STABILITY OF G-FRAMES, APPROXIMATE DUALS AND RESOLUTIONS OF THE IDENTITY IN HILBERT C^*-MODULES

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In this paper, we consider the stability of g-frames and some concepts related to g-frames on Hilbert C^*-modules, such as approximate duals and (a,m)-approximate duals, under different kinds of perturbations. We also obtain some results for the perturbations of fusion frames and resolutions of the identity in Hilbert C^*-modules. Moreover, some new resolutions of the identity are constructed using morphisms of Hilbert C^*-modules.

Keywords: g-Frame, approximate dual, Hilbert C^*-module.

MSC2020: Primary 42C15, 46L08 Secondary 46H25, 47A05.

1. **Introduction and preliminaries**

A Hilbert C^*-module is a generalization of a Hilbert space by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Frank and Larson in [10] presented a general approach to the frame theory in Hilbert C^*-modules. Also fusion frames and g-frames in Hilbert C^*-modules were introduced in [13].
Different kinds of perturbations for frames, g-frames and fusion frames in Hilbert spaces have been introduced (see [4, 5, 7, 6, 25, 24, 16, 14]). After generalizing the frame theory to Hilbert C^*-modules, some authors studied perturbations of frames and g-frames in Hilbert C^*-modules (see [11, 23, 19]). In this paper, we get some new results in perturbations of frames, g-frames and fusion frames in Hilbert C^*-modules.

Suppose that $\mathfrak{A}$ is a unital C^*-algebra and $\mathcal{X}$ is a left $\mathfrak{A}$-module such that the linear structures of $\mathfrak{A}$ and $\mathcal{X}$ are compatible. $\mathcal{X}$ is a pre-Hilbert $\mathfrak{A}$-module if $\mathcal{X}$ is equipped with an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle\colon \mathcal{X}\times\mathcal{X}\longrightarrow\mathfrak{A}$, that is sesquilinear, positive definite and respects the module action. In other words

(i) $\langle\alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$, for each $\alpha,\beta\in\mathbb{C}$ and $x,y,z\in\mathcal{X}$;
(ii) $\langle ax,y\rangle=a\langle x,y\rangle$, for each $a\in\mathfrak{A}$ and $x,y\in\mathcal{X}$;
(iii) $\langle x,y\rangle=\langle y,x\rangle^*$, for each $x,y\in\mathcal{X}$;
(iv) $\langle x,x\rangle\geq 0$, for each $x\in\mathcal{X}$ and if $\langle x,x\rangle=0$, then $x=0$.

For each $x\in\mathcal{X}$, we define $\|x\|=\|\langle x,x\rangle\|^{\frac{1}{2}}$. If $\mathcal{X}$ is complete with the norm $\|\cdot\|$, it is called a *Hilbert $\mathfrak{A}$-module* or a *Hilbert C^*-module* over $\mathfrak{A}$.
Some typical examples of Hilbert C^*-modules are as follows.
• Every Hilbert space is a left Hilbert $\mathbb{C}$-module.
• Every C^*-algebra $\mathfrak{A}$ is a Hilber $\mathfrak{A}$-module with the inner product $\langle a,b\rangle=ab^*$ for $a,b\in\mathfrak{A}$.

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• For a Hilber space $\mathcal{H}$, the space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a Hilbert $\mathcal{B}(\mathcal{H})$-module via $\langle S,T\rangle = ST^*$.

For each a in a C^*-algebra $\mathfrak{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and we define $|x| = \langle x,x\rangle^{\frac{1}{2}}$, for each $x \in \mathcal{X}$. The *center* of $\mathfrak{A}$ is denoted by $\mathcal{Z}(\mathfrak{A})$ and is defined by

$$\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}.$$

We note that $\mathcal{Z}(\mathfrak{A})$ is a commutative C^*-subalgebra of $\mathfrak{A}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathfrak{A}$-modules. The operator $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is called *adjointable* if there exists an operator $T^*\colon \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Every adjointable operator T is automatically bounded and $\mathfrak{A}$-linear (that is, $T(ax) = aT(x)$ for each $x \in \mathcal{X}$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from $\mathcal{X}$ into $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X},\mathcal{Y})$. Note that $\mathcal{L}(\mathcal{X},\mathcal{X})$ is a C^*-algebra and we denote it by $\mathcal{L}(\mathcal{X})$, for more details see [17].

