NULL CONTROLLABILITY FOR A HEAT EQUATION WITH PERIODIC BOUNDARY CONDITIONS

Isil Oner[1]

This paper deals with the null controllability of the heat equation with periodic boundary conditions. The null controllability problem is reduced to the exponential moment problem by using the Fourier series expansion. The moment problem is solved by utilizing an appropriate biorthogonal family of functions.

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1. Introduction

The controllability of parabolic differential equations has attracted a remarkable interest in control theory over the last five decades (see [11, 7, 6, 5, 19, 14, 15]).

The controllability problem for linear parabolic equations in one-dimensional space is first addressed by Fattorini and Russel [8, 7] in which *the moment method* is employed for the solution of the problem in one-dimensional space. Lebeau and Rabbiano [13] solved the null controllability of the heat equation with a constant coefficient. Later, Fursikov and Imanuvilov [10] demonstrated that the null controllability of a general second order parabolic equation can be treated by utilizing a new approach based on *Carleman estimates.* The null controllability problem of the systems governed by semilinear parabolic equations is solved in [9]. Moreover, L. Pandolfi in [16] reduced the controllability problem of the heat equation with memory to a suitable *moment problem* with respect to the Riesz system. Along these lines, S. Avdonin and L. Pandolfi [1] addressed simultaneous temperature and flux controllability problems for heat equations with memory by using the *moment method,* where they introduced L-bases and Riesz bases especially suited to heat equations with memory. The papers listed above considered the controllability problem under the Dirichlet and Neumann boundary conditions.

Up to our knowledge, there are limited results on the null controllability problem with periodic boundary conditions. For instance, O. Yu. Imanuvilov considered the controllability problem for the Boussinesq system with periodic boundary conditions [12]. Beauchard and Zuazua [2] studied the null controllability problem of the Kolmogorov equation under the periodic boundary conditions. Using *the method of the moments*, S. Chowdhury, D. Mitra [4] proved that the linearized compressible Navier-Stokes equations with periodic boundary conditions are null controllable.

In this paper, *the moment method* is applied to the one-dimensional heat equation with periodic boundary conditions and the necessary and sufficient conditions for null controllability of the equation are obtained. Here, the objective is to control the heat differences at the boundaries.

[1]Department of Mathematics in Faculty of Science, Gebze Technical University, 41400 Gebze, Kocaeli, Turkey, e-mail: `ioner@gtu.edu.tr`

The outline of this paper can be summarized as follows. In Section 2, we provide the necessary and sufficient conditions for the null controllability of the systems with periodic boundary conditions. This is achieved by reducing the null controllability problem to the moment problem. Finally, in Section 3 we introduce the main results.

2. **Null controllability for a heat equation with periodic boundary conditions**

We focus on the following one-dimensional heat equation:

$$\begin{cases} y_t - y_{xx} + cy = 0, & 0 < x < 1, \quad 0 < t < T, \\ y(1,t) - y(0,t) = u(t), & 0 < t < T, \\ y_x(1,t) - y_x(0,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $c > 0$ is a real number, $y^0(x) \in L^2(0,1)$ and $u(t) \in L^2(0,T)$.

System (1) is called a null-controllable in time T if for every $y^0 \in L^2(0,1)$ there exits $u(t) \in L^2(0,T)$ such that the corresponding solution of (1) satisfies

$$y(x,T) = 0 \text{ for all } x \in (0,1).$$

Lemma 2.1. *The system* (1) *is null controllable, in time* $T > 0$ *if and only if for any* $y^0 \in L^2(0,1)$ *there exist* $u(t) \in L^2(0,T)$ *such that*

$$\int_0^1 y^0(x)\varphi(x,0)dx - \int_0^T u(t)\varphi_x(0,t)dt = 0 \tag{2}$$

holds for all $\varphi^0 \in L^2(0,1)$*, where* $\varphi(x,t)$ *is the solution of the following backward adjoint problem.*

$$\begin{cases} \varphi_t + \varphi_{xx} - c\varphi = 0, & 0 < x < 1, \quad 0 < t < T, \\ \varphi(1,t) - \varphi(0,t) = 0, & 0 < t < T, \\ \varphi_x(1,t) - \varphi_x(0,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1. \end{cases} \tag{3}$$

