FIXED POINTS OF SUZUKI TYPE QUASI-CONTRACTIONS

B. Mohammadi[1], Simona Dinu[2], Sh. Rezapour[3]

To generalize the celebrated Banach contraction theorem, many authors have introduced various type contraction inequalities. In 2008*, Suzuki introduced a new method and then this method was extended by some authors. In* 2012*, Samet, Vetro and Vetro introduced α-ψ-contractive mappings and gave some results on fixed point of mappings. Their results generalized some previous fixed point results. In this paper, by using the idea of Samet, Vetro and Vetro, we introduce fixed point results of Suzuki type quasi-contractive selfmaps and multifunctions.*

Keywords: α-admissible, fixed point, multifunction, quasi-contraction.

1. Introduction

In 1969, Subrahmanyam proved that a metric space X is complete if and only if every Kannan mapping on X has a fixed point, [38]. In 2008, Suzuki introduced a new type of mappings and a generalization of the Banach contraction principle in which the completeness can be also characterized by the existence of fixed points of these mappings, [39]; for some generalizations, see [1]. The notion of quasi-contraction was provided by Ćirić in 1974, [11]. Later, several researchers published some papers about quasi-contractions (see for example, [3], [17], [19], [22], [31]). In 2012, Samet, Vetro and Vetro introduced the notion of α-ψ-contractive mapping and gave some results on fixed point of mappings, [32]. In this paper, by combining different ideas of the above listed papers and providing a simple method, we give some fixed point results about Suzuki type quasi-contractive selfmaps and multifunctions, [12], [26], [27], [28]. These results complement several fixed point results for different kinds of contractions on some spaces such as: ordered metric spaces [5, 6, 9, 21, 30, 34, 40], G-metric spaces [7, 10, 35, 36], convex metric spaces [29], metric spaces endowed with a graph [8, 20], fuzzy metric spaces [25], partial metric spaces [33], quasi-partial metric spaces [37].

Denote by Ψ the family of nondecreasing functions $\psi\colon [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$, for each $t > 0$. We know that $\psi(t) < t$, for all $t > 0$.

Let (X, d) be a metric space, $\alpha\colon X \times X \to [0, \infty)$ a mapping, F a selfmap on X and $T\colon X \to 2^X$ a multifunction. We say that T is α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ we have $\alpha(y, z) \geq 1$ for all $z \in Ty$. Also,

[1]PhD Student, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran, e-mail: babakmohammadi28@yahoo.com (Corresponding author)

[2]PhD Student, Department of Mathematics & Informatics, University “Politehnica” of Bucharest, Romania, e-mail: simogrigo@yahoo.com

[3]Professor, Department of Mathematics, Azarbaidjan Shahid Madani University, Azarshahr, Tabriz, Iran, e-mail: sh.rezapour@azaruniv.edu

we say that F is α-quasi-admissible whenever $\alpha(x, Fx) \geq 1$ implies $\alpha(Fx, F^2x) \geq 1$ for all $x \in X$. Finally, we say that X satisfies the condition (C_α) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k (see [2] for the idea of this notion).

Recall that T is continuous whenever $H(Tx_n, Tx) \to 0$ for all sequence $\{x_n\}$ in X with $x_n \to x$, where H is the Hausdorff metric.

Throughout this paper, we use the nondecreasing function $\theta\colon [0,1) \to (1/2, 1]$ which is given by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases}$$

2. Some results on selfmaps

Now, we are ready to state and prove our main results. First, we give the following theorem by following techniques of similar results in literature.

Theorem 2.1. *Let (X, d) be a complete metric space, $r \in [0, 1)$ and T a selfmap on X such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rK(x, y)$ for all $x, y \in X$, where $K(x, y) = \max\{d(x, y), d(x, Tx), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. Then T has a unique fixed point.*

Proof. Since $\theta(r)d(x, Tx) \leq d(x, Tx)$ for all $x \in X$, we have

$$d(Tx, T^2x) \leq r \max\{d(x, Tx), d(x, Tx), \frac{d(x, T^2x) + d(Tx, Tx)}{2}\}$$
$$= r \max\{d(x, Tx), \frac{d(x, T^2x)}{2}\}.$$

Hence, $d(Tx, T^2x) \leq rd(x, Tx)$ or $d(Tx, T^2x) \leq r\frac{d(x,T^2x)}{2}$.

If $d(Tx, T^2x) \leq r\frac{d(x,T^2x)}{2}$, then $2d(Tx, T^2x) \leq rd(x, Tx) + rd(Tx, T^2x)$ and so

$$d(Tx, T^2x) \leq (2 - r)d(Tx, T^2x) \leq rd(x, Tx).$$

Hence, in each case we have $d(Tx, T^2x) \leq rd(x, Tx)$ for all $x \in X$.

