to q in M is called a minimizing geodesic if its length equals to $d(p, q)$.

A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for any $t \in R$, Hopf-Rinow theorem asserts that if M is complete then any pair of points in M can be joined by a minimizing geodesic. A complete simply connected Riemannian manifold of non-positive sectional curvature is named a Hadamard manifold. Throughout this paper, we assume that M is a Hadamard manifold, the following results are well known and will be useful.

Let $\gamma(t) : [a, b] \to R$, the parallel transport $P_{\gamma,\gamma(a),\gamma(b)} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ on the tangent bundle TM on the $\gamma(t)$ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(\nu) = F(\gamma(b)), \forall a, b \in R, \nu \in T_{\gamma(a)}M,$$

where F is a unique vector field such that $F(\gamma(a)) = \nu$ and $\nabla_{\gamma'(t)} F = 0, \forall t \in [a, b]$.

If $\gamma(t) : [a, b] \to R$ is a minimizing geodesic joining a to b, $P_{\gamma,b,a}$ is denoted by $P_{b,a}$ and $P^{-1}_{b,a} = P_{a,b}$ generally. Recall that, for $a, b \in R$, for all $u, v \in T_{\gamma(a)}M$, we have

$$\langle P_{\gamma(b),\gamma(a)}u, P_{\gamma(b),\gamma(a)}v\rangle = \langle u, v\rangle.$$

Definition 2.1. *The vector field $T : K \to TM$ is called monotone, if*

$$\langle Tx, exp^{-1}_x y\rangle + \langle Ty, exp^{-1}_y x\rangle \leq 0, \forall x, y \in K.$$

Definition 2.2. *The vector field $T : K \to TM$ is called pseudomonotone, if*

$$\langle Tx, exp^{-1}_x y\rangle \geq 0 \text{ implies that } \langle Ty, exp^{-1}_y x\rangle \leq 0, \forall x, y \in K.$$

Definition 2.3. *The vector field $T : K \to TM$ is called Γ-Lipschitz continuous, if there exists $\Gamma > 0$ such that*

$$\|P_{x,y}Ty - Tx\| \leq \Gamma d(x, y), \forall x, y \in K.$$

Definition 2.4. *The mapping $T : K \to K$ is called nonexpansive, if the following inequality holds*

$$d(Tx, Ty) \leq d(x, y).$$

Definition 2.5. *The mapping $T : K \to K$ is said to be contractive, if there exists a constant $\alpha \in (0, 1)$ and the following inequality holds*

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Lemma 2.1 ([37])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real number such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and δ_n is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\delta_n| < \infty$ or $\limsup_{n\to\infty} \frac{\delta_{n+1}}{\gamma_n} \leq 0$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([2])**.** *Let $\triangle(p, q, r)$ be a geodesic triangle in a Hadamard manifold M, then there exists $p', q', r' \in R^2$ such that*

$$d(p, q) = \|p' - q'\|, d(q, r) = \|q' - r'\|, d(r, p) = \|r' - p'\|.$$

Remark 2.1. *The triangle $\triangle(p', q', r')$ is said to be the comparison triangle of the geodesic triangle $\triangle(p, q, r)$, which is unique up to isometry of M.*

Lemma 2.3 ([19]). *Let $\triangle(p, q, r)$ be a geodesic triangle in a Hadamard manifold M, and $\triangle(p', q', r')$ is its comparison triangle.*

(i) Let $\alpha, \beta, \gamma\,(\alpha', \beta', \gamma')$ be the angles of $\triangle(p, q, r)\,(\triangle(p', q', r'))$ at the vertices $p, q, r\,(p', q', r')$. Then the following inequalities hold:

$$\alpha \leq \alpha',\ \ \beta \leq \beta',\ \ \gamma \leq \gamma'.$$

(ii) Let z be a point in the geodesic joining p to q, and z' is its comparison point in the interval $[p', q']$. Suppose that $d(z, p) = \|z' - p'\|$ and $d(z, q) = \|z' - q'\|$. Then the the following inequality holds:

$$d(z, r) \leq \|z - r'\|$$

.

Lemma 2.4 ([24]). *Let $d : M \times M \to R$ be the distance function. Then d is a convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$, the following inequality holds for all $t \in [0, 1]$:*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1 - t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1))$$

.