Proof. Let $u \in L^2(0,T)$ be arbitrary, and y be a solution of (1). Let us denote by φ the solution of the adjoint problem. Multiplying (1) by φ and integrating the resulting equation over $(0,1) \times (0,T)$, one gets the following equation.

$$\int_0^1 y(x,T)\varphi^0(x)dx - \int_0^1 y^0(x)\varphi(x,0)dx + \int_0^T u(t)\varphi_x(0,t)dt = 0 \tag{4}$$

If equation (2) holds, then $\int_0^1 y(x,T)\varphi^0(x)dx = 0$ for all $\varphi^0(x) \in L^2(0,1)$ and $y(x,T) = 0$. This would mean that system (1) is null-controllable and $u(t)$ is a control for the system.

Conversely, suppose that $u(t)$ is a control for system (1). Then, $y(x,T) = 0$. From (4) it may be concluded that equation (2) holds, which completes the proof. □

We use the basis of $L^2(0,1)$ formed by the eigenfunctions of the following second order auxiliary spectral problem of (3).

$$\begin{cases} X''(x) + (\lambda - c)X = 0, & 0 < x < 1 \\ X(0) = X(1) \\ X'(0) = X'(1) \end{cases} \tag{5}$$

which is self-adjoint in $L^2(0,1)$. The eigenvalues and normalized eigenfunctions of this auxiliary spectral problem are

$$\lambda_0 = c \quad (c > 0), \quad X_0(x) = 1,$$
$$\lambda_n = (2n\pi)^2 + c, \quad X_{2n-1}(x) = \sqrt{2}\cos(2n\pi x), \quad X_{2n}(x) = \sqrt{2}\sin(2n\pi x)$$

for $n = 1, 2, \ldots$ Therefore, any initial data $\varphi^0(x) \in L^2(0,1)$ can be expressed in terms of the eigenfunctions as a Fourier series.

Now, we can prove the following Lemma.

Lemma 2.2. *The system* (1) *is null controllable in time* $T > 0$ *if and only if for any* $y^0 \in L^2(0,1)$ *with Fourier expansion*

$$y^0(x) = \beta_0 + \sum_{n=1}^{\infty} \left[\beta_{2n-1}\sqrt{2}\cos(2n\pi x) + \beta_{2n}\sqrt{2}\sin(2n\pi x)\right]$$

there exists a function $f(t) \in L^2(0,T)$ *such that*

$$\begin{aligned} \int_0^T f(t)e^{-\lambda_1 t}dt &= \frac{\beta_0\eta_0 e^{-\lambda_0 T}}{\eta_2 2\sqrt{2}\pi} \\ \int_0^T f(t)e^{-\lambda_n t}dt &= \frac{e^{-\lambda_{n-1}T}[\beta_{2n-3}\eta_{2n-3} + \beta_{2n-2}\eta_{2n-2}]}{\eta_{2n}2\sqrt{2}\pi n} \quad n = 2, 3, \ldots \end{aligned} \tag{6}$$

with $\eta_{2n} = \int_0^1 \varphi^0(x)\sqrt{2}\sin(2n\pi x)dx \neq 0$ *for* $n = 1, 2\ldots$

Proof. It follows from Lemma 2.1 that u is a control for system (1) if and only if equation (2) holds. Since $\{1, \sqrt{2}\cos(2n\pi x), \sqrt{2}\sin(2n\pi x)\}_{n=1}^{\infty}$, is an orthonormal basis for $L^2(0,1)$, the equation (2) holds if and only if it holds for any orthonormal basis. Hence, the solution of (3) can be represented as follows.