Now, fix $u \in X$ and define a sequence $\{u_n\}$ by $u_n = T^n u$ for all $n \geq 1$. By using the above inequality, we have $d(u_n, u_{n+1}) \leq r^n d(u, Tu)$ for all n. Thus, $\sum d(u_n, u_{n+1}) < \infty$ and so $\{u_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $u_n \to z$.

Now, we show that $d(z, Tx) \leq rd(z, x)$, for all $x \in X$ with $x \neq z$.

Let $z \neq x \in X$ be given. Choose a natural number n_0 such that $d(z, u_n) \leq \frac{1}{3}d(z, x)$ for all $n \geq n_0$. Then, we obtain

$$\theta(r)d(u_n, Tu_n) \leq d(u_n, u_{n+1}) \leq d(z, u_n) + d(z, u_{n+1}) \leq \frac{2}{3}d(z, x)$$
$$= d(z, x) - \frac{1}{3}d(z, x) \leq d(z, x) - d(z, u_n) \leq d(u_n, x).$$

Thus, $d(u_{n+1}, Tx) \leq r \max\{d(u_n, x), d(u_n, u_{n+1}), \frac{1}{2}(d(u_n, Tx) + d(u_{n+1}, x))\}$, for all $n \geq n_0$. Hence, $d(z, Tx) \leq r \max\{d(z, x), \frac{1}{2}(d(z, Tx) + d(z, x))\}$. This implies that

$d(z,Tx) \le rd(z,x)$ or $d(z,Tx) \le \frac{r}{2}(d(z,Tx)+d(z,x))$. If the second case holds, that is, $d(z,Tx) \le \frac{r}{2}(d(z,Tx)+d(z,x))$, then $2d(z,Tx) \le rd(z,Tx)+rd(z,x))$ and so $d(z,Tx) \le (2-r)d(z,Tx) \le rd(z,x)$. Thus, in each case we have $d(z,Tx) \le rd(z,x)$. Hence, we proved the claim.

Now, we claim that there exists a natural number j such that $T^j z = z$. If $T^j z \ne z$ for all j, then $d(T^{j+1}z,z) \le r^j d(Tz,z)$ for all j. Let $0 \le r \le \frac{\sqrt{5}-1}{2}$. Then, $2r^2 < 1$. If $d(T^2z,z) < d(T^2z,T^3z)$, then

$$d(z,Tz) \le d(z,T^2z) + d(T^2z,Tz) < d(T^2z,T^3z) + d(T^2z,Tz)$$
$$\le r^2 d(Tz,z) + rd(Tz,z) \le d(z,Tz)$$

which is a contradiction. Thus, $d(T^2z,z) \ge d(T^2z,T^3z) = \theta(r)d(T^2z,T^3z)$ and so

$$d(T^3z,Tz) \le r\max\{d(T^2z,z), d(T^2z,T^3z), \frac{1}{2}(d(T^2z,Tz)+d(z,T^3z))\}$$
$$\le \max\{r^2d(Tz,z), r^3d(z,Tz), \frac{r^2+r^3}{2}d(z,Tz)\} = r^2d(Tz,z).$$

Hence,

$$d(z,Tz) \le d(z,T^3z)+d(T^3z,Tz) \le r^2d(Tz,z)+r^2d(Tz,z) = 2r^2d(z,Tz) < d(z,Tz)$$

which is a contradiction again.

Now, assume $\frac{\sqrt{5}-1}{2} < r < 2^{-1/2}$. Then, $2r^2 < 1$.

If $d(T^2z,z) < \theta(r)d(T^2z,T^3z)$, then

$$d(z,Tz) \le d(z,T^2z) + d(T^2z,Tz) < \theta(r)d(T^2z,T^3z) + d(T^2z,Tz)$$
$$\le \theta(r)r^2d(Tz,z) + rd(Tz,z) = d(z,Tz)$$

which is a contradiction. Thus, $\theta(r)d(T^2z,T^3z) \le d(T^2z,z)$.

Similar to the previous case, we can prove $d(z,Tz) \le 2r^2d(z,Tz) < d(z,Tz)$, which is a contradiction again.

Now, let $2^{-1/2} \le r < 1$. Then, it is easy to see that for each $x,y \in X$ we have $\theta(r)d(x,Tx) \le d(x,y)$ or $\theta(r)d(Tx,T^2x) \le d(Tx,y)$. Thus, for each natural number n we have $\theta(r)d(u_{2n},u_{2n+1}) \le d(u_{2n},z)$ or $\theta(r)d(u_{2n+1},u_{2n+2}) \le d(u_{2n+1},z)$. Hence, $d(u_{2n+1},Tz) \le rK(u_{2n},z)$ or $d(u_{2n+2},Tz) \le rK(u_{2n+1},z)$ for all n. Therefore, we have at least one of the following cases:

(1) There exists a subsequence $\{n_k\}$ such that $d(u_{2n_k+1},Tz) \le rK(u_{2n_k},z)$,

(2) There exists a subsequence $\{n_k\}$ such that $d(u_{2n_k+2},Tz) \le rK(u_{2n_k+1},z)$.