Let P_K denotes the projection onto K, and for a point $p \in M$, $P_K(p)$ is defined by

$$P_K(p) = \{p_0 \in K | d(p, p_0) \leq d(p, q),\ \forall q \in K\}.$$

Lemma 2.5 ([36]). *For any point $p \in M$, $P_K(p)$ is a singleton and the following inequality holds*

$$\langle \exp^{-1}_{P_K(p)} p, \exp^{-1}_{P_K(p)} q\rangle \leq 0,\ \forall q \in K.$$

Lemma 2.6 ([18]). *Let $x^* \in M$ and $\{x_n\} \subset M$ with $x_n \to x^*$ as $n \to \infty$. Then the following conclusions hold:*

(i) For any $y \in M$, then $\exp^{-1}_{x_n} y \to \exp^{-1}_{x^*} y$ and $\exp^{-1}_{y} x_n \to \exp^{-1}_{y} x^*$ as $n \to \infty$.

(ii) If $v_n \in T_{x_n}M$ and $v_n \to v^*$ as $n \to \infty$, then $v^* \in T_{x^*}M$.

(iii) Let $\eta_n, \nu_n \in T_{x_n}M$ and $\eta_*, \nu^* \in T_{x^*}M$ if $\eta_n \to \eta^*$ and $\nu_n \to \nu^*$ as $n \to \infty$, then $\langle \eta_n, \nu_n\rangle \to \langle \eta^*, \nu^*\rangle$ as $n \to \infty$.

Lemma 2.7 ([18]). *If $x, y \in M$ and $w \in T_yM$, then*

$$\langle w, -\exp^{-1}_{y} x\rangle = \langle w, P_{y,x} \exp^{-1}_{x} y\rangle = \langle P_{y,x}w, \exp^{-1}_{y} x\rangle.$$

Lemma 2.8 ([8]). *If $x, y, z \in M$ and $w \in T_xM$, then*

$$\langle w, \exp^{-1}_{x} y\rangle \leq \langle w, \exp^{-1}_{x} z\rangle + \langle w, P_{x,z} \exp^{-1}_{z} y\rangle$$

Lemma 2.9 ([24]). *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle in M. Then*

(i) $d^2(x_1, x_2) + d^2(x_2, x_3) - 2\langle \exp^{-1}_{x_2} x_1, \exp^{-1}_{x_2} x_3\rangle \leq d^2(x_3, x_1)$,

(ii) $d^2(x_1, x_2) \leq \langle \exp^{-1}_{x_1} x_3, \exp^{-1}_{x_1} x_2\rangle + \langle \exp^{-1}_{x_2} x_3, \exp^{-1}_{x_2} x_1\rangle$.

(iii) If γ is the angle at x_1, then we have

$$\langle \exp^{-1}_{x_1} x_2, \exp^{-1}_{x_1} x_3\rangle = d(x_2, x_1)d(x_1, x_3) \cos \gamma.$$

3. Main results

Let $x_0 \in M$, $\{\alpha_n\} \subset (0,1)$, $f : M \to M$ a contraction with coefficient α and $S : K \to K$ a nonexpansive mapping with $Fix(S) \neq \emptyset$. Let the vector field T be pseudomonotone and Γ-Lipschitz continuous. Now, we define the following iteration scheme. For an initial point $x_0 \in K$, $\{x_n\}$ is a sequence generated by the form

$$x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)} Sy_n, \tag{6}$$

$$y_n = \exp_{z_n} \mu_n(P_{z_n,x_n}Tx_n - Tz_n), \tag{7}$$

$$\langle P_{z_n,x_n}Tx_n - \frac{1}{\mu_n}\exp^{-1}_{z_n} x_n, \exp^{-1}_{z_n} y\rangle \geq 0, \forall y \in K, \tag{8}$$

where the iterative equality (6) is equivalent to the following equality:

$$x_{n+1} = \gamma_n(1-\alpha_n), \quad n \geq 0, \tag{9}$$

where $\gamma_n : [0,1] \to M$ is the geodesic joining $f(x_n)$ to sy_n (i.e. $\gamma(0) = f(x_n)$ and $\gamma(1) = Sy_n$) and $\{\alpha_n\} \subset (0,1)$ and $\{\mu_n\}$ satisfies

(H1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(H2) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$;

(H3) either $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ or $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

(H4) $0 < \mu' \leq \mu_n \leq \mu'' < \frac{1}{\Gamma}$, $\Gamma > 0$.