$$\varphi(x,t) = \eta_0 e^{-\lambda_0(T-t)} + \sum_{n=1}^{\infty} e^{-\lambda_n(T-t)}\left[\eta_{2n-1}\sqrt{2}\cos(2n\pi x) + \eta_{2n}\sqrt{2}\sin(2n\pi x)\right],$$

where

$$\eta_0 = \int_0^1 \varphi^0(x)dx, \text{ and } \eta_{2n-1} = \int_0^1 \varphi^0(x)\sqrt{2}\cos(2n\pi x)dx$$
$$\eta_{2n} = \int_0^1 \varphi^0(x)\sqrt{2}\sin(2n\pi x)dx \quad \text{for } n = 1, 2, \ldots$$

Utilizing $\varphi(x,t)$ and $y^0(x)$ from (2), we obtain

$$\int_0^T u(t)e^{-\lambda_1(T-t)}\eta_2 2\sqrt{2}\pi dt = \beta_0\eta_0 e^{-\lambda_0 T}$$

and

$$\int_0^T u(t)e^{-\lambda_n(T-t)}\eta_{2n}2\sqrt{2}\pi n dt = e^{-\lambda_{n-1}T}[\beta_{2n-3}\eta_{2n-3} + \beta_{2n-2}\eta_{2n-2}] \quad n = 2, 3, \ldots$$

After replacing $T - t$ by t in the last integrals and choosing $u(T-t) = f(t)$, the proof is completed. □

In order to determine control $u(t)$, we choose a function $f(t)$ which satisfies (6). This is a moment problem in $L^2(0,T)$ with respect to the family $\Lambda = \{e^{-\lambda_n t}\}_{n\geq 0}$. Suppose that we can construct $\{\Psi_m\}_{m\geq 0}$ of functions biorthogonal to the set Λ in $L^2(0,T)$ such that

$$\int_0^T e^{-\lambda_n t}\Psi_m(t)dt = \delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

for all $m, n = 0, 1, 2 \dots$. Then, the moment problems (6) have solutions by setting

$$f(t) = \frac{\beta_0 \eta_0 e^{-\lambda_0 T}}{\eta_2 2\sqrt{2}\pi} \Psi_1 + \sum_{m=2}^{\infty} \frac{e^{-\lambda_{m-1} T} [\beta_{2m-3}\eta_{2m-3} + \beta_{2m-2}\eta_{2m-2}]}{\eta_{2m} 2\sqrt{2} m\pi} \Psi_m.$$

If this series converges in $L^2(0,T)$, we will obtain the solution of (6).

Now, we are in a position to prove the main result of this paper.

Theorem 2.1. *Given any $T > 0$, suppose that there exists a sequence $\{\Psi_m(t)\}_{m \geq 0}$ in $L^2(0,T)$ biorthogonal to the set Λ such that*

$$\|\Psi_m\|_{L^2(0,T)} \leq K e^{m\rho}, \quad \forall m \geq 0 \tag{7}$$

holds, where K and ρ are two positive constants. Then, system (1) *is null-controllable in time T.*

Proof. According to Lemma (2.2), system (1) is null controllable in time T if for any $y^0 \in L^2(0,1)$ with Fourier expansion

$$y^0(x) = \beta_0 + \sum_{n=1}^{\infty} \left[\beta_{2n-1}\sqrt{2}\cos(2n\pi x) + \beta_{2n}\sqrt{2}\sin(2n\pi x)\right],$$

there exists a function $f(t) \in L^2(0,T)$ for which equation (6) holds. Choose

$$f(t) = \frac{\beta_0 \eta_0 e^{-\lambda_0 T}}{\eta_2 2\sqrt{2}\pi} \Psi_1 + \sum_{m=2}^{\infty} \frac{e^{-\lambda_{m-1} T} [\beta_{2m-3}\eta_{2m-3} + \beta_{2m-2}\eta_{2m-2}]}{\eta_{2m} 2\sqrt{2} m\pi} \Psi_m. \tag{8}$$