If (1) holds, then we get

$$d(u_{2n_k+1},Tz) \le r\max\{d(u_{2n_k},z), d(u_{2n_k},u_{2n_k+1}), \frac{1}{2}(d(u_{2n_k},Tz)+d(z,u_{2n_k+1}))\}$$

for all k and so $d(z,Tz) \le \frac{r}{2}d(z,Tz)$ and so $z = Tz$. This is a contradiction.

If (2) holds, then we get

$$d(u_{2n_k+2},Tz) \le r\max\{d(u_{2n_k+1},z), d(u_{2n_k+1},u_{2n_k+2}),$$
$$\frac{1}{2}(d(u_{2n_k+1},Tz)+d(z,u_{2n_k+2}))\}$$

for all k. Hence, $d(z,Tz) \le \frac{r}{2}d(z,Tz)$. Thus, $z = Tz$. This is a contradiction. Therefore, we proved the second claim, that is, there exists a natural number j such that $T^j z = z$. But, we showed that $\{T^n u\}$ is a Cauchy sequence for all $u \in X$. Thus, $\{T^n z\}$ is a Cauchy sequence.

Now, consider the subsequences $\{T^{nj} z\}_{n\geq 1}$ and $\{T^{nj+1} z\}_{n\geq 1}$ of $\{T^n z\}$. Note that, $T^{nj} z \to z$ and $T^{nj+1} z \to Tz$. Thus, $Tz = z$ and so T a fixed point. It is easy to show that T has a unique fixed point. □

The following example shows us the difference between Theorem 2.1 and Theorem 2 in [39], that is, there are some mappings in which we can use Theorem 2.1 while we can not apply Theorem 2 of [39] for the maps.

Example 2.1. Let $X = \{(0,0), (5,0), (0,5), (6,0), (0,6), (5,6), (6,5)\}$ and

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Now, we define the selfmap T on X by $T(x_1, x_2) = (x_1, 0)$ whenever $x_1 \leq x_2$ and $T(x_1, x_2) = (0, x_2)$ whenever $x_1 > x_2$. Suppose there exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Put $x = (6, 0)$ and $y = (5, 6)$. Then $\theta(r)d(x, Tx) = 6\theta(r) \leq 7 = d(x, y)$. Hence $d(Tx, Ty) \leq rd(x, y)$ and so $5 = d((0, 0), (5, 0)) \leq 7r$. Therefore, $1 > r \geq \frac{5}{7} > 2^{-\frac{1}{2}}$ and so

$$\theta(r) = \frac{1}{1 + r} \leq \frac{1}{1 + \frac{5}{7}} = \frac{7}{12}.$$

Put $x = (6, 0)$ and $y = (6, 5)$. Then $\theta(r)d(x, Tx) = 6\theta(r) \leq (\frac{7}{12})6 \leq 5 = d(x, y)$. But, $d(Tx, Ty) = 5 = d(x, y)$ and so $d(Tx, Ty) > rd(x, y)$. Therefore, T does not satisfy the condition of Theorem 2 of [39].

Now, we show that one can use Theorem 2.1 for the selfmap T.

In this respect, let $x, y \in \{(0, 0), (6, 0), (0, 6), (5, 0), (0, 5)\}$. Then, we have $d(Tx, Ty) = 0 \leq rK(x, y)$. If $x = (5, 6)$ and $y \in \{(0, 0), (6, 0), (0, 6), (5, 0), (0, 5)\}$, then $d(Tx, Ty) = d((5, 0), (0, 0)) = 5$ and $d(x, Tx) = 6$. So, $K(x, y) \geq 6$. Now, put $r = \frac{10}{11}$. Then, $2^{-\frac{1}{2}} \leq r < 1$ and $\theta(r) = \frac{1}{1+r} = \frac{1}{1+10/11} = 11/21$. Thus, in this case we obtain $d(Tx, Ty) = 5 \leq \frac{10}{11}K(x, y) = rK(x, y)$. Now, suppose that $x = (6, 5)$ and $y \in \{(0, 0), (6, 0), (0, 6), (5, 0), (0, 5)\}$. Then, $d(Tx, Ty) = d((0, 5), (0, 0)) = 5$ and $d(x, Tx) = 6$. Thus, $K(x, y) \geq 6$ and so $d(Tx, Ty) = 5 \leq rK(x, y)$. Finally, suppose that $y \in \{(5, 6), (6, 5)\}$ and $x \in \{(0, 0), (6, 0), (0, 6), (5, 0), (0, 5)\}$. Then, we have $d(Tx, Ty) = 5$, $K(x, y) \geq \frac{d(x,Ty)+d(y,Tx)}{2} \geq \frac{d(y,Tx)}{2} = \frac{11}{2}$. This implies that $d(Tx, Ty) = 5 \leq rK(x, y)$. Therefore, we can use Theorem 2.1 for the selfmap T while we cannot apply Theorem 2 of [39] for the map.