In this paper, we focus on finitely and countably generated Hilbert C^*-modules over unital C^*-algebras. A Hilbert $\mathfrak{A}$-module $\mathcal{X}$ is *finitely generated* if there exists a finite set $\{x_1,\dots,x_n\} \subseteq \mathcal{X}$ such that every element $x \in \mathcal{X}$ can be expressed as an $\mathfrak{A}$-linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$-module $\mathcal{X}$ is *countably generated* if there exists a countable set $\{x_i\}_{i\in\mathbb{I}} \subseteq \mathcal{X}$ such that $\mathcal{X}$ equals the norm-closure of $\mathfrak{A}$-linear hull of $\{x_i\}_{i\in\mathbb{I}}$.

Let $\mathcal{X}$ be a Hilbert $\mathfrak{A}$-module. A family $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{X}$ is a *frame* for $\mathcal{X}$, if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in \mathcal{X}$,

$$A\langle x,x\rangle \leq \sum_{i\in\mathbb{I}} \langle x, f_i\rangle\langle f_i, x\rangle \leq B\langle x,x\rangle. \tag{1}$$

The numbers A and B are called the lower and upper bound of the frame, respectively. In this case, we call it an (A,B) frame. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (1) converges in norm, the frame is called *standard*.

Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ be standard Bessel sequences in $\mathcal{X}$. Then we say that $\mathcal{G}$ (resp. $\mathcal{F}$) is an *alternate dual* or a *dual* of $\mathcal{F}$ (resp. $\mathcal{G}$), if $x = \sum_{i\in\mathbb{I}} \langle x, f_i\rangle g_i$ or equivalently $x = \sum_{i\in\mathbb{I}} \langle x, g_i\rangle f_i$, for each $x \in \mathcal{X}$.

For more results about frames in Hilbert C^*-modules, see [10, 2].

Let $\{\mathcal{X}_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert $\mathfrak{A}$-modules. A sequence $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{X},\mathcal{X}_i)\colon i \in \mathbb{I}\}$ is called a *g-frame* for $\mathcal{X}$ with respect to $\{\mathcal{X}_i\colon i \in \mathbb{I}\}$ if there exist real constants $A_\Lambda, B_\Lambda > 0$ such that for each $x \in \mathcal{X}$,

$$A_\Lambda\langle x,x\rangle \leq \sum_{i\in\mathbb{I}} \langle \Lambda_i x, \Lambda_i x\rangle \leq B_\Lambda\langle x,x\rangle.$$

A_Λ and B_Λ are g-frame bounds of Λ. In this case, we call it an (A_Λ, B_Λ) g-frame. The g-frame is *standard* if for each $x \in \mathcal{X}$, the sum converges in norm. If only the second-hand inequality is required, Λ is called a *g-Bessel sequence*. If $A_\Lambda = B_\Lambda$, the g-frame is called *tight* and if $A_\Lambda = B_\Lambda = 1$, the g-frame is called *Parseval*.

If $\{\mathcal{X}_i\colon i \in \mathbb{I}\}$ is a sequence of Hilbert $\mathfrak{A}$-modules, then

$$\oplus_{i\in\mathbb{I}}\mathcal{X}_i = \left\{x = \{x_i\}_{i\in\mathbb{I}} : x_i \in \mathcal{X}_i \text{ and } \sum_{i\in\mathbb{I}} \langle x_i, x_i\rangle \text{ is norm convergent in } \mathfrak{A}\right\},$$

is a Hilbert $\mathfrak{A}$-module with pointwise operations and $\mathfrak{A}$-valued inner product

$$\langle x, y\rangle = \sum_{i\in\mathbb{I}} \langle x_i, y_i\rangle,$$

where $x = \{x_i\}_{i\in\mathbb{I}}$ and $y = \{y_i\}_{i\in\mathbb{I}}$.