Proposition 3.1. *Let K be a nonempty closed geodesic convex subset of M, $S : K \to K$ a nonexpansive mapping with $Fix(S) \neq \emptyset$, and $f : K \to K$ a contraction with coefficient α. Let the vector field T be pseudomonotone and Γ-Lipschitz continuous. Let the sequence $\{x_n\}$ be generated by the equations (6)-(8). Let $\{\alpha_n\} \subset (0,1)$ and $\{\mu_n\}$ be two sequences satisfying the conditions (H1)-(H4), then*

$$d^2(y_n, \bar{x}) \leq d^2(x_n, \bar{x}) - (1 - \Gamma^2\mu_n^2)d^2(x_n, z_n), \forall \bar{x} \in \boldsymbol{S}.$$

Proof. The proof is similar with the proof of Lemma 5 in [16]. We only need to do the following operation:

Replacing x_{n+1} in Lemma 5 in [16] by y_n.

Replacing y_n in Lemma 5 in [16] by z_n.

Replacing x in Lemma 5 in [16] by $\bar{x}$.

The required conclusion is completed.

□

Theorem 3.1. *Let K be a nonempty closed geodesic convex subset of M, $S : K \to K$ a nonexpansive mapping with $Fix(S) \neq \emptyset$, and $f : K \to K$ a contraction with coefficient α. Let the vector field T be pseudomonotone and Γ-Lipschitz continuous. Let the sequence $\{x_n\}$ be generated by the equations (6)-(8). Let $\{\alpha_n\} \subset (0,1)$ and $\{\mu_n\}$ be two sequences satisfying the conditions (H1)-(H4), Then the sequence $\{x_n\}$ converges to $\tilde{x}$, where $\tilde{x}$ is the unique solution of the variation inequality*

$$\langle \exp^{-1}_{\tilde{x}} f(\tilde{x}), \exp^{-1}_{\tilde{x}} x\rangle \leq 0, \forall x \in \boldsymbol{S}. \tag{10}$$

Proof. The proof is divided into five steps.

Step 1. We show $\{x_n\}$ is bounded.

Take $x \in \mathbf{S}$ and fix n, by the Proposition 3.1, we have

$$d^2(y_n, x) \leq d^2(x_n, x) - (1 - \Gamma^2\mu_n^2)d^2(x_n, z_n), \forall x \in \mathbf{S}.$$

By the condition (H4), we know $1-\Gamma^2\mu_n^2>0$, then

$$d^2(y_n,x)\le d^2(x_n,x), forn\ge 0.$$

By the convexity of the Riemannian distance in Lemma 2.4 and the nonexpansive of S, we have that

$$\begin{aligned}
d(x_{n+1},x) &= d(\gamma_n(1-\alpha_n),x)\\
&\le \alpha_n d(\gamma_n(0),x)+(1-\alpha_n)d(\gamma_n(1),x)\\
&= \alpha_n d(f(x_n),x)+(1-\alpha_n)d(Sy_n,x)\\
&\le \alpha_n d(f(x_n),f(x))+\alpha_n d(f(x),x)+(1-\alpha_n)d(y_n,x)\\
&\le \alpha_n\alpha d(x_n,x)+\alpha_n d(f(x),x)+(1-\alpha_n)d(x_n,x)\\
&= [1-(1-\alpha)\alpha_n]d(x_n,x)+\alpha_n d(f(x),x)\\
&\le \max\{d(x_n,x),\frac{1}{1-\alpha}d(f(x),x)\}.
\end{aligned}$$

By induction

$$d(x_n,x)\le \max\{d(x_0,x),\frac{1}{1-\alpha}d(f(x),x)\},\forall n\ge 0.$$

Then $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$ and $\{Sy_n\}$.

Step 2. We show $\lim_{n\to\infty} d(y_n,z_n)=0$.

