ϕ-INJECTIVITY AND CHARACTER INJECTIVITY OF BANACH MODULES

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In this paper, some homological notions of Banach A-modules such as ϕ-injectivity, 0-injectivity and character injectivity are discussed, where ϕ is a non-zero multiplicative linear functional on Banach algebra A. We characterize 0-injectivity of Banach A-modules in terms of ϕ-injectivity over the unitization of the Banach algebra A. With some examples, we show difference between these notions. As a consequence, we show that for $LUC(G)$, the space of all left uniformly continuous functions on locally compact group G, as a $L^1(G)$-module these notions are equivalent. This leads to a generalization of some known results.

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1. introduction

Projectivity and flatness of Banach modules were originally introduced and studied by A. Ya. Helemskii. It is a well known theorem due to Helemskii that if A is an amenable (contractible) Banach algebra, then every Banach A-module E is flat (projective) [5, Theorem VII.2.29]. However, the converse is a long standing open problem yet. Recently, Nasr-Isfahani and Soltani Renani in [11] introduced and studied the notions of ϕ-injectivity and ϕ-flatness for Banach modules which is a somewhat restrictive category of Banach modules and their morphisms, where $\phi : A \longrightarrow \mathbb{C}$ is a non-zero homomorphism. They obtained an important connection between the notions ϕ-flatness and ϕ-amenability introduced by Kaniuth, Lau and Pym in [6]; see also [7] and [9]. Indeed, it is shown in [11, Proposition 3.1] that the Banach algebra A is left ϕ-amenable if and only if every Banach left A-module E is ϕ-flat. This result gives a positive answer to the above open problem raised by Helemskii in this homology setting based on character ϕ.

Afterwards, the authors in [3] studied some hereditary properties of ϕ-injectivity for Banach A-modules related to the closed ideals of Banach algebra A. It is shown that if J is a left invariant complemented ideal in A, then ϕ-injectivity of J and A/J is equivalent to the ϕ-injectivity of A as Banach left A-module. As a main result, it is shown in [3, Theorem

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3.2] that the semigroup algebra $\ell^1(\mathbb{N}_\wedge)$ as a Banach left $\ell^1(\mathbb{N}_\wedge)$-module is ϕ-injective for each character ϕ, although is not injective. The present paper is as a continuation of this latter work and organized as follows:

In section 2, after recalling some background definitions and notations we introduce the notions of 0-injectivity and character injectivity of Banach modules. From this follows the relation between 0-injectivity and ϕ-injectivity of Banach modules over the unitization of Banach algebra A. Using this fact, we can construct many examples of Banach modules which are not ϕ-injective. In section 3, we list some examples on harmonic analysis and show that the notions 0-injectivity, character injectivity and injectivity are different. Moreover, we concentrate on character injectivity of some classes of Banach modules over the group algebra $L^1(G)$. Indeed, we prove that the notions of injectivity and character injectivity are equivalent for some classes of Banach modules. This result generalizes a known result due to Nasr-Isfahani and Soltani Renani.

2. ϕ-injectivity and 0-injectivity

Suppose that E and F are Banach spaces. We denote by $B(E,F)$ the set of bounded linear operators from E into F. For a Banach algebra A, we also denote by **A-mod** the category of Banach left A-modules. For each $E, F \in$ **A-mod**, let ${}_AB(E,F)$ be the closed subspace of $B(E,F)$ consisting of the left A-module morphisms. An operator $T \in B(E,F)$ is called *admissible* if there exists $S \in B(F,E)$ such that $T \circ S \circ T = T$. A Banach left A-module E is called *injective* if for each $F, K \in$ **A-mod** and admissible monomorphism $T \in {}_AB(F,K)$, the induced map ${}_AB(K,E) \longrightarrow {}_AB(F,E)$ is onto. Throughout the paper, we regard the space $B(A,E)$ as a Banach A-bimodule with the following module actions:

$$(a \cdot T)(b) = T(ba), \qquad (T \cdot a)(b) = T(ab) \quad (T \in B(A,E), a, b \in A).$$

Let $\Delta(A)$ be the set of all characters of the Banach algebra A, $\phi \in \Delta(A)$ and $E \in$ **A-mod**. We set,

$$I(\phi, E) = \operatorname{span}\{a \cdot x - \phi(a)x : a \in A, x \in E\}.$$

We recall that ϕ-injectivity of Banach modules was first introduced by Nasr-Isfahani and Soltani Renani in [11].