For a standard g-Bessel sequence Λ, the operator $T_\Lambda\colon \oplus_{i\in\mathbb{I}} \mathcal{X}_i \longrightarrow \mathcal{X}$ which is defined by $T_\Lambda(\{g_i\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} \Lambda_i^* g_i$ is called the *synthesis operator* of Λ. T_Λ is adjointable and $T_\Lambda^*(x) = \{\Lambda_i x\}_{i\in\mathbb{I}}$. The operator $S_\Lambda\colon \mathcal{X} \longrightarrow \mathcal{X}$ which is defined by $S_\Lambda x = T_\Lambda T_\Lambda^*(x) = \sum_{i\in\mathbb{I}} \Lambda_i^*\Lambda_i(x)$, is called the *operator* of Λ. If Λ is a standard (A_Λ, B_Λ) g-frame, then $A_\Lambda \cdot \mathrm{Id}_{\mathcal{X}} \leq S_\Lambda \leq B_\Lambda \cdot \mathrm{Id}_{\mathcal{X}}$. Recall that if $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{I}}$ are standard g-Bessel sequences such that $\sum_{i\in\mathbb{I}} \Gamma_i^*\Lambda_i x = x$ or equivalently $\sum_{i\in\mathbb{I}} \Lambda_i^*\Gamma_i x = x$, for each $x \in \mathcal{X}$, then Γ (resp. Λ) is called a *g-dual* of Λ (resp. Γ).

For more results about g-frames in Hilbert C^*-modules, see [13, 26].
In this paper, all C^*-algebras are unital and all Hilbert C^*-modules are finitely or countably generated.

2. (a, m)-approximate duals, perturbations and adjointable operators

In this section, we consider the stability of (a, m)-approximate duals under perturbations and the construction of new (a, m)-approximate duals using adjointable operators.

Recall that $\ell^\infty(\mathbb{I}, \mathfrak{A})$ is the set

$$\Big\{\{a_i\}_{i\in\mathbb{I}} \subseteq \mathfrak{A} : \|\{a_i\}_{i\in\mathbb{I}}\|_\infty = \sup\{\|a_i\| : i \in \mathbb{I}\} < \infty\Big\}.$$

throughout this paper, m is always a sequence $\{m_i\}_{i\in\mathbb{I}} \in \ell^\infty(\mathbb{I}, \mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in \mathbb{I}$. Each sequence with these properties is called a *symbol.* We recall the following two definitions from [15].

Definition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathfrak{A}$-modules, and let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{X}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}} \subseteq \mathcal{Y}$ be standard Bessel sequences. It was proved in* [15] *that the operator $M_{m,\mathcal{G},\mathcal{F}} \colon \mathcal{X} \longrightarrow \mathcal{Y}$ which is defined by $M_{m,\mathcal{G},\mathcal{F}}(x) = \sum_{i\in\mathbb{I}} m_i\langle x, f_i\rangle g_i$, is adjointable. $M_{m,\mathcal{G},\mathcal{F}}$ is called the* Bessel multiplier *for the Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ with symbol m. If $m_i = \mathbf{1}_{\mathfrak{A}}$, for each $i \in \mathbb{I}$, then we denote $M_{m,\mathcal{G},\mathcal{F}}$ by $M_{\mathcal{G},\mathcal{F}}$.*

In this paper, $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ are standard Bessel sequences in a Hilbert C^*-module $\mathcal{X}$, so $M_{m,\mathcal{G},\mathcal{F}} \in \mathcal{L}(\mathcal{X})$.