First, we show that $\lim_{n\to\infty} d(x_n,z_n)=0$. From Proposition 3.1, we know easily

$$\begin{aligned}
d^2(x_n,z_n) &\le \frac{1}{1-\Gamma^2\mu_n^2}d^2(x_n,x)\\
&\le \frac{1}{1-\Gamma^2\mu''^2}d^2(x_n,x)\\
&= (\bar\Gamma)^2 d^2(x_n,x),
\end{aligned}$$

where $\bar\Gamma=\frac{1}{\sqrt{1-\Gamma^2\mu''^2}}$, then

$$\begin{aligned}
d(x_n,z_n) &\le \bar\Gamma d(x_n,x)\\
&\le \bar\Gamma\{[1-(1-\alpha)\alpha_{n-1}]d(x_{n-1},x)+\alpha_{n-1}d(f(x),x)\}\\
&\le \bar\Gamma[(1-\bar\alpha_{n-1})d(x_{n-1},x)+\alpha_{n-1}d(f(x),x)].
\end{aligned}$$

Let $m\le n$, by induction, we have

$$\begin{aligned}
d(x_n,z_n) &\le \bar\Gamma\Pi_{j=m}^{n-1}(1-\bar\alpha_j)d(x_m,x)+\bar\Gamma\Sigma_{j=m}^{n-1}\{\alpha_j\Pi_{i=j+1}^{n-1}(1-\bar\alpha_j)\}d(f(x),x)\\
&\le \bar\Gamma C_1\Pi_{j=m}^{n-1}(1-\bar\alpha_j)+\bar\Gamma\Sigma_{j=m}^{n-1}\{\alpha_j\Pi_{i=j+1}^{n-1}(1-\bar\alpha_j)\}d(f(x),x)
\end{aligned}$$

where $\Pi_{j=m}^{n}\alpha(j)=1$ as $m>n$.

By taking $n\to\infty$, we have

$$d(x_n,z_n)\le \bar\Gamma C_1\Pi_{j=m}^{\infty}(1-\bar\alpha_j)+\bar\Gamma\Sigma_{j=m}^{\infty}\{\alpha_j\Pi_{i=j+1}^{\infty}(1-\bar\alpha_j)\}d(f(x),x).$$

From condition (H1) and (H2), we get

$$\lim_{m\to\infty}\Pi_{j=m}^{\infty}(1-\bar\alpha_j)=0 \tag{11}$$

and

$$\lim_{m\to\infty}\Sigma_{j=m}^{\infty}\{\alpha_j\Pi_{i=j+1}^{\infty}(1-\bar\alpha_j)\}=0. \tag{12}$$

Now, adding the above equalities (11) and (12) and taking $m\to\infty$, we get

$$\lim_{m\to\infty} d(x_n,z_n)=\lim_{n\to\infty} d(x_n,z_n)=0. \tag{13}$$

Secondly, we show that $\lim_{n\to\infty} d(y_n, z_n) = 0$.

From the algorithm (7), and T is Γ-Lipschitz continuous, we get

$$\begin{aligned} d(y_n, z_n) &= \|\exp_{z_n}^{-1} y_n\| \\ &= \mu_n \|P_{z_n,x_n} T x_n - T z_n\| \\ &\le \mu_n \|T x_n - T z_n\| \\ &\le \mu_n \Gamma d(x_n, z_n) \\ &\le d(x_n, z_n). \end{aligned} \tag{14}$$

By the squeeze theorem, we obtain

$$\lim_{n\to\infty} d(y_n, z_n) = 0.$$

Step 3. Since $\{x_n\}$ is bounded, we can assume that there exists some subsequence $\{x_{n_k}\}$ of $\{x_n\}$, and $\lim_{k\to\infty} x_{n_k} = \hat{x}$. In this step, we show that $\hat{x} \in \mathbf{S}$.