Definition 2.1. *Let A be a Banach algebra, $\phi \in \Delta(A)$ and $E \in$ **A-mod**. We say that E is ϕ-injective if, for each $F, K \in$ **A-mod** and admissible monomorphism $T : F \longrightarrow K$ with $I(\phi, K) \subseteq \operatorname{Im}(T)$, the induced map $T_E : {}_AB(K,E) \longrightarrow {}_AB(F,E)$ defined by $T_E(R) = R \circ T$ is onto.*

Let $E \in$ **A-mod** and $A^\sharp = A \times \mathbb{C}$ denotes the unitization of A. Then E is a Banach left $A^\sharp$-module with the following left module action:

$$(a, \lambda) \cdot x = a \cdot x + \lambda x \qquad (a \in A, \lambda \in \mathbb{C}, x \in E).$$

According to [11], ${}_\phi B(A^\sharp, E)$ is the set of all $T \in B(A^\sharp, E)$ satisfy

$$T(ab - \phi(b)a) = a \cdot T(b - \phi(b)e^\sharp) \qquad (a, b \in A),$$

where $e^\sharp = (0,1)$ denotes the unit of $A^\sharp$. For each $a, b, c \in A$ and $T \in_\phi B(A^\sharp, E)$,

$$\begin{aligned}(c \cdot T)(ab - \phi(b)a) &= T(abc - \phi(b)ac)\\ &= T(a(bc - \phi(b)c)).\end{aligned}$$

By definition, since $\phi(bc - \phi(b)c) = 0$ and $T \in_\phi B(A^\sharp, E)$ we conclude that,

$$\begin{aligned}(c \cdot T)(ab - \phi(b)a) &= a \cdot T(bc - \phi(b)c)\\ &= a \cdot (c \cdot T)(b - \phi(b)e^\sharp).\end{aligned}$$

This follows that ${}_\phi B(A^\sharp, E)$ is a closed A-submodule of $B(A^\sharp, E)$. Moreover, *the canonical morphism* $\Pi : E \longrightarrow B(A^\sharp, E)$ is regarded as follows:

$$\Pi(x)(a) = a \cdot x \qquad (x \in E, a \in A^\sharp).$$

It is straightforward to check that $\mathrm{Im}(\Pi) \subseteq_\phi B(A^\sharp, E)$. According to this fact, the canonical map Π can be denoted by ${}_\phi\Pi$ as a map from E into ${}_\phi B(A^\sharp, E)$. The following theorem gives a characterization for ϕ-injectivity in terms of a coretraction problem; see [11, Theorem 2.4].

Theorem 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. For $E \in$ **A-mod** the following statements are equivalent.*

(i) *E is ϕ-injective.*

(ii) *There exists a left A-module morphism ${}_\phi\rho :_\phi B(A^\sharp, E) \longrightarrow E$ such that is a left inverse for the canonical morphism ${}_\phi\Pi$.*

In this section, we first give the definition of 0-injectivity and character injectivity of Banach left A-module. Also, we obtain relation between 0-injectivity of Banach left A-modules and ϕ-injectivity of $A^\sharp$-modules.

Similar to [11], for each $E \in$ **A-mod** we can define,

$${}_0B(A^\sharp, E) = \{T \in B(A^\sharp, E) : T(ab) = a \cdot T(b) \text{ for all } a, b \in A\},$$

which is a closed left Banach A-submodule of $B(A^\sharp, E)$.