By providing an easy proof, one can obtain the next result.

We say that a selfmap T is a Suzuki type quasi-contraction whenever T satisfy the main condition of Theorem 2.1 or Theorem 2.2.

Theorem 2.2. *Let (X, d) be a complete metric space, $\psi \in \Psi$, $\alpha\colon X \times X \to [0, \infty)$ a mapping and T an α-quasi-admissible selfmap on X such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$$

for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If T is continuous or X has the property (C_α) and ψ is upper semi-continuous, then T has a fixed point.

Corollary 2.3. *Let $(X, d, \leq)$ be an ordered complete metric space, $\psi \in \Psi$ and T a selfmap on X such that $d(Tx, Ty) \leq \psi(M(x, y))$ for all comparable $x, y \in X$. Assume that there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable, X has the property (C_α) and ψ is upper semi-continuous. Suppose that Tx and T^2x are comparable whenever x and Tx so are. Then T has a fixed point.*

Proof. Define the mapping $\alpha\colon X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$, whenever x and y are comparable and $\alpha(x, y) = 0$, otherwise. Then by using Theorem 2.2, T has a fixed point. □

Corollary 2.4. *Let (X, d) be a complete metric space, $A \subseteq X$, $\psi \in \Psi$ and T a selfmap on X such that $d(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$ which are comparable with at least one element of A. Assume that there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable with at least one element of A, X has the property (C_α) and ψ is upper semi-continuous. Suppose that Tx and T^2x are comparable with at least one element of A whenever x and Tx so are. Then T has a fixed point.*

Proof. Define the mapping $\alpha\colon X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$, whenever x and y are comparable with at least one element of A and $\alpha(x, y) = 0$ otherwise. Then by using Theorem 2.2, T has a fixed point. □

Let (X, d) be a metric space and T a selfmap on X. We say that X has the property (E) whenever for each sequence $\{x_n\}$ in X with $\theta(r)d(x_n, Tx_n) \leq d(x_n, x_{n+1})$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\theta(r)d(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, x)$$

for all k.

Corollary 2.5. *Let (X, d) be a complete metric space, $r \in [0, 1)$, T a selfmap on X such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rM(x, y)$ for all $x, y \in X$. If T is continuous or X has the property (E), then T has a unique fixed point.*

Proof. Define $\alpha\colon X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$ whenever $\theta(r)d(x, Tx) \leq d(x, y)$ and $\alpha(x, y) = 0$ otherwise. Since $\theta(r)d(x, Tx) \leq d(x, Tx)$ for all $x \in X$, $\alpha(x, Tx) \geq 1$ for all $x \in X$ and so T is α-admissible. Also, it implies that $d(Tx, Ty) \leq rM(x, y)$ for all $x, y \in X$. Define $\psi(t) = rt$ for all $t \geq 0$. Then by using Theorem 2.2, T has a fixed point. If x and y are two fixed points of T, then $\theta(r)d(x, Tx) = 0 \leq d(x, y)$ and so $d(x, y) = d(Tx, Ty) \leq rM(x, y) = rd(x, y)$. Thus, $x = y$. □

Now by using a similar proof, one can obtain next corollary.

Corollary 2.6. *Let (X, d) be a complete metric space, $r \in [0, 1)$, T a selfmap on X such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rK(x, y)$ for all $x, y \in X$. If T is continuous or X has the property (E), then T has a unique fixed point.*

3. Some results on multifunctions

In this section, we suppose that (X, d) is a metric space, 2^X the family of non-empty subsets of X, $CB(X)$ is the set of all closed and bounded subsets of X, $T\colon X \to 2^X$ a multifunction and $D(x, Ty) = \inf_{z \in Ty} d(x, z)$ for all $x, y \in X$. Also, we use the notations

$$K(x, y) = \max\Big\{d(x, y), D(x, Tx), \frac{D(x, Ty) + D(y, Tx)}{2}\Big\},$$

and

$$M(x,y)=\max\Big\{d(x,y),D(x,Tx),D(y,Ty),\frac{D(x,Ty)+D(y,Tx)}{2}\Big\},$$

in this section.

The proof of next result is similar to proof of Theorem 2.1 in [4], but this result is a different one. In fact, this result is a multifunction version of Theorem 2.2.