Since $\|\Psi_m\|_{L^2(0,T)} \leq K e^{m\rho}$, for all $m \geq 0$, we deduce that

$$\begin{aligned}
&\sum_{m=2}^{\infty} \Big\| \frac{e^{-\lambda_{m-1} T} [\beta_{2m-3}\eta_{2m-3} + \beta_{2m-2}\eta_{2m-2}]}{\eta_{2m} 2\sqrt{2} m\pi} \Psi_m \Big\|_{L^2(0,T)} \\
&\leq \sum_{m=2}^{\infty} \frac{1}{2\sqrt{2} m\pi} \Big| \frac{\beta_{2m-3}\eta_{2m-3} + \beta_{2m-2}\eta_{2m-2}}{\eta_{2m}} \Big| e^{-\lambda_{m-1} T} \|\Psi_m\|_{L^2(0,T)} \\
&\leq K_1 \sum_{m=2}^{\infty} \frac{1}{2\sqrt{2} m\pi} \Big| \frac{\beta_{2m-3}\eta_{2m-3} + \beta_{2m-2}\eta_{2m-2}}{\eta_{2m}} \Big| e^{-\lambda_{m-1} T + m\rho} < \infty
\end{aligned}$$

i.e., $f(t)$ converges in $L^2(0,T)$. Hence, (8) implies that $f(t)$ satisfies (6) and this ends the proof. □

It is clear from Theorem 2.1 that the system (1) is null controllable if the biorthogonal sequence exists and satisfies (7). To this end, we first prove the existence of a biorthogonal sequence by using Muntz Theorem. Then, we calculate the estimations of $\|\Psi_m\|_{L^2(0,T)}$ for $m \geq 0$.

Fattoroni and Russell's well-known result states that if the exponential moment problem is solvable for $T = \infty$, then it is solvable for every time $T > 0$ (see [8]). Therefore, we begin by finding the estimations of $\|\Psi_m\|_{L^2(0,\infty)}$ for $m \geq 0$. Later, using these estimations we obtain estimations of $\|\Psi_m\|_{L^2(0,T)}$ for $m \geq 0$. The methods of proof the following theorems are inspired by the methods utilized in [8, 19].

The Muntz theorem will be applied to prove the existence of the biorthogonal sequence (see e.g., Theorem 15.26 in [17]). Let $E(\Lambda, T)$ denote the minimal closed subspace of $L^2(0,T)$ spanned by the functions $p_n(t) = e^{-\lambda_n t}$ for $n \geq 0$. Since,

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{(2\pi n)^2 + c} < \infty, \tag{9}$$

it follows from the Muntz Theorem that $E(\Lambda, T)$ is a proper space of $L^2(0,T)$ generated by Λ. Moreover, let $E(m,\Lambda,T)$ denote closed subspace spanned by $\{p_n(t)|n \neq m\}$. If (9) is true, then $E(m,\Lambda,T)$ is the minimal closed subspace of $E(\Lambda,T)$.

If there is a unique element $r_m \in E(m,\Lambda,T)$ such that

$$\|p_m - r_m\|_{L^2(0,T)} = \min_{r\in E(m,\Lambda,T)} \|p_m - r\|_{L^2(0,T)},$$

then the functions

$$\Psi_m(t) = \frac{p_m - r_m}{\|p_m - r_m\|^2_{L^2(0,T)}} \tag{10}$$

all lie in $E(\Lambda,T)$ and provide biorthogonal set $\{\Psi_m(t)\}$ for the set $\{e^{-\lambda_m t}\}$ in $L^2(0,T)$. Also, it is easy to check that $\|\Psi_m(t)\|_{L^2(0,T)}$ is minimal.