Definition 2.2. *Let A be a Banach algebra and $E \in$ **A-mod**. We say that*

(i) *E is 0-injective if there exists a map ${}_0\rho :_0 B(A^\sharp, E) \longrightarrow E$ such that is a left A-module morphism and a left inverse for the canonical morphism ${}_0\Pi$.*

(ii) *E is character injective if E is ϕ-injective for each $\phi \in \Delta(A) \cup \{0\}$.*

Proposition 2.1. *Let A be a Banach algebra and $E \in$ **A-mod**. Then E is 0-injective if and only if for each $F, K \in$ **A-mod** and admissible monomorphism $T : F \longrightarrow K$ for which $A \cdot K = \mathrm{span}\{a \cdot k; a \in A, k \in K\} \subseteq \mathrm{Im}(T)$, the induced map $T_E :_A B(K, E) \longrightarrow_A B(F, E)$ is onto.*

Proof. It is clear that ${}_0\Pi$ is an admissible monomorphism from E into ${}_0B(A^\sharp, E)$. First, suppose that for each $F, K \in$ **A-mod** and admissible monomorphism $T : F \longrightarrow K$ for which

$$A \cdot K = \mathrm{span}\{a \cdot k : a \in A, k \in K\} \subseteq \mathrm{Im}(T),$$

the induced map $T_E :_A B(K, E) \longrightarrow_A B(F, E)$ is onto. Take $F = E$, $K =_0 B(A^\sharp, E)$ and $T =_0\Pi$. Then $A \cdot K \subseteq \mathrm{Im}({}_0\Pi)$ and $a \cdot T =_0\Pi(T(a))$, for each $a \in A$ and $T \in K$. Hence, for the identity map $I_E \in_A B(F,E) =_A B(E,E)$, there exists $\rho \in_A B(K,E) =_A B({}_0B(A^\sharp,E),E)$ such that $\rho \circ {}_0\Pi = \rho \circ T = I_E$. It follows that E is 0-injective.

Conversely, let E be 0-injective. Suppose that $F, K \in$ **A-mod** and $T : F \longrightarrow K$ is an admissible monomorphism such that, $A \cdot K \subseteq \mathrm{Im}(T)$. Let $W \in_A B(F,E)$ and define the map $R : K \longrightarrow {}_0B(A^\sharp,E)$ by

$$R(k)(a) = W \circ T'(a \cdot k) \quad (k \in K, a \in A^\sharp),$$

where $T' \in B(K,F)$ satisfies $T \circ T' \circ T = T$. We show that R is well defined, i.e., $R(k) \in {}_0B(A^\sharp,E)$ for each $k \in K$. So, we will show that $R(k)(ab) = a \cdot R(k)(b)$ for each $a, b \in A$. By assumption $A \cdot K \subseteq \mathrm{Im}(T)$ and so there exist $f, f' \in F$ such that $b \cdot k = T(f)$ and $ab \cdot k = T(f')$. Therefore,

$$\begin{aligned} a \cdot R(k)(b) &= a \cdot W \circ T'(b \cdot k) = a \cdot W \circ T'(T(f)) \\ &= a \cdot W(f) = W(a \cdot f) \end{aligned}$$

and,

$$R(k)(ab) = W \circ T'(ab \cdot k) = W \circ T'(T(f')) = W(f').$$

On the other hand,

$$T(a \cdot f) = a \cdot T(f) = a \cdot (b \cdot k) = ab \cdot k = T(f').$$

Hence,

$$a \cdot f = T' \circ T(a \cdot f) = T' \circ T(f') = f'.$$

Therefore, $a \cdot f = f'$ and this implies that $a \cdot R(k)(b) = R(k)(ab)$. Moreover, for each $b \in A^\sharp$ we have

$$\begin{aligned} R(a \cdot k)(b) &= W \circ T'(b \cdot (a \cdot k)) = W \circ T'(ba \cdot k) \\ &= R(k)(ba) = (a \cdot R(K))(b). \end{aligned}$$

It follows that $R(a \cdot k) = a \cdot R(k)$. Now, take $S = {}_0\rho \circ R \in {}_AB(K,E)$. Since $R \circ T = {}_0\Pi \circ W$, we conclude that $S \circ T = W$ and the proof is complete. □

Recall that for a Banach algebra A, $\phi_\infty : A^\sharp \longrightarrow \mathbb{C}$ denotes the character on $A^\sharp$ defined by $\phi_\infty(a,\lambda) = \lambda$. In the following theorem, we obtain the relation between 0-injectivity of Banach A-modules and ϕ_∞-injectivity of Banach modules over the unitization of Banach algebra A.