First, we show that $\hat{x} \in Fix(S)$.

$$\begin{aligned} d(x_{n+1}, Sy_n) &= d(\gamma_n(1-\alpha_n), Sy_n) \\ &\le \alpha_n d(\gamma_n(0), Sy_n) + (1-\alpha_n) d(\gamma_n(1), Sy_n) \\ &\le \alpha_n d(f(x_n), Sy_n) + (1-\alpha_n) d(Sy_n, Sy_n) \\ &\le \alpha_n d(f(x_n), Sy_n), \end{aligned}$$

by the boundedness of $f(x_n)$ and Sy_n, and the condition (H1), we get

$$\lim_{n\to\infty} d(x_{n+1}, Sy_n) = 0. \tag{15}$$

Then,

$$d(Sy_{n_k}, \hat{x}) \le d(Sy_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, \hat{x}),$$

by the squeeze theorem, we obtain

$$\lim_{n\to\infty} Sy_{n_k} = \hat{x}.$$

And

$$d(y_{n_k}, \hat{x}) \le d(y_{n_k}, z_{n_k}) + d(z_{n_k}, x_{n_k}) + d(x_{n_k}, \hat{x}),$$

by the squeeze theorem,(14) and (13), we obtain

$$\lim_{k\to\infty} y_{n_k} = \hat{x}. \tag{16}$$

Then, we have

$$\begin{aligned} d(\hat{x}, S\hat{x}) &\le d(\hat{x}, x_{n_k+1}) + d(x_{n_k+1}, Sy_{n_k}) + d(Sy_{n_k}, S\hat{x}) \\ &\le d(\hat{x}, x_{n_k+1}) + d(x_{n_k+1}, Sy_{n_k}) + d(y_{n_k}, \hat{x}) \end{aligned}$$

by the squeeze theorem,(15) and (16), we obtain

$$d(\hat{x}, S\hat{x}) = \lim_{k\to\infty} d(\hat{x}, S\hat{x}) = 0,$$

that is to say $\hat{x} \in Fix(S)$.

Secondly, we show that $\hat{x} \in VIP(T, K)$.

By the algorithm (8), we get

$$\langle \mu_{n_k} P_{z_{n_k}, x_{n_k}} T x_{n_k} - \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} y \rangle \ge 0, \forall y \in K,$$

by Lemma 2.8, we have

$$\begin{aligned}0 &\geq \langle \exp_{z_{n_k}}^{-1} x_{n_k} - \mu_{n_k} P_{z_{n_k},x_{n_k}} T x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle \\ &= \langle \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle - \mu_{n_k}\langle P_{z_{n_k},x_{n_k}} T x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle \\ &= \langle \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle - \mu_{n_k}\langle P_{z_{n_k},x_{n_k}} T x_{n_k}, \exp_{z_{n_k}}^{-1} x_{n_k}\rangle \\ &- \mu_{n_k}\langle P_{z_{n_k},x_{n_k}} T x_{n_k}, P_{z_{n_k},x_{n_k}} \exp_{x_{n_k}}^{-1} y\rangle \\ &= \langle \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle - \mu_{n_k}\langle P_{z_{n_k},x_{n_k}} T x_{n_k}, \exp_{z_{n_k}}^{-1} x_{n_k}\rangle \\ &- \mu_{n_k}\langle T x_{n_k}, \exp_{x_{n_k}}^{-1} y\rangle.\end{aligned}$$

It follows from the above inequality and Lemma 2.7 that

$$\begin{aligned}\langle T x_{n_k}, \exp_{x_{n_k}}^{-1} y\rangle &\geq \frac{1}{\mu_{n_k}}\langle \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle - \langle P_{z_{n_k},x_{n_k}} T x_{n_k}, \exp_{z_{n_k}}^{-1} x_{n_k}\rangle \\ &\geq \frac{1}{\mu_{n_k}}\langle \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} y\rangle + \langle T x_{n_k}, \exp_{x_{n_k}}^{-1} z_{n_k}\rangle,\end{aligned}$$

by taking $k \to \infty$ and Lemma 2.6, we get

$$\langle T\hat{x}, \exp_{\hat{x}}^{-1} y\rangle \geq 0, \forall y \in K.$$

That is to say $\hat{x} \in VIP(T, K)$.

Thus, $\hat{x} \in \mathbf{S}$.

Step 4. We show that

$$\limsup_{n\to\infty}\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} Sy_n\rangle \leq 0,$$

where $\tilde{x}$ satisfies the variational inequality (10).

Since $\{y_n\}$ is bounded by step1, $\limsup_{n\to\infty}\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} Sy_n\rangle$ exists, and by the definition of the upper limit, we can find a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} Tx_n\rangle = \lim_{k\to\infty}\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} Tx_{n_k}\rangle$$

Without loss of generality, since $\{y_n\}$ is bounded, we can assume that $y_{n_k} \to \bar{x} \in M$ as $k \to \infty$.