Now, we can evaluate the norm of the biorthogonal sequence $L^2(0,T)$ for $T=\infty$. Let $E^n := E^n(\Lambda,\infty)$ is the subspace generated by $\Lambda^n := \{e^{-\lambda_k t}\}_{0\le k\le n}$ in $L^2(0,T)$ and $E^n_m := E^n(m,\Lambda,\infty)$ is the subspace generated by $\Lambda^n_m := \{e^{-\lambda_k t}\}_{\substack{0\le k\le n\\ k\neq m}}$ in $L^2(0,T)$. Note that E^n and E^n_m are finite dimensional subspaces and

$$E(\Lambda,\infty) = \bigcup_{n=0}^{\infty} E^n \qquad E(\Lambda_m,\infty) = \bigcup_{n=0}^{\infty} E^n_m.$$

For each $n \ge 0$, there exists a unique orthogonal family $\{\Psi^n_m\}_{0\le m\le n} \subset E^n$ to the family of Λ^n, where

$$\Psi^n_m = \frac{p_m - r_m}{\|p_m - r_m\|^2_{L^2(0,\infty)}}, \tag{11}$$

and r^n_m is the orthogonal projection of p_m over E^n_m. If

$$\Psi^n_m = \sum_{k=0}^{n} c^m_k p_k, \tag{12}$$

then multiplying (12) by p_l and by integrating over $(0,\infty)$, it follows that

$$\delta_{ml} = \sum_{k=0}^{n} c^m_k \int_0^\infty p_k p_l dt \qquad 0 \le m, l \le n. \tag{13}$$

Also, multiplying (12) by Ψ^n_m and by integrating over $(0,\infty)$, we obtain

$$\|\Psi^n_m\|^2_{L^2(0,\infty)} = c^m_m.$$

If $G = [g^l_k]$ is the Gramm matrix of the family Λ such that

$$g^l_k = \int_0^\infty p_k(t)p_l(t)dt, \qquad 0 \le k, l \le n,$$

then, from (13) it follows that the elements of the inverse G give us c^m_k. If $|G|$ is determinant of the matrix G and $|G_m|$ is determinant of the matrix G_m obtained by changing the $m-th$ column of G by the $m-th$ vector of canonical basis, then using Cramer Rule, we get

$$c^m_m = \frac{|G_m|}{|G|}.$$

Therefore,

$$\|\Psi^n_m\|^2_{L^2(0,\infty)} = \sqrt{\frac{|G_m|}{|G|}}.$$

We calculate the elements of G as follows. For the sake of simplicity in computation, we can choose $c = 2s\pi^2$ where s is any positive number. Then,

$$g_k^n = \int_0^\infty p_k(t)p_n(t)dt = \int_0^\infty e^{-(4n^2\pi^2+4k^2\pi^2+2c)t}dt = \frac{1}{4n^2\pi^2 + 4k^2\pi^2 + 4s\pi^2}.$$

In order the calculate the determinants of G and G_m, we invoke the following lemma.

Lemma 2.3 (Cauchy's Lemma [3])**.**

$$\prod_{(i,j)} (x_i + y_j) \begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix} = \prod_{i<j} (x_j - x_i)(y_j - y_i).$$

By using Cauchy's Lemma, the determinants can be calculated as follows:

$$|G| = \frac{\prod\limits_{0\le i<j\le n} \left(4j^2\pi^2 - 4i^2\pi^2\right)^2}{\prod\limits_{0\le i,j\le n} \left(4i^2\pi^2 + 4j^2\pi^2 + 4s\pi^2\right)} \qquad |G_m| = \frac{\prod\limits_{\substack{0\le i<j\le n\\ i,j\neq m}} \left(4j^2\pi^2 - 4i^2\pi^2\right)^2}{\prod\limits_{\substack{0\le i,j\le n\\ i,j\neq m}} \left(4i^2\pi^2 + 4j^2\pi^2 + 4s\pi^2\right)}.$$

Since

$$\frac{|G_m|}{|G|} = 4\pi^2(2m^2+s) \prod_{\substack{k=0\\k\neq m}}^{n} \frac{\left(m^2+k^2+s\right)^2}{\left(m^2 - k^2\right)^2},$$

we have

$$\|\Psi_m^n\|_{L^2(0,\infty)} = \sqrt{\frac{|G_m|}{|G|}} = 2\pi\sqrt{2m^2+s} \prod_{\substack{k=0\\k\neq m}}^{n} \frac{m^2+k^2+s}{|m^2 - k^2|} \tag{14}$$

Now, we are in a position to prove that the following Lemma.