Theorem 2.2. *Let A be a Banach algebra and $E \in$ **A-mod**. Then the following are equivalent:*

(i) *$E \in$ **A-mod** is 0-injective.*
(ii) *E as a left $A^\sharp$-module is ϕ_∞-injective.*

Proof. (i)⇒(ii) Suppose that $E \in$ **A-mod** is 0-injective. Thus, there exists a left A-module morphism $\rho : {}_0B(A^\sharp,E) \longrightarrow E$ such that $\rho \circ {}_0\Pi = I_E$. Note that for each $T \in {}_{\phi_\infty}B((A^\sharp)^\sharp,E)$, we have $T_{|_{A^\sharp}} \in {}_0B(A^\sharp,E)$. Now, we define the map $\widetilde{\rho} : {}_{\phi_\infty}B((A^\sharp)^\sharp,E) \longrightarrow E$ by

$$\widetilde{\rho}(T) = \rho(T_{|_{A^\sharp}}) \qquad (T \in {}_{\phi_\infty}B((A^\sharp)^\sharp,E)).$$

Hence, for each $T \in {}_{\phi_\infty}B((A^\sharp)^\sharp, E)$ and $(a,\lambda) \in A^\sharp$,

$$\begin{aligned}\widetilde{\rho}((a,\lambda)\cdot T) &= \rho((a,\lambda)\cdot T_{|_{A^\sharp}}) = \rho(a \cdot T_{|_{A^\sharp}}) + \lambda\rho(T_{|_{A^\sharp}})\\ &= a\cdot\rho(T_{|_{A^\sharp}}) + \lambda\rho(T_{|_{A^\sharp}}) = (a,\lambda)\cdot\rho(T_{|_{A^\sharp}})\\ &= (a,\lambda)\cdot\widetilde{\rho}(T).\end{aligned}$$

On the other hand, for each $x \in E$ we have

$$(\widetilde{\rho}\circ {}_{\phi_\infty}\Pi)(x) = \rho({}_{\phi_\infty}\Pi(x)_{|_{A^\sharp}}) = \rho({}_0\Pi(x)) = x.$$

This follows that E as a left $A^\sharp$-module is ϕ_∞-injective.

(ii)⇒(i) Suppose that E as a left $A^\sharp$-module is ϕ_∞-injective. So, there exists a left $A^\sharp$-module morphism $\rho : {}_{\phi_\infty}B((A^\sharp)^\sharp, E) \longrightarrow E$ such that $\rho \circ {}_{\phi_\infty}\Pi = I_E$. Now, define the map $\widetilde{\rho} : {}_0B(A^\sharp, E) \longrightarrow E$ by

$$\widetilde{\rho}(T) = \rho(\widetilde{T}) \qquad (T \in {}_0B(A^\sharp, E)),$$

where the map $\widetilde{T} : (A^\sharp)^\sharp \longrightarrow E$ is defined by

$$\widetilde{T}(((a,\lambda),\mu)) = T(a, \lambda+\mu) \qquad (a \in A, \lambda, \mu \in \mathbb{C}).$$

First, note that for each $(a,\lambda),(b,\mu) \in A^\sharp$ we have

$$\begin{aligned}(a,\lambda)\cdot\widetilde{T}((b,\mu) - \phi_\infty(b,\mu)e_{(A^\sharp)^\sharp}) &= (a,\lambda)\cdot\widetilde{T}(((b,\mu),-\mu))\\ &= (a,\lambda)\cdot T(b,0)\\ &= (a,0)\cdot T((b,0)) + \lambda T((b,0)).\end{aligned}$$

Since $T \in {}_0B(A^\sharp, E)$, we have

$$\begin{aligned}(a,\lambda)\cdot\widetilde{T}((b,\mu) - \phi_\infty(b,\mu)e_{(A^\sharp)^\sharp}) &= T((a,0)(b,0)) + \lambda T((b,0))\\ &= T((ab+\lambda b, 0))\\ &= \widetilde{T}(((ab+\lambda b),0),0))\\ &= \widetilde{T}((ab+\lambda b+\mu a, \lambda\mu) - (\mu a, \mu\lambda))\\ &= \widetilde{T}((a,\lambda)(b,\mu) - \phi_\infty(b,\mu)(a,\lambda)).\end{aligned}$$

This follows that $\widetilde{T} \in {}_{\phi_\infty}B((A^\sharp)^\sharp, E)$ and so $\widetilde{\rho}$ is well defined.