Theorem 3.1. *Let (X,d) be a complete metric space, $\alpha\colon X\times X\to[0,\infty)$ a function, $\psi\in\Psi$ a strictly increasing map and $T\colon X\to CB(X)$ an α-admissible multifunction such that $\alpha(x,y)H(Tx,Ty)\le\psi(M(x,y))$ for all $x,y\in X$ and there exist $x_0\in X$ and $x_1\in Tx_0$ with $\alpha(x_0,x_1)\ge 1$. If T is continuous or X has the property (C_α) and ψ is upper semi-continuous, then T has a fixed point.*

Proof. If $x_1=x_0$, then we have nothing to prove.

Let $x_1\neq x_0$. Then, we have

$$D(x_1,Tx_1)\le\alpha(x_0,x_1)H(Tx_0,Tx_1)$$
$$\le\psi(\max\{d(x_0,,x_1),D(x_0,Tx_0),D(x_1,Tx_1),\frac{D(x_0,Tx_1)+D(x_1,Tx_0)}{2}\})$$
$$=\psi(\max\{d(x_0,x_1),D(x_1,Tx_1),\frac{D(x_0,Tx_1)}{2}\})$$
$$\le\psi(\max\{d(x_0,x_1),D(x_1,Tx_1),\frac{d(x_0,x_1)+D(x_1,Tx_1)}{2}\})$$
$$=\psi(\max\{d(x_0,x_1),D(x_1,Tx_1)\}).$$

If $\max\{d(x_0,x_1),D(x_1,Tx_1)\}=D(x_1,Tx_1)$, then it is easy to see that it follows $D(x_1,Tx_1)\le\psi(D(x_1,Tx_1))$, and we get $D(x_1,Tx_1)=0$. Thus, $d(x_0,x_1)=0$, which is a contradiction. Hence, we obtain $\max\{d(x_0,x_1),D(x_1,Tx_1)\}=d(x_0,x_1)$ and so $D(x_1,Tx_1)\le\psi(d(x_0,x_1))$.

If $x_1\in Tx_1$, then x_1 is a fixed point of T. Let $x_1\notin Tx_1$ and $q>1$. Then,

$$0<D(x_1,Tx_1)\le q\psi(d(x_0,x_1)).$$

Put $t_0=d(x_0,x_1)$. Then, $t_0>0$ and $D(x_1,Tx_1)<q\psi(t_0)$. Hence, there exists $x_2\in Tx_1$ such that $d(x_1,x_2)<q\psi(t_0)$ and so $\psi(d(x_1,x_2))<\psi(q\psi(t_0))$. It is clear that $x_2\neq x_1$.

Put $q_1=\frac{\psi(q\psi(t_0))}{\psi(d(x_1,x_2))}$. Then $q_1>1$ and we have

$$D(x_2,Tx_2)\le\alpha(x_1,x_2)H(Tx_1,Tx_2)$$
$$\le\psi(\max\{d(x_1,x_2),D(x_1,Tx_1),D(x_2,Tx_2),\frac{D(x_1,Tx_2)+D(x_2,Tx_1)}{2}\})$$
$$=\psi(\max\{d(x_1,x_2),D(x_2,Tx_2),\frac{D(x_1,Tx_2)}{2}\})\le\psi(\max\{d(x_1,x_2),D(x_2,Tx_2)\}).$$

Similarly, we have $\max\{d(x_1,x_2),D(x_2,Tx_2)\}=d(x_1,x_2)$ and so we obtain $D(x_2,Tx_2)\le\psi(d(x_1,x_2))$.

If $x_2\in Tx_2$, then x_2 is a fixed point of T.

Let $x_2\notin Tx_2$. Then, $0<D(x_2,Tx_2)\le q\psi(d(x_1,x_2))<q_1\psi(d(x_1,x_2))$. Hence, there exists $x_3\in Tx_2$ such that $d(x_2,x_3)<q_1\psi(d(x_1,x_2))=\psi(q\psi(t_0))$. It is clear that $x_3\neq x_2$ and $\psi(d(x_2,x_3))<\psi^2(q\psi(t_0))$.

Put $q_2 = \frac{\psi^2(q\psi(t_0))}{\psi(d(x_2,x_3))}$. Then, $q_2 > 1$. Also, we have

$$D(x_3, Tx_3) \leq \alpha(x_2, x_3)H(Tx_2, Tx_3)$$
$$\leq \psi(\max\{d(x_2, x_3), D(x_2, Tx_2), D(x_3, Tx_3), \frac{D(x_2, Tx_3) + D(x_3, Tx_2)}{2}\})$$
$$= \psi(\max\{d(x_2, x_3), D(x_3, Tx_3), \frac{D(x_2, Tx_3)}{2}\}) \leq \psi(\max\{d(x_2, x_3), D(x_3, Tx_3)\}).$$

By continuing this process, we finally obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) \leq \psi^{n-1}(q\psi(t_0))$, for all n.