Lemma 2.4. *The norm of biorthogonal sequence* $(\Psi_m(t))_{m\ge 0}$ *to the family* Λ *in* $L^2(0,\infty)$ *given in* (10) *satisfies the following result.*

$$\|\Psi_m(t)\|_{L^2(0,\infty)} = 2\pi\sqrt{2m^2+s} \prod_{\substack{k=0\\k\neq m}}^{\infty} \frac{m^2+k^2+s}{|m^2 - k^2|} \tag{15}$$

Proof. By evaluating the limit in (14) as $n \to \infty$, we obtain (15). Also, the product

$$\prod_{\substack{k=0\\k\neq m}}^{\infty} \frac{m^2+k^2+s}{|m^2 - k^2|}.$$

converges for all $m \geq 0$. Indeed, the product can be rewritten in the new form as follows.

$$\begin{aligned}
1 \leq \prod_{\substack{k=0\\k\neq m}}^{\infty} \frac{m^2+k^2+s}{|m^2 \ -k^2|} &= \exp\Bigg[\sum_{\substack{k=0\\k\neq m}}^{\infty} \ln\Big(\frac{m^2+k^2+s}{|m^2 \ -k^2|}\Big)\Bigg] \\
&= \exp\Bigg[\sum_{k=0}^{m-1} \ln\Big(1+\frac{2k^2+s}{m^2-k^2}\Big) + \sum_{k=m+1}^{\infty} \ln\Big(1+\frac{2m^2+s}{k^2-m^2}\Big)\Bigg] \\
&\leq \exp\Bigg[\sum_{k=0}^{m-1} \ln\Big(1+\frac{2m^2+s}{m^2-k^2}\Big) + \sum_{k=m+1}^{\infty} \ln\Big(1+\frac{2m^2+s}{k^2-m^2}\Big)\Bigg] \\
&= \exp\Bigg[\sum_{\substack{k=0\\k\neq m}}^{\infty} \ln\Big(1+\frac{2m^2+s}{|m^2-k^2|}\Big)\Bigg]
\end{aligned}$$

Since $\ln(1+x) < x$ for all $x > 0$, we conclude that

$$\prod_{\substack{k=0\\k\neq m}}^{\infty} \frac{m^2+k^2+s}{|m^2 \ -k^2|} \leq \exp\Big((2m^2+s)\sum_{\substack{k=0\\k\neq m}}^{\infty} \frac{1}{|m^2-k^2|}\Big) < \infty.$$

Applying this result in (15), we conclude that $\lim_{n\to\infty} \|\Psi_m^n(t)\|_{L^2(0,\infty)}$ exists. Also, it is easy to check that

$$\lim_{n\to\infty} \|\Psi_m^n(t)\|_{L^2(0,\infty)} = \lim_{n\to\infty} \|\Psi_m(t)\|_{L^2(0,\infty)}. \tag{16}$$

This completes the proof. □

Remark 2.1. *The general estimations of $\|\Psi_m\|_{L^2(0,\infty)}$ have already been calculated by H.O. Fattoroni and D. L. Russell. They have shown in* [7] *that if the λ_n are real and satisfy the following asymptotic relationship*

$$\lambda_n = K(n+\alpha)^\zeta + o(n^{\zeta-1}) \quad (n\to\infty)$$

where $K > 0, \zeta > 1$ and α is real, then there exists constants $\hat{K}, K_\zeta$ such that

$$\|\Psi_n(t)\|_{L^2(0,\infty)} \leq \hat{K}\exp[(K_\zeta + o(1))\lambda_n^{1/\zeta}] \quad (n \geq 1)$$

where $o(1)$ represents a term tending to zero as n goes to infinity.The computation of the constant K_ζ is given in [8]*. Since $\lambda_n = (2n\pi)^2 + c$, it can be seen that we can find an estimation for the case which is taking into account in this paper. For the sake of completeness, we will calculate the details for this particular case as follows.*

The following lemma provides an upper bound for the norm of $\{\Psi_m(t)\}_{m\geq 0}$ in $L^2(0,\infty)$.