On the other hand, for each $a \in A$ and $T \in {}_0B(A^\sharp, E)$,

$$\widetilde{\rho}(a\cdot T) = \rho(\widetilde{a\cdot T}) = \rho(a\cdot\widetilde{T}) = a\cdot\rho(\widetilde{T}) = a\cdot\widetilde{\rho}(T),$$

and

$$(\widetilde{\rho}\circ {}_0\Pi)(x) = \rho(\widetilde{{}_0\Pi(x)}) = \rho({}_{\phi_\infty}\Pi(x)) = x \qquad (x \in E).$$

Now, we conclude that $E \in$ **A-mod** is 0-injective and the proof is complete. □

The following is an analogous result of [12, Proposition 3.1] which holds for ϕ-injectivity of Banach modules.

Proposition 2.2. *Let A be a Banach algebra, $\phi \in \Delta(A)$, $S \subseteq A^\sharp$ and $E \in$ **A-mod**. Suppose that $T \in {}_\phi B(A^\sharp, E)$ satisfies the following relation*

$$T(ab) = a \cdot T(b) \qquad (a \in A^\sharp, b \in S).$$

If E is ϕ-injective, then there exists $x_0 \in E$ such that $T(b) = b \cdot x_0$ for each $b \in S$.

Proof. Since E is ϕ-injective there exists $\rho \in_A B({}_\phi B(A^\sharp, E), E)$ such that $\rho \circ {}_\phi \Pi = I_E$. Now, the result follows by taking $x_0 = \rho(T)$. □

Proposition 2.3. *Let A be a Banach algebra, $\phi \in \Delta(A)$ and B be a subalgebra of A. Then,*

(i) *if A as a left $B^\sharp$-module is ϕ_∞-injective, then there exists $a_0 \in A$ such that $b = ba_0$ for all $b \in B$.*

(ii) *if A as a left $A^\sharp$-module is ϕ_∞-injective, then A has a right identity.*

(iii) *if A is a Banach algebra with a left approximate identity and A as a left $A^\sharp$-module is ϕ_∞-injective, then A is unital.*

Proof. (i) Let $T : (B^\sharp)^\sharp \to A$ defined by

$$T((b, \lambda), \mu) = b \qquad (b \in B, \lambda, \mu \in \mathbb{C}).$$

For each $a, b \in B$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned} T(((a,\lambda),0)((b,\mu),0) - \phi_\infty((b,\mu),0)((a,\lambda),0)) &= ab + \mu a + \lambda b - \mu a \\ &= ab + \lambda b, \end{aligned}$$

and

$$((a,\lambda),0) \cdot T(((b,\mu),0) - \phi_\infty((b,\mu),0)((0,0),1)) = ((a,\lambda),0) \cdot b = ab + \lambda b.$$

Therefore, $T \in {}_{\phi_\infty} B((B^\sharp)^\sharp), A)$. On the other hand, for all $a, b \in B$ and $\lambda, \mu \in \mathbb{C}$ we have

$$T(((a,\lambda),\mu)((b,0),0)) = ab + \lambda b + \mu b = ((a,\lambda),\mu) \cdot T(((b,0),0)).$$

By Proposition 2.2 with $E = A$ and $S = B$, we conclude that there exists $a_0 \in A$ such that $b = ba_0$ for all $b \in B$.

(ii) This easily follows from the part (i) with $B = A$.