Let $m > n$. Then,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(t_0))$$

and so $\{x_n\}$ is a Cauchy sequence in X. Hence, there exists $x^\star \in X$ such that $x_n \to x^\star$.

If T is continuous, then

$$D(x^\star, Tx^\star) = \lim_{n\to\infty} D(x_{n+1}, Tx^\star) \leq \lim_{n\to\infty} H(Tx_n, Tx^\star) = 0$$

and so $x^\star \in Tx^\star$.

If X has the property (C_α) and ψ is upper semi-continuous, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^\star) \geq 1$ for all k. Thus,

$$D(x^\star, Tx^\star) = \lim_{n\to\infty} D(x_{n_k+1}, Tx^\star) \leq \lim_{n\to\infty} \alpha(x_{n_k}, x^\star)H(Tx_{n_k}, Tx^\star)$$
$$\leq \lim_{n\to\infty} \psi(\max\{d(x_{n_k}, x^\star), D(x_{n_k}, Tx_{n_k}), D(x^\star, Tx^\star), \frac{D(x_{n_k}, Tx^\star) + D(x^\star, Tx_{n_k})}{2}\})$$
$$\leq \psi(D(x^\star, Tx^\star)).$$

Hence, $D(x^\star, Tx^\star) = 0$ and so $x^\star \in Tx^\star$. □

Let (X, d) be a metric space, $\psi \in \Psi$ and T a multifunction on X. We say that X has the property (F) whenever for each sequence $\{x_n\}$ in X with

$$D(x_n, Tx_n) \leq d(x_n, x_{n+1}) + \psi(d(x_n, x_{n+1}))$$

and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$D(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, x) + \psi(d(x_{n_k}, x))$$

for all k.

Corollary 3.2. *Let (X, d) be a complete metric space, $\psi \in \Psi$ a strictly increasing map and $T\colon X \to CB(X)$ a multifunction such that $D(x, Tx) \leq d(x, y) + \psi(d(x, y))$ implies $H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$. If T is continuous or X has the property (F) and ψ is upper semi-continuous, then T has a fixed point.*

Proof. It is sufficient we define the map $\alpha\colon X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ whenever $D(x, Tx) \leq d(x, y) + \psi(d(x, y))$ and $\alpha(x, y) = 0$ otherwise and use Theorem 3.1. □

Corollary 3.3. *Let (X, d) be a complete metric space, r a real number in $[0, 1)$ and $T\colon X \to CB(X)$ a multifunction such that $\frac{1}{1+r}d(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq rM(x, y)$ for all $x, y \in X$. If T is continuous or X has the property (F), then T has a fixed point.*

Note that, $K(x,y) \le M(x,y)$ for all $x,y \in X$. We use this fact in next result.

Theorem 3.4. *Let (X,d) be a complete metric space, $\psi \in \Psi$ a strictly increasing and upper semi-continuous map such that $\psi(\frac{a+b}{2}) \le \frac{\psi(a)}{2} + \frac{\psi(b)}{2}$ for all $a,b \ge 0$, $T\colon X \to CB(X)$ a multifunction such that $D(x,Tx) \le d(x,y) + \psi(d(x,y))$ implies $H(Tx,Ty) \le \psi(K(x,y))$ for all $x,y \in X$. Then T has a fixed point.*

Proof. Define the map $\alpha\colon X \times X \to [0,\infty)$ by

$\alpha(x,y) = 1$ if $D(x,Tx) \le d(x,y) + \psi(d(x,y))$ or $x = y$, and $\alpha(x,y) = 0$, otherwise. Then, it is easy to check that T is α-admissible.

Fix $x_0 \in X$ and $x_1 \in Tx_0$. Then,

$$D(x_0,Tx_0) \le d(x_0,x_1) \le d(x_0,x_1) + \psi(d(x_0,x_1)),$$

and so $\alpha(x_0,x_1) = 1$. Also, note that $\alpha(x,y)H(Tx,Ty) \le \psi(M(x,y))$ for all x,y in X. By using Theorem 3.1, it is sufficient we show that X has the property (C_α).