Definition 2.2. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{I}}$ be standard g-Bessel sequences for $\mathcal{X}$ with respect to $\{\mathcal{X}_i\}_{i\in\mathbb{I}}$. Then it was shown in* [15] *that the operator $M_{m,\Gamma,\Lambda} \colon \mathcal{X} \longrightarrow \mathcal{X}$ which is defined by $M_{m,\Gamma,\Lambda}(x) = \sum_{i\in\mathbb{I}} m_i\Gamma_i^*\Lambda_i(x)$ is adjointable. $M_{m,\Gamma,\Lambda}$ is called the* g-Bessel multiplier *for the g-Bessel sequences Λ and Γ with symbol m. If $m_i = \mathbf{1}_{\mathfrak{A}}$, for each $i \in \mathbb{I}$, then $M_{m,\Gamma,\Lambda}$ is denoted by $M_{\Gamma,\Lambda}$.*

We recall the definitions of approximate duals and approximate g-duals in Hilbert C^*-modules from [18] (we mention that approximate duals for Hilbert spaces were introduced in [9]).

Definition 2.3. (i) *Two standard g-Bessel sequences Λ and Γ are approximately dual g-frames if $\|Id_{\mathcal{X}} - M_{\Gamma,\Lambda}\| < 1$. In this case, we say that Γ is an approximate g-dual of Λ.*

(ii) *Two standard Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames if $\|Id_{\mathcal{X}} - M_{\mathcal{G},\mathcal{F}}\| < 1$. In this case, we say that $\mathcal{G}$ is an approximate dual of $\mathcal{F}$.*

Note that if $a \in \mathcal{Z}(\mathfrak{A})$ and $T \in \mathcal{L}(\mathcal{X})$, then the operator $aT \colon \mathcal{X} \longrightarrow \mathcal{X}$ which is defined by $(aT)(x) = aT(x)$ is adjointable with $(aT)^* = a^*T^*$.
Now we state the definition of (a, m)-approximate duals from [20].

Definition 2.4. *Let m be a symbol and $a \in \mathcal{Z}(\mathfrak{A})$.*

(i) *Let Λ and Γ be standard g-Bessel sequences. Then we say that Γ is an (a, m)-approximate g-dual (resp. (a, m)-g-dual) of Λ if $\|Id_{\mathcal{X}} - aM_{m,\Gamma,\Lambda}\| < 1$ (resp. $Id_{\mathcal{X}} = aM_{m,\Gamma,\Lambda}$).*

(ii) *Let $\mathcal{F}$ and $\mathcal{G}$ be standard Bessel sequences. Then we say that $\mathcal{G}$ is an (a,m)-approximate dual (resp. (a,m)-dual) of $\mathcal{F}$ if $\|Id_{\mathcal{X}} - aM_{m,\mathcal{G},\mathcal{F}}\| < 1$ (resp. $Id_{\mathcal{X}} = aM_{m,\mathcal{G},\mathcal{F}}$).*

Note that if $a = \mathbf{1}_{\mathfrak{A}}$, $m_i = \mathbf{1}_{\mathfrak{A}}$, for each $i \in \mathbb{I}$, then (a,m)-approximate duality coincides with the concept of approximate duality stated in Definition 2.3.
If Γ is an (a,m)-approximate g-dual of Λ, then using Newmann series, we get $M_{m,\Gamma,\Lambda}^{-1} = a\sum_{n=0}^{\infty}(\mathrm{Id}_{\mathcal{X}} - aM_{m,\Gamma,\Lambda})^n$, and for each $x \in \mathcal{X}$, we have the following reconstruction formula:

$$x = M_{m,\Gamma,\Lambda}M_{m,\Gamma,\Lambda}^{-1}x = a\sum_{n=0}^{\infty} M_{m,\Gamma,\Lambda}(\mathrm{Id}_{\mathcal{X}} - aM_{m,\Gamma,\Lambda})^n x.$$

The following result is a generalization of Proposition 3.7 in [18] to (a,m)-approximate duals.