Lemma 2.5. *There exist two positive constants K and ρ such that for any $m \geq 1$*

$$\|\Psi_m(t)\|_{L^2(0,\infty)} \leq Ke^{m\rho}$$

where $\rho = c_1 + 2\sqrt{c_1}\arctan(\frac{1}{\sqrt{c_1}})$ and $K = 2\pi(\frac{(1+c_1)^{c_1+2}}{c_1^{c_1}})\sqrt{2m^2+s}$ and $c_1 = 2 + \frac{s}{m^2}$. Also, the following relation holds for $m = 0$.

$$\|\Psi_0(t)\|_{L^2(0,\infty)} \leq 2\pi\sqrt{s}e^{s/3}.$$

Proof. We distinguish two cases. Firstly, suppose that $m \geq 1$. Note that

$$\prod_{\substack{k=0\\k\neq m}}^{\infty} \frac{m^2+k^2+s}{|m^2 \ -k^2|} \leq \exp\Bigg[\sum_{\substack{k=0\\k\neq m}}^{\infty} \ln\Big(1+\frac{2m^2+s}{|m^2 \ -k^2|}\Big)\Bigg].$$

Then,

$$\sum_{\substack{k=0\\k\neq m}}^{\infty}\ln\Big(1+\frac{2m^2+s}{|m^2\ -k^2|}\Big)\leq\int_0^m\ln\Big(1+\frac{2m^2+s}{m^2\ -x^2}\Big)dx+\int_m^{2m}\ln\Big(1+\frac{2m^2+s}{x^2-m^2}\Big)dx$$

$$+\int_{2m}^{\infty}\ln\Big(1+\frac{2m^2+s}{x^2-m^2}\Big)dx$$

$$=m\Big[\int_0^1\ln\Big(1+\frac{c_1}{1\ -x^2}\Big)dx+\int_1^2\ln\Big(1+\frac{c_1}{x^2-1}\Big)dx$$

$$+\int_2^{\infty}\ln\Big(1+\frac{c_1}{x^2-1}\Big)dx\Big]$$

$$=m(I_1+I_2+I_3)$$

where $c_1=2+\dfrac{s}{m^2}$.

These integrals can be evaluated as follows.

$$I_1=\int_0^1\ln\Big(1+\frac{c_1}{1\ -x^2}\Big)dx=\int_0^1\ln\Big(1+\frac{c_1}{(1-x)(1+x)}\Big)dx$$

$$\leq\int_0^1\ln\Big(1+\frac{c_1}{(1-x)}\Big)dx=(x-1)\ln\Big(1+\frac{c_1}{1-x}\Big)|_0^1+\int_0^1\frac{c_1}{1+c_1-x}dx$$

$$=\ln\frac{(1+c_1)^{c_1+1}}{c_1^{c_1}}.$$

$$I_2=\int_1^2\ln\Big(1+\frac{c_1}{x^2-1}\Big)dx\leq\int_1^2\ln\Big(1+\frac{c_1}{(x-1)^2}\Big)dx$$

$$=(x-1)\ln\Big(1+\frac{c_1}{(x-1)^2}\Big)|_1^2+\int_1^2\frac{2c_1}{(x-1)^2+c_1}$$

$$=\ln(1+c_1)+2\sqrt{c_1}\arctan(\frac{1}{\sqrt{c_1}}).$$

For the third one, since $\ln(1+x)<x$ for all $x>0$, we have

$$I_3=\int_2^{\infty}\ln\Big(1+\frac{c_1}{x^2-1}\Big)dx\leq\int_2^{\infty}\ln\Big(1+\frac{c_1}{(x-1)^2}\Big)$$

$$\leq\int_2^{\infty}\Big(\frac{c_1}{(x-1)^2}\Big)=c_1.$$

The result is obtained as follows.

$$\|\Psi_m(t)\|_{L^2(0,\infty)}\leq Ke^{m\rho}$$

where $\rho=c_1+2\sqrt{c_1}\arctan(\frac{1}{\sqrt{c_1}})$ and $K=2\pi(\dfrac{(1+c_1)^{c_1+2}}{c_1^{c_1}})\sqrt{2m^2+s}$.