Let $\{x_n\}$ be a sequence in X such that $x_n \to z$ and $\alpha(x_n,x_{n+1}) \ge 1$ for all n. If there exists a natural number N such that $x_{n+1} = x_n$ for all $n \ge N$, then we have nothing to prove. Thus, we can suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k+1} \ne x_{n_k}$ and

$$D(x_{n_k},Tx_{n_k}) \le d(x_{n_k},x_{n_k+1}) + \psi(d(x_{n_k},x_{n_k+1})).$$

First, we show that $D(z,Tx) \le \psi(d(z,x))$ for all $x \in X\backslash\{z\}$. Since $x_{n_k} \to z$, there exists a natural number K such that $d(z,x_{n_k}) \le \frac{1}{3}d(z,x)$ for all $k \ge K$. Then,

$$d(x_{n_k},x_{n_k+1}) \le d(z,x_{n_k}) + d(z,x_{n_k+1}) \le \frac{2}{3}d(z,x)$$
$$= d(z,x) - \frac{1}{3}d(z,x) \le d(z,x) - d(z,x_{n_k}) \le d(x_{n_k},x)$$

for all $k \ge K$. Also, we have

$$D(x_{n_k},Tx_{n_k}) \le d(x_{n_k},x_{n_k+1}) + \psi(d(x_{n_k},x_{n_k+1})) \le d(x_{n_k},x) + \psi(d(x_{n_k},x)) \to 0.$$

On the other hand, we have

$$d(x_{n_k},Tx) - d(x_{n_k},Tx_{n_k}) \le H(Tx_{n_k},Tx)$$
$$\le \psi(\max\{d(x_{n_k},x), D(x_{n_k},Tx_{n_k}), \frac{1}{2}(D(x_{n_k},Tx) + D(x,Tx_{n_k}))\})$$
$$\le \psi(\max\{d(x_{n_k},x), D(x_{n_k},Tx_{n_k}), 1/2(D(x_{n_k},Tx) + d(x_{n_k},x) + D(x_{n_k},Tx_{n_k}))\}).$$

and so $D(z,Tx) \le \psi(\max\{d(z,x), \frac{1}{2}(D(z,Tx) + d(z,x))\})$. Hence,

$$D(z,Tx) \le \psi(d(z,x))$$

or $D(z,Tx) \le \psi(1/2(D(z,Tx)+d(z,x)))$. If $D(z,Tx) \le \psi(1/2(D(z,Tx)+d(z,x)))$, then $D(z,Tx) \le 1/2\psi(D(z,Tx)) + 1/2\psi(d(z,x))$ and so

$$2D(z,Tx) \le \psi(D(z,Tx)) + \psi(d(z,x)).$$

Thus, $D(z,Tx) \le 2D(z,Tx) - \psi(D(z,Tx)) \le \psi(d(z,x))$, and this proves the claim. Therefore, we obtain $D(x,Tx) \le d(x,z) + D(z,Tx) \le d(x,z) + \psi(d(x,z))$ for all $x \in X\backslash\{z\}$.

Now, we can assume that $x_{n_k} \ne z$ for all k, because $\alpha(x_{n_k},z) = 1$ whenever $x_{n_k} = z$. Hence, $D(x_{n_k},Tx_{n_k}) \le d(x_{n_k},z) + \psi(d(x_{n_k},z))$ for all k. Thus, we get $\alpha(x_{n_k},z) \ge 1$, for all k, and so X has the property (C_α). □

Corollary 3.5. *Let (X, d) be a complete metric space, $r \in [0, 1)$ and a multifunction $T\colon X \to CB(X)$ such that $\frac{1}{1+r}D(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq rK(x, y)$ for all $x, y \in X$. Then T has a fixed point.*

4. **Conclusion**

In this article, we introduce fixed point results of Suzuki type quasi-contractive selfmaps and multifunctions. Our results are extensions of several results as in relevant items from the reference section of this paper, as well as in the literature in general. For stability results related to our fixed point research, please see [18], [29].