Proposition 2.1. *Let Λ be a standard g-Bessel sequence and $\Psi = \{\psi_i\}_{i\in\mathbb{I}}$ be an (a,m)-approximate g-dual (resp. an (a,m)-g-dual) of Λ with upper bound C. If Γ is a sequence such that $\Gamma - \Lambda := \{\Gamma_i - \Lambda_i\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence with upper bound K and $\|a\|^2\|m\|_\infty^2 CK < (1 - \|\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Lambda}\|)^2$ (resp. $\|a\|^2\|m\|_\infty^2 CK < 1$), then Ψ is an (a,m)-approximate g-dual of Γ and Γ is a standard g-frame.*

Proof. Similar to the proof of Proposition 3.7 in [18], we get Γ is a standard g-Bessel sequence. Now using the Cauchy-Schwarz inequality in Hilbert C^*-modules, for each $x \in \mathcal{X}$, we have

$$\begin{aligned}
\|(\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Gamma})x\| &\leq \|(\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Lambda})x\| + \|a(M_{m,\Psi,\Lambda} - M_{m,\Psi,\Gamma})x\| \\
&\leq \|(\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Lambda})x\| \\
&+ \|a\|\|m\|_\infty \sup_{\|y\|=1}\left\{\Big\|\sum_{i\in\mathbb{I}}|(\Lambda_i - \Gamma_i)x|^2\Big\|^{\frac{1}{2}}\Big\|\sum_{i\in\mathbb{I}}|\psi_i y|^2\Big\|^{\frac{1}{2}}\right\} \\
&\leq \left(\|\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Lambda}\| + \|a\|\|m\|_\infty\sqrt{CK}\right)\|x\|.
\end{aligned}$$

Hence

$$\|\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Gamma}\| \leq \|\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Lambda}\| + \|a\|\|m\|_\infty\sqrt{CK} < 1.$$

Also, if Ψ is an (a,m)-g-dual of Λ, then $aM_{m,\Psi,\Lambda} = \mathrm{Id}_{\mathcal{X}}$ and we have

$$\|\mathrm{Id}_{\mathcal{X}} - aM_{m,\Psi,\Gamma}\| \leq \|a\|\|m\|_\infty\sqrt{CK} < 1.$$

Now Theorem 3.5 in [20] implies that Γ is a standard g-frame. □

Proposition 2.2. *Suppose that $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H})\colon i \in \mathbb{I}\}$ and $\Gamma = \{\Gamma_i \in \mathcal{B}(\mathcal{H})\colon i \in \mathbb{I}\}$ are two g-Bessel sequences such that Λ_i's and Γ_i's are normal operators. Then*

(i) *$\Lambda - \Gamma := \{\Lambda_i - \Gamma_i\}_{i\in\mathbb{I}}$ and $\Lambda^* - \Gamma^* := \{\Lambda_i^* - \Gamma_i^*\}_{i\in\mathbb{I}}$ are g-Bessel sequences with the same upper bound. If $B_{\Lambda-\Gamma} = \varepsilon = B_{\Lambda^*-\Gamma^*}$, then*

$$\sum_{j\in\mathbb{I}} ||(T_\Lambda^*\Lambda_j^* - T_\Gamma^*\Gamma_j^*)f||^2 \leq (\sqrt{B_\Lambda} + \sqrt{B_\Gamma})^2\varepsilon\|f\|^2,$$

for each $f \in \mathcal{H}$.

(ii) *Let Λ and Γ be Parseval g-frames. Suppose that $a \in \mathbb{C}$, $m \in \ell^\infty(\mathbb{I})$, $B_{\Lambda-\Gamma} = \varepsilon = B_{\Lambda^*-\Gamma^*}$ and $\Phi = \{\phi_i \in \mathcal{B}(\mathcal{H})\colon i \in \mathbb{I}\}$ is an (a,m)-g-dual for Λ^*. If $4\varepsilon|a|^2\|m\|_\infty^2 B_\Phi < 1$, then $\{T_\Lambda^*\phi_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate g-dual of $\{T_\Gamma^*\Gamma_i^*\}_{i\in\mathbb{I}}$.*

Proof. (i) Since Λ_i's and Γ_i's are normal operators, $\Lambda^* = \{\Lambda_i^*\}_{i\in\mathbb{I}}$ and $\Gamma^* = \{\Gamma_i^*\}_{i\in\mathbb{I}}$ are two g-Bessel sequences with upper bounds B_Λ and B_Γ, respectively. Also it is easy to see that

$$\sum_{i\in\mathbb{I}} ||(\Lambda_i^* - \Gamma_i^*)f||^2 \leq (\sqrt{B_\Lambda} + \sqrt{B_\Gamma})^2\|f\|^2.$$