By step 3, we can easily get $\hat{x} \in \mathbf{S}$.

Hence, by Lemma 2.6, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} Sy_n\rangle &= \lim_{k\to\infty}\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} Sy_{n_k}\rangle \\ &= \langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} \hat{x}\rangle \\ &\leq 0.\end{aligned}$$

This proof is completed.

Step 5. We Show $\lim_{n\to\infty} x_n = \tilde{x}$.

By Lemma 2.1, it suffices to verify that

$$d^2(x_{n+1}, \tilde{x}) \leq (1 - \tilde{\alpha_n}) \leq d^2(x_n, \tilde{x}) + \tilde{\alpha_n}\tilde{\beta_n}, \forall n \geq 0.$$

To this aim, we fix $n \geq 0$ and set $u = f(x_n)$, $p = Sy_n$, $q = \tilde{x}$. Consider the geodesic triangle $\triangle(u, p, q)$ and its comparison triangle $\triangle(u', p', q')$, then

$$d(f(x_n), \tilde{x}) = d(u, q) = \|u' - q'\|,$$

and

$$d(Sy_n, \tilde{x}) = d(p, q) = \|p' - q'\|.$$

Then the iterative algorithm (6) can be written as

$$\begin{aligned} x_{n+1} &= \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)} Sy_n \\ &= \exp_u(1-\alpha_n)\exp^{-1}_u p, \quad n \geq 0, \end{aligned}$$

The comparison point of x_{n+1} is denoted by x'_{n+1} as follows:

$$x'_{n+1} = \alpha_n u' + (1-\alpha_n)p', \quad n \geq 0.$$

Let β and β' denote the angles at q and q', respectively. And we know $\beta < \beta'$ by Lemma 2.3, so $\cos\beta' \leq \cos\beta$.

Then, from Lemma 2.3, we get

$$\begin{aligned} d^2(x_{n+1},\tilde{x}) &\leq \|x'_{n+1} - q'\|^2 \\ &= \|\alpha_n(u'-q') + (1-\alpha_n)(p'-q')\|^2 \\ &= \alpha_n^2\|u'-q'\|^2 + (1-\alpha_n)^2\|p'-q'\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\|u'-q'\|\|p'-q'\|\cos\beta' \\ &\leq \alpha_n^2 d^2(f(x_n),\tilde{x}) + (1-\alpha_n)^2 d^2(Sy_n,\tilde{x}) \\ &\quad + 2\alpha_n(1-\alpha_n)d(f(x_n),\tilde{x})d(Sy_n,\tilde{x})\cos\beta \\ &\leq (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + \alpha_n^2 d^2(f(x_n),\tilde{x}) \\ &\quad + 2\alpha_n(1-\alpha_n)[d(f(x_n),f(\tilde{x})) + d(f(\tilde{x}),\tilde{x})]d(Sy_n,\tilde{x})\cos\beta \\ &\leq (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n(1-\alpha_n)\alpha d^2(x_n,\tilde{x}) \\ &\quad + \alpha_n^2 d^2(f(x_n),\tilde{x}) + 2\alpha_n(1-\alpha_n)\langle \exp^{-1}_{\tilde{x}} f(\tilde{x}), \exp^{-1}_{\tilde{x}} Sy_n\rangle \\ &\leq (1-2\alpha_n+\alpha_n^2+2\alpha\alpha_n(1-\alpha_n))d^2(x_n,\tilde{x}) \\ &+ \quad \alpha_n[\alpha_n d^2(f(x_n),\tilde{x}) + 2(1-\alpha_n)\langle \exp^{-1}_{\tilde{x}} f(\tilde{x}), \exp^{-1}_{\tilde{x}} Sy_n\rangle] \\ &= (1-\tilde{\alpha}_n)d^2(x_n,\tilde{x}) + \tilde{\alpha}_n \cdot \tilde{\beta}_n, \end{aligned}$$

where

$$\tilde{\alpha}_n = 2\alpha_n - \alpha_n^2 - 2\alpha\alpha_n(1-\alpha_n),$$

and

$$\tilde{\beta}_n = \frac{\alpha_n d^2(f(x_n),\tilde{x}) + 2(1-\alpha_n)\langle \exp^{-1}_{\tilde{x}} f(\tilde{x}), \exp^{-1}_{\tilde{x}} Sy_n\rangle}{2-\alpha_n-2\alpha(1-\alpha_n)}.$$