REFERENCES

1. S. M. A. Aleomraninejad, Sh. Rezapour, N. Shahzad, *On generalizations of the Suzuki's method*, Appl. Math. Lett. 24 (2011) 1037–1040.
2. S. M. A. Aleomraninejad, Sh. Rezapour, N. Shahzad, *Some fixed point results on a metric space with a graph*, Topology Appl. 159 (2012) 659–663.
3. A. Amini-Harandi, *Fixed point theory for set-valued quasi-contraction maps in metric spaces*, Appl. Math. Lett. 24 (2011) 1791–1794.
4. J. H. Asl, Sh. Rezapour, N. Shahzad, *On fixed points of α-ψ-contractive multifunctions*, Fixed Point Theory Appl. Vol. 2012, ID 2012:212.
5. H. Aydi, E. Karapınar, M. Postolache, *Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces*, Fixed Point Theory Appl. Vol. 2012, ID: 2012:44.
6. H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa, N. Tahat, *Theorems for Boyd-Wong type contractions in ordered metric spaces*, Abstr. Appl. Anal. Vol. 2012, ID: 359054.
7. H. Aydi, M. Postolache, W. Shatanawi, *Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl. 63 (2012), No. 1, 298–309.
8. I. Beg, A. R. Butt, S. Radojevic, *The contraction principle for set valued mappings on a metric space with a graph*, Comput. Math. Appl. 60 (2010) 1214–1219.
9. S. Chandok, M. Postolache, *Fixed point theorem for weakly Chatterjea-type cyclic contractions*, Fixed Point Theory Appl. Vol. 2013, ID: 2013:28, 9 pp.
10. S. Chandok, Z. Mustafa and M. Postolache, *Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces*, U. Politeh. Buch. Ser. A (in printing).
11. L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. 45 (1974) 267–273.
12. S. Dhompongsa, H. Yingtaweesittikul, *Fixed point for multivalued mappings and the metric completeness*, Fixed Point Theory Appl. Vol. 2009, ID 972395, 15 pp.
13. F. Echenique, *A short and constructive proof of Tarski's Fixed point theorem*, Internat. J. Game Theory 33 (2005) No. 2, 215–218.
14. R. Espinola, W. A. Kirk, *Fixed point theorems in R-trees with applications to graph theory*, Topology Appl. 153 (2006) 1046–1055.
15. R. Espinola, P. Lorenzo, A. Nicolae, *Fixed points, selections and common fixed points for nonexpansive-type mappings*, J. Math. Anal. Appl. 382 (2011) 503–515.
16. G. Gwozdz-Lukawska, J. Jachymski, *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl. 356 (2009) 453–463.
17. R. H. Haghi, Sh. Rezapour, N. Shahzad, *On fixed points of quasi-contraction type multifunctions*, Appl. Math. Lett. 25 (2012) 843–846.
18. R. H. Haghi, M. Postolache, Sh. Rezapour, *On T-stability of the Picard iteration for generalized φ-contraction mappings*, Abstr. Appl. Anal. Vol. 2012, ID: 658971, 7 pp.
19. D. Ilic, V. Rakocevic, *Quasi-contraction on a cone metric space*, Appl. Math. Lett. 22 (2009) 728–731.
20. J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. 136 (2008) No. 4, 1359–1373.
21. J. Jachymski, *Equivalent conditions for generalized contractions on (ordered) metric spaces*, Nonlinear Anal. 74 (2011) 768–774.

22. Z. Kadelburg, S. Radenovic, V. Rakocevic, *Remarks on "quasi-contraction on a cone metric space"*, Appl. Math. Lett. 22 (2009) 1674–1679.
23. M. Kikkawa, T. Suzuki, *Some similarity between contractions and Kannan Mappings*, Fixed Point Theory Appl. (2008) ID 649749.
24. M. Kikkawa, T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal. 69 (2008) 2942–2949.
25. D. Miheţ, *A Banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets Syst. 144 (2004) 431–439.
26. B. Mohammadi, Sh. Rezapour, N. Shahzad, *Some results on fixed points of α-ψ-quasi-contractive multifunctions*, Submitted.
27. G. Mot, A. Petrusel, *Fixed point theory for a new type of contractive multivalued operators*, Nonlinear Anal. 70 (2009) 3371–3377.
28. A. Nicolae, D. O'Regan, A. Petruşel, *Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph*, Georgian Math. J. 18 (2011) 307–327.
29. M. O. Olatinwo, M. Postolache, *Stability results for Jungck-type iterative processes in convex metric spaces*, Appl. Math. Comput. 218 (2012), No. 12, 6727–6732.
30. D. O'Regan, A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. 341 (2008) 1241–1252.
31. Sh. Rezapour, R. H. Haghi, N. Shahzad, *Some notes on fixed points of quasi-contraction maps*, Appl. Math. Lett. 23 (2010) 498–502.
32. B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. 75 (2012) 2154–2165.
33. W. Shatanawi, M. Postolache, *Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces*, Fixed Point Theory Appl. Vol. 2013, ID: 2013:54, 17 pp.
34. W. Shatanawi, M. Postolache, *Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces*, Fixed Point Theory Appl. Vol. 2013, ID: 2013:60, 13 pp.
35. W. Shatanawi, M. Postolache, *Some fixed point results for a G-weak contraction in G-metric spaces*, Abstr. Appl. Anal. Vol. 2012, ID: 815870, 19 pp.
36. W. Shatanawi, S. Chauhan, M. Postolache, M. Abbas, S. Radenović, *Common fixed points for contractive mappings of integral type in G-metric spaces*, J. Adv. Math. Stud. 6 (2013), No. 1, 53–72.
37. W. Shatanawi, Ariana Pitea, *Some coupled fixed point theorems in quasi-partial metric spaces*, Fixed Point Theory Appl. Vol. 2013, ID: 2013:153.
38. P. V. Subrahmanyam, *Completeness and fixed points*, Monatsh. Math. 74 (1969) No. 4, 325–330.
39. T. Suzuki, *A generalized Banach contraction principle that characterizes metric completness*, Proc. Amer. Math. Soc. 136 (2008) 1861–1869.
40. L. Zhilong, *Fixed point theorems in partially ordered complete metric spaces*, Math. Comput. Modeling 54 (2011) 69–72.