Hence $\Lambda^* - \Gamma^*$ is a g-Bessel sequence with upper bound $(\sqrt{B_\Lambda}+\sqrt{B_\Gamma})^2$. The result for $\Lambda - \Gamma$ is obtained similarly. Now for each $f \in \mathcal{H}$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{I}} \|(T_\Lambda^*\Lambda_j^* - T_\Gamma^*\Gamma_j^*)f\|^2 = \sum_{j\in\mathbb{I}}\sum_{i\in\mathbb{I}} \|\Lambda_i\Lambda_j^* f - \Gamma_i\Gamma_j^* f\|^2 \\
\leq\ & \sum_{j\in\mathbb{I}}\sum_{i\in\mathbb{I}} \|\Lambda_i(\Lambda_j^* - \Gamma_j^*)f\|^2 + \sum_{j\in\mathbb{I}}\sum_{i\in\mathbb{I}} \|(\Lambda_i - \Gamma_i)\Gamma_j^* f\|^2 \\
+\ & 2\Big(\sum_{j\in\mathbb{I}}\sum_{i\in\mathbb{I}} \|\Lambda_i(\Lambda_j^* - \Gamma_j^*)f\|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\in\mathbb{I}}\sum_{i\in\mathbb{I}} \|(\Lambda_i - \Gamma_i)\Gamma_j^* f\|^2\Big)^{\frac{1}{2}} \\
\leq\ & B_\Lambda B_{\Lambda^*-\Gamma^*}\|f\|^2 + B_{\Lambda-\Gamma}B_\Gamma\|f\|^2 + 2\sqrt{B_\Lambda B_\Gamma}B_{\Lambda-\Gamma}\|f\|^2 \\
=\ & (\sqrt{B_\Lambda} + \sqrt{B_\Gamma})^2\varepsilon\|f\|^2.
\end{aligned}$$

(ii) Because Λ and Γ are Parseval, $B_\Lambda = 1 = B_\Gamma$, $T_\Lambda T_\Lambda^* = \mathrm{Id}_{\mathcal{H}}$ and B_Φ is an upper bound for $\{T_\Lambda^*\phi_i\}_{i\in\mathbb{I}}$. Now for each $f \in \mathcal{H}$, we get

$$\sum_{i\in\mathbb{I}} am_i\phi_i^* T_\Lambda T_\Lambda^*\Lambda_i^* f = \sum_{i\in\mathbb{I}} am_i\phi_i^*\Lambda_i^* f = f.$$

Thus $\{T_\Lambda^*\phi_i\}_{i\in\mathbb{I}}$ is an (a, m)-g-dual of $\{T_\Lambda^*\Lambda_i^*\}_{i\in\mathbb{I}}$. Also

$$|a|^2\|m\|_\infty^2(\sqrt{B_\Lambda} + \sqrt{B_\Gamma})^2\varepsilon B_\Phi = 4\varepsilon|a|^2\|m\|_\infty^2 B_\Phi < 1.$$

Now the result follows from part (i) and Proposition 2.1. □

Here, we introduce (a, m, T)-duals in Hilbert C^*-modules.

Definition 2.5. *Let $\{f_i\}_{i\in\mathbb{I}}$ be a standard Bessel sequence for $\mathcal{X}$ and let T be an invertible operator in $\mathcal{L}(\mathcal{X})$. A standard Bessel sequence $\{g_i\}_{i\in\mathbb{I}}$ is called an (a, m, T)-dual of $\{f_i\}_{i\in\mathbb{I}}$ if*

$$f = \sum_{i\in\mathbb{I}} am_i\langle Tf, f_i\rangle g_i,$$

for each $f \in \mathcal{X}$.