It is easily seen that $\lim_{n\to\infty}\tilde{\alpha}_n = 0$ and $\sum_{n=0}^{\infty}\tilde{\alpha}_n = \infty$. And $\limsup_{n\to\infty}\tilde{\beta}_n \leq 0$ by the condition (H1), step 1 and step 4. By Lemma 2.1, we get $\lim_{n\to\infty} x_n = \tilde{x}$. □

4. **Conclusion**

In this paper, an iterative algorithm for approximating a common element of the set of fixed points of a nonexpansive mapping and the solutions of variational inequality on Hadamard manifolds has been proposed, and we have proved the sequence generated by the suggested algorithm strongly converges to the common solution of problem (5). The results present in this paper not only extended some recent results, and but also solved the common element of the fixed point of a nonexpansive mapping and the solutions of variational inequality on Hadamard manifolds.

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REFERENCES

[1] T. D. Benavides and B. Gavira, *The fixed point property for multivalued nonexpansive mappings*, J. Math. Anal. Appl. **328** (2007), 1471-1483.

[2] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer Verlag, Berlin, Heidelberg, New york, 1999.

[3] F. E. Browder and W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197-228.

[4] L. C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19**(2018), 487–502.

[5] L. C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20**(2019), 113–133.

[6] Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148**(2)(2011), 318-335.

[7] S. Chang, *The Mann and Ishikawa iterative application of solution to variational inclusion with accretive type mappings*, Comput. Math. Appl. **37** (1999), 17-24.

[8] J. Chen, S. Liu and X. Chang, *Modified tseng's extragradient methods for variational inequality on Hadamard manifolds*, Applicable Analysis, **100** (12)(2021), 2627-2640.

[9] Q. L. Dong, Y. Peng and Y. Yao, *Alternated inertial projection methods for the split equality problem*, J. Nonlinear Convex Anal., **22**(2021), 53–67.

[10] F. Facchinei and J. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. II. Springer Series in Operations Research. Springer, New York, 2003.

[11] Q. Fan, J. Peng and H. He, *Weak and strong convergence theorems for the split common fixed point problem with demicontractive operators*, Optimization, **70** (5-6)(2021), 1409-1423.

[12] K. Goebel, W.A. Kirk, *Topics in metric fixed point theory, in: Cambridge Studies in Advanced Mathematics*, vol. **28**, Cambridge University Press, Cambridge, 1990.

[13] Y. Hu and W. Song, *Weak sharp solutions for variational inequalities in Banach spaces*, J. Math. Anal. Appl. **374** (1)(2011), 118-132.

[14] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. 41(1974)147-150.

[15] A. Iusem and M. Nasri, *Korpelevich's method for variational inequality problems in Banach spaces*, J. Glob. Optim. **50**(1)(2011), 59-76.

[16] K. Khammahawong, K. Poom and P. Chaipinya, *New Tseng's extragradient methods for pseudomonotone variational inequality problems in Hadamard manifolds*, Fixed Point Theory Algorithms Sci Eng, **2021**(2021), Artticle Number 5.

[17] G. Korpelevič, *An extragradient method for finding saddle points and for other problems*, Ekon. Mat. Metody, **12**(4)(1976), 747-756.

[18] C. Li, G. Lopez and V. Marquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. Lond. Math. Soc. **79(3)**(2009), 663-683.

[19] C. Li, G. Lopez and V. Marquez, *Iterative algorithms for nonexpansive mappings on Hadamard manifolds*, Tanwanese Journal of Mathematics, **14**(2)(2010), 541-559.

[20] W. R. Mann,*Mean value methods in iterations*, Proc. Amer. Math. Soc. **4**(1953), 506-510.

[21] G. Marino, H.K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **318** (1)(2006), 43-52.

[22] S. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal. **52**(5)(2003),1491-1498.

[23] D. R. Sahu, A. Pitea and M. Verma, *A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems*, Numer. Algor., **83**(2020), 421–449.