(iii) Let (e_α) be a left approximate identity for A. According the clause (ii), suppose that a_0 is a right identity for A. Hence $e_\alpha = e_\alpha a_0 \longrightarrow a_0$ and for each $a \in A$, we have

$$e_\alpha a \longrightarrow a, \qquad e_\alpha a \longrightarrow a_0 a.$$

It follows that a_0 is an identity of A. □

Theorem 2.3. *Let A be a unital Banach algebra. Then $A \in$ **A-mod** is 0-injective.*

Proof. Let e be the identity of A. Define the map $\rho : {}_0B(A^\sharp, A) \longrightarrow A$ by

$$\rho(T) = T(e) \qquad (T \in {}_0B(A^\sharp, A)).$$

It is obvious that ρ is a left inverse for ${}_0\Pi$, because for each $a \in A$ we have,

$$(\rho \circ {}_0\Pi)(a) = ({}_0\Pi(a))(e) = ea = a.$$

Moreover, for each $a \in A$ and $T \in_0 B(A^\sharp, A)$,

$$\rho(a \cdot T) = (a \cdot T)(e) = T(ea) = T(a)$$
$$= T(ae) = a \cdot T(e) = a \cdot \rho(T).$$

It follows that ρ is a left A-module morphism. Therefore, $A \in$ **A-mod** is 0-injective. □

As a consequence, we characterize 0-injectivity of $A \in$ **A-mod**, in the case where A has a left approximate identity.

Corollary 2.1. *Let A be a Banach algebra with a left approximate identity. Then the following are equivalent:*

(i) $A \in$ **A-mod** *is 0-injective.*
(ii) *A has an identity.*

Proof. (i)⇒(ii) Apply Theorem 2.2 and Proposition 2.3(iii).
(ii)⇒(i) This follows from Theorem 2.3. □

Example 2.1. *Consider the Banach algebra $A = A_\phi(X)$* [11, Example 2.5] *where X is a Banach space with* $\dim(X) > 2$ *and $\phi \in X' \setminus \{0\}$ and the multiplication is defined by*

$$ab = \phi(a)b \qquad (a, b \in A).$$

This Banach algebra has a left identity but does not have any right identity. Using Corollary 2.1, it follows that A is not 0-injective in **A-mod**. *Moreover, $A \in$* **A-mod** *is ϕ-injective* [11, Example 2.5].

We finish this section with a result on left invariantly complemented ideals. Let X be a Banach left A-module and Y be a Banach A-submodule of X. Following [4, Definition 6.3], we say that Y is *left* (*right*) *invariantly complemented* in X if there exists a $P \in_A B(X, Y)$ ($P \in B_A(X, Y)$) such that $P^2 = P$ and $P(X) = Y$.

Proposition 2.4. *Let A be a Banach algebra and I be a closed ideal of A. Then,*

(i) *if A/I as a left $A^\sharp$-module is ϕ_∞-injective, then I has a right modular identity.*
(ii) *if I is left invariantly complemented and I as a left $A^\sharp$-module is ϕ_∞-injective, then there exists $b_0 \in I$ such that $a = ab_0$ for all $a \in I$.*
(iii) *if I is left invariantly complemented and I'' as a left $A^\sharp$-module is ϕ_∞-injective, then I has a bounded right approximate identity.*

Proof. (i) Consider the operator $T : (A^\sharp)^\sharp \to A/I$ defined by

$$T(((a, \lambda), \mu)) = a + I \qquad (a \in A, \lambda, \mu \in \mathbb{C}).$$

Now, apply Proposition 2.2.

(ii) Since I is left invariantly complemented there exists a projection $P \in_A B(A, A)$ such that $P(A) = I$. Define $T : (A^\sharp)^\sharp \longrightarrow I$ by $T(((a, \lambda), \mu)) = P(a)$. For each $b \in I$, $a \in A$ and scalars λ, μ we have

$$T(((a, \lambda), \mu)((b, 0), 0)) = P(ab + \lambda b + \mu b) = ab + \lambda b + \mu b = ((a, \lambda), \mu) \cdot T(((b, 0), 0)).$$

Moreover,

$$\begin{aligned} T(((a,\lambda),0)((b,\mu),0) - \phi_\infty((b,\mu),0)((a,\lambda),0)) &= P(ab+\mu a+\lambda b) - P(\mu a) \\ &= P(ab+\lambda b), \end{aligned}$$

and

$$\begin{aligned} ((a,\lambda),0)\cdot T(((b,\mu),0) - \phi_\infty((b,\mu),0)((0,0),1)) &= ((a,\lambda),0)\cdot P(b) \\ &= a\cdot P(b) + P(\lambda b) \\ &= P(ab+\lambda b). \end{aligned}$$

Now, using Proposition 2.2 the proof is complete.