For the second case, suppose that $m=0$. By using equation (15), we obtain

$$\|\Psi_0(t)\|_{L^2(0,\infty)}=\pi^2\sqrt{s}\prod_{k=1}^{\infty}\frac{k^2+s}{k^2}\leq\exp\Big[\sum_{k=1}^{\infty}\ln\Big(1+\frac{s}{k^2}\Big)\Big].$$

Since

$$\sum_{k=1}^{\infty}\ln\Big(1+\frac{s}{k^2}\Big)\Big]\leq\int_1^{\infty}\ln\Big(1+\frac{s}{x^2}\Big)dx\leq\int_1^{\infty}\Big(\frac{s}{x^2}\Big)dx=s/3,$$

we conclude that

$$\|\Psi_0(t)\|_{L^2(0,\infty)} \leq 2\pi\sqrt{s}e^{s/3}.$$

□

Now, we can estimate the norm of the biorthogonal sequence $\{\Psi_m(t)\}_{m\geq 0}$ for $T<\infty$. To this end, we need to use the following result given in [7].

Theorem 2.2. *Let Λ be the family of exponential functions $\{e^{-\lambda_n t}\}_{n\geq 0}$ and let T be arbitrary in $(0,\infty)$. Then, R_T maps $E(\Lambda,\infty)$ onto $E(\Lambda,T)$ in a one-to-one fashion and thus has an inverse $(R_T)^{-1}: E(\Lambda,T)\to E(\Lambda,\infty)$. Moreover, there exists a positive constant C which only depends on T such that $\|(R_T)^{-1}\|\leq C$.*

Proof. The proof of the theorem can be found [7] or [18]. □

In view of the above theorem, we have

$$\begin{aligned} R_T :& E(\Lambda,\infty)\to E(\Lambda,T)\\ R_T(v) &= v|_{[0,T]}. \end{aligned}$$

If $p_n(t)=e^{-\lambda_n t}$ for $t>0$ and $n\geq 0$, then

$$R_T(p_n(t)) = p_n(t)|_{[0,T]}.$$

Also,

$$\begin{aligned} \delta_{n,m} &= (p_n,\Psi_m(t))_{L^2(0,\infty)} = (R_T^{-1}R_T p_n,\Psi_m(t.))_{L^2(0,\infty)}\\ &= (p_n,(R_T^{-1})^*\Psi_m(t))_{L^2(0,T)}. \end{aligned}$$

Therefore, the family $\{(R_T^{-1})^*\Psi_m(t)\}_{m\geq 0}$ is biorthogonal to $\{e^{-\lambda_n t}\}_{n\geq 0}$ in $L^2(0,T)$. Because of the uniqueness of the biorthogonal sequence in $E(\Lambda,T)$ and $\|(R_T^{-1})^*\|=\|R_T^{-1}\|$, it follows that

$$\|\Psi_m(t)\|_{L^2(0,T)} = \|(R_T^{-1})^*\Psi_m(t)\|_{L^2(0,T)} \leq \|R_T^{-1}\|\|\Psi_m(t)\|_{L^2(0,\infty)}.$$

This shows that inequality (7) holds and the proof of Theorem 2.1 is completed.

3. **Conclusion**

This paper studies the null controllability of the heat equation with periodic boundary conditions. The null controllability problem of the system (1) is reduced to moment problems by utilizing the Fourier series expansion. Then, using the solution of these moment problems, it is proved that the system is null controllable.

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