[24] T. Sakai, Riemanian Geometry, *Translations of Mathematical Monographs 149, American Mathematical Society*, Providence, RI, 1996.

[25] G. Stampacchia, *Formes biliniaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris **258**(1964), 4413-4416.

[26] W. Takahashi, *Nonlinear Functional Analysis-Fixed Point Theory and its Applications*, Yokohama Publishers inc, Yokohama, 2000.

[27] G. Tang and N. Huang, *Korpelevich method for variational inequality problems on Hadamard manifolds*, J. Glob. Optim. **54**(3)(2012), 493-509.

[28] G. Tang, X. Wang and H. Liu, *A projection-type method for variational inequalities on Hadamard manifolds and verification of solution existence*, Optimization, **64** (5)(2015), 1081-1096.

[29] G. Tang, L. Zhou and N. Huang, *The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds*, Optim. Lett. **7**(4)(2013), 779-790.

[30] B. S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., **275**(2016), 147–155.

[31] B. S. Thakur, D. Thakur and M. Postolache, *A new iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat, **30**(2016), 2711–2720.

[32] D. Thakur, B. S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings*, J. Inequal. Appl., **2014**(2014), Art. No. 328.

[33] D. Thong and D. Hieu, *Inertial extragradient algorithms for strongly pseudomonotone variational inequalities*, J. Comput. Appl. Math. **341** (2018),80-98.

[34] D. Thong and D. Hieu, *Weak and strong convergence theorems for variational inequality problems*, Numer. Algorithms, **78**(4)(2018), 1045-1060.

[35] G. I. Usurelu and M. Postolache, *Algorithm for generalized hybrid operators with numerical analysis and applications*, J. Nonlinear Variational Anal. (in printing).

[36] R. Walter, *On the metric projection onto convex sets in Riemannian spaces*, Aechivder Mathematik, **25** (1974), 91-98.

[37] H. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298**(2004), 279-291.

[38] H. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. **314**(2006), 631-643.

[39] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18**(2017), 875–882.

[40] Y. Yao, H. Li and M. Postolache, *Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions*, Optim., in press, DOI: 10.1080/02331934.2020.1857757.

[41] Y. Yao, Y.C. Liou and M. Postolache, *Self-adaptive algorithms for the split problem of the demicontractive operators*, Optim., **67**(2018), 1309-1319.

[42] Y. Yao, Y. C. Liou and J. C. Yao, *Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015**(2015), Article ID 127.

[43] Y. Yao, Y. C. Liou and J. C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843–854.

[44] Y. Yao, M. Postolache and Y. C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013**(2013), Art. No. 201.

[45] Y. Yao, M. Postolache, Y. C. Liou and Z. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., **10**(2016), 1519–1528.

[46] Y. Yao, M. Postolache and J. C. Yao, *Iterative algorithms for the generalized variational inequalities*, U.P.B. Sci. Bull., Series A, **81**(2019), 3–16.

[47] Y. Yao, M. Postolache and J. C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7**(2019), Art. No. 61.

[48] Y. Yao, M. Postolache and J. C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U.P.B. Sci. Bull., Series A, **82(1)**(2020), 3–12.

[49] Y. Yao, M. Postolache and Z. Zhu, *Gradient methods with selection technique for the multiple-sets split feasibility problem*, Optim., **69**(2020), 269–281.

[50] Y. Yao, X. Qin and J. C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19**(2018), 407–415.

[51] Y. Yao, J. C. Yao, Y.C. Liou and M. Postolache, *Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms*, Carpathian J. Math., **34**(2018), 459–466.

[52] C. Zhang, Z. Zhu, Y. Yao and Q. Liu, *Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities*, Optim., **68**(2019), 2293–2312.

[53] X. Zhao, M. A. Kobis, Y. Yao and J. C. Yao, *A projected subgradient method for nondifferentiable quasiconvex multiobjective optimization problem*, J. Optim. Theory Appl., **190**(2021), 82–107.

[54] X. Zhao, J. C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U. P. B. Sci. Bull., Series A, **82(3)**(2020), 43–52.

[55] X. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., **69**(2020), 1987–2002.

[56] H. Zhou, *Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces*, Nonlinear Analysis, **69**(2008), 456-462.

[57] L. J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, **83(1)**(2021), 3–14.