(iii) Suppose that $T : (A^\sharp)^\sharp \to I''$ is defined by $T(((a,\lambda),\mu)) = \widehat{P(a)}$. Hence similar to the proof of part (ii), there exists $\Phi \in I''$ such that for all $a \in I$,

$$a\cdot\Phi = \widehat{P(a)} = \widehat{a}.$$

Now, the result follows from the Goldstine and Mazur Theorems; see [1, A.3.29, (i), (ii)]. □

3. **Some examples on harmonic analysis**

In this section, with some examples we show difference between 0-injectivity, character injectivity and injectivity of Banach modules.

Example 3.1. *Let $\ell^1(\mathbb{N}_\wedge)$ be the semigroup algebra on semigroup $S = (\mathbb{N},\wedge)$ with the following product:*

$$\mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}, \quad (m,n) \longrightarrow m\wedge n = \min\{m,n\}.$$

It has been proved in [3, Theorem 3.2] *that $\ell^1(\mathbb{N}_\wedge)$ as a Banach left $\ell^1(\mathbb{N}_\wedge)$-module is ϕ-injective, for each $\phi \in \Delta(\ell^1(\mathbb{N}_\wedge))$. Since $\ell^1(\mathbb{N}_\wedge)$ is a non-unital Banach algebra with a bounded approximate identity, by Corollary 2.1, we conclude that $\ell^1(\mathbb{N}_\wedge)$ as a Banach left $\ell^1(\mathbb{N}_\wedge)$-module is not 0-injective.*

Example 3.2. *Let $\ell^1(\mathbb{N}_\vee)$ be the semigroup algebra on semigroup $S = (\mathbb{N},\vee)$ with the following product:*

$$\mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}, \quad (m,n) \longrightarrow m\vee n = \max\{m,n\}.$$

It is proved in [3, Theorem 3.3] *that $\ell^1(\mathbb{N}_\vee)$ as a Banach left $\ell^1(\mathbb{N}_\vee)$-module is ϕ-injective for each $\phi \in \Delta(\ell^1(\mathbb{N}_\vee))$. Moreover, since $\ell^1(\mathbb{N}_\vee)$ is unital we conclude that $\ell^1(\mathbb{N}_\vee)$ as a Banach left $\ell^1(\mathbb{N}_\vee)$-module is character injective. Recall that by a modification of* [12, Theorem 4.8] *for right modules, we conclude that $c_0(\mathbb{N}_\vee)$ as a Banach right $\ell^1(\mathbb{N}_\vee)$-module is not flat. Thus $\ell^1(\mathbb{N}_\vee)$ as a Banach left $\ell^1(\mathbb{N}_\vee)$-module is not injective.*

In the sequel, we concentrate on character injectivity of some classes of Banach modules over the group algebra $L^1(G)$ and the measure algebra $M(G)$.

Example 3.3. *For a locally compact group G, by Corollary 2.1, we conclude that $L^1(G) \in$ $\mathbf{L^1(G)}$**-mod** is 0-injective if and only if G is discrete. Moreover, it follows that $M(G) \in$ **M(G)-mod** is 0-injective. Recall that $M(G) \in$ **M(G)-mod** is injective if and only if G is* *amenable* [13, Table 3.1, Page 50]. *Similar to* [13, Theorem 3.1.2], $M(G) \in$ $\mathbf{L^1(G)}$**-mod** *is 0-injective.*

For a locally compact group G, the space of all bounded left uniformly continuous functions $LUC(G)$, is a closed submodule of $L^\infty(G)$ as a Banach $M(G)$-bimodule. Thus, we can regard $LUC(G)'$ as a Banach $M(G)$-bimodule with the dual module actions; for more details see [1] and [10]. It is shown in [10, Theorem 2.6] that $LUC(G)'$ as the Banach left (right) $M(G)$-module is injective if and only if G is amenable . In the following, we prove that $LUC(G)'$ as the Banach left $M(G)$-module and $L^1(G)$-module is always 0-injective.

Theorem 3.1. *Let G be a locally compact group. Then we have*

(i) $LUC(G)' \in$ $\mathbf{L^1(G)}$**-mod** *is 0-injective.*

(ii) $LUC(G)' \in$ $\mathbf{M(G)}$**-mod** *is 0-injective.*

Proof. (i) By Theorem 2.2, it suffices to show that $LUC(G)'$ as a Banach left $L^1(G)^\sharp$-module is ϕ_∞-injective. Since $\ker \phi_\infty = L^1(G)$ has a bounded approximate identity, it follows from [6, Proposition 2.1] that $L^1(G)^\sharp$ is ϕ_∞-amenable. Now, by [11, Proposition 3.1] we conclude that $LUC(G)' \in$ $\mathbf{L^1(G)^\sharp}$**-mod** is ϕ_∞-injective and the proof is complete.

The clause (ii) follows by a similar argument. □

In the following, we show that for some classes of Banach modules the notions 0-injectivity, character injectivity and injectivity are equivalent. This leads to a generalization of [10, Theorem 2.4].

Theorem 3.2. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) $LUC(G) \in$ $\mathbf{L^1(G)}$**-mod** *is injective.*

(ii) $LUC(G) \in$ $\mathbf{L^1(G)}$**-mod** *is character injective.*

(iii) $LUC(G) \in$ $\mathbf{L^1(G)}$**-mod** *is 0-injective.*

(iv) *G is discrete.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are obvious. By [10, Theorem 2.5] , the statement (i) and (iv) are equivalent.

(iii)⇒(iv) Let $LUC(G)$ as a Banach left $L^1(G)$-module be 0-injective. Thus, there exists a left inverse $\rho \in {}_{L^1(G)}B({}_0B(L^1(G)^\sharp, LUC(G)), LUC(G))$ for the canonical morphism ${}_0\Pi$. Define the map $Q : L^\infty(G) \longrightarrow {}_0B(L^1(G)^\sharp, LUC(G))$ by

$$Q(g)((f,\lambda)) = f \cdot g \qquad (\lambda \in \mathbb{C}, f \in L^1(G), g \in L^\infty(G)).$$

It is easy to check that Q is well defined. On the other hand, we have

$$Q(h)((f,0)) = {}_0\Pi(h)((f,0)) \qquad (h \in LUC(G), f \in L^1(G)).$$

Now, we show that $\rho \circ Q : L^\infty(G) \to LUC(G)$ is a projection. Let (e_α) be a left bounded approximate identity for $LUC(G)$ in $L^1(G)$. Thus for all $h \in LUC(G)$ we have,

$$\rho(Q(h) - h) = \lim_\alpha e_\alpha \cdot \rho(Q(h) - {}_0\Pi(h)) = \lim_\alpha \rho(e_\alpha Q(h) - e_\alpha\, {}_0\Pi(h)) = 0.$$

Now, since $LUC(G)$ is a closed subspace of $C_b(G)$ which contains $C_0(G)$, the result follows from [8, Theorem 4]. □

We finish the paper with the following open problem.

Let G be a locally compact group. It is shown in [2, Theorem 4.9] that $L^1(G) \in$ $\mathbf{L^1(G)}$**-mod** is injective if and only if G is amenable and discrete. Now, suppose that $L^1(G) \in \mathbf{L^1(G)}$**-mod** is character injective. By Corollary 2.1, it follows that $L^1(G)$ has an identity and so G is discrete, but we do not know whether G is amenable.

Open problem: Are the following statements equivalent?

(i) $L^1(G) \in \mathbf{L^1(G)}$**-mod** is character injective.

(ii) G is amenable and discrete.

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