Definition 2.6. *Let T be an invertible operator in $\mathcal{L}(\mathcal{X})$ and let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}, \Gamma = \{\Gamma_i\}_{i\in\mathbb{I}}$ be standard g-Bessel sequences. We say that Γ is an (a, m, T)-g-dual of Λ if $\{\Gamma_i\}_{i\in\mathbb{I}}$ and $\{am_i\Lambda_i T\}_{i\in\mathbb{I}}$ are g-duals, equivalently*

$$\sum_{i\in\mathbb{I}} am_i\Gamma_i^*\Lambda_i Tf = f = \sum_{i\in\mathbb{I}} a^*m_i^*T^*\Lambda_i^*\Gamma_i f,$$

for each $f \in \mathcal{X}$.

Remark 2.1. *Let $\{f_i\}_{i\in\mathbb{I}}, \{g_i\}_{i\in\mathbb{I}}$ be standard Bessel sequences for $\mathcal{X}$ and let T be an invertible opertor on $\mathcal{X}$. Assume that Λ_i and Γ_i are functionals defined by $\Lambda_i(x) = \langle x, f_i\rangle$ and $\Gamma_i(x) = \langle x, g_i\rangle$, respectively. Since*

$$\sum_{i\in\mathbb{I}} am_i\Gamma_i^*\Lambda_i Tx = \sum_{i\in\mathbb{I}} am_i\langle Tx, f_i\rangle g_i,$$

$\{g_i\}_{i\in\mathbb{I}}$ is an (a, m, T)-dual of $\{f_i\}_{i\in\mathbb{I}}$ if and only if Γ is an (a, m, T)-g-dual of Λ.

The next theorem and corollary are generalizations of Proposition 4.1 in [8] to (a,m)-approximate duals in Hilbert C^*-modules.

Theorem 2.1. *If Γ is an (a,m)-approximate g-dual of Λ, then Γ is an $(a,m,(aM_{m,\Gamma,\Lambda})^{-1})$-g-dual of Λ.*

Proof. Since $\|\mathrm{Id}_{\mathcal{X}} - aM_{m,\Gamma,\Lambda}\| < 1$, by Neumann algorithm, $aM_{m,\Gamma,\Lambda}$ is invertible. Now for each $f \in \mathcal{X}$, we have

$$f = aM_{m,\Gamma,\Lambda}(aM_{m,\Gamma,\Lambda})^{-1}f = \sum_{i\in\mathbb{I}} am_i\Gamma_i^*\Lambda_i(aM_{m,\Gamma,\Lambda})^{-1}f.$$

This means that $\{\Gamma_i\}_{i\in\mathbb{I}}$ and $\{am_i\Lambda_i(aM_{m,\Gamma,\Lambda})^{-1}\}_{i\in\mathbb{I}}$ are g-duals, equivalently, Γ is an $(a,m,(aM_{m,\Gamma,\Lambda})^{-1})$-g-dual of Λ. □

Corollary 2.1. *If $\mathcal{G}$ is an (a,m)-approximate dual of $\mathcal{F}$, then $\mathcal{G}$ is an $(a,m,(aM_{m,\mathcal{G},\mathcal{F}})^{-1})$-dual of $\mathcal{F}$.*

In [22], using special bounded operators on Hilbert spaces, new approximate duals are constructed. Here, we obtain analogous results for (a,m)-approximate duals in Hilbert C^*-modules. First, we recall the following definition from [17].

Definition 2.7. *An element T in $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is called a partial isometry if $\mathcal{Y}_0 = ran(T)$ is complemented in $\mathcal{F}$ (i.e., $\mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{Y}_0^{\perp}$) and there exists a complemented submodule $\mathcal{X}_0$ of $\mathcal{X}$ such that T is isometric from $\mathcal{X}_0$ onto $\mathcal{Y}_0$ and $T(\mathcal{X}_0^{\perp}) = \{0\}$.*

Proposition 2.3.
- (i) *Assume that Γ and Λ are two standard g-Bessel sequences such that Γ_i is a partial isometric operator, for each $i \in \mathbb{I}$. Then Γ is an (a,m)-approximate g-dual (resp. (a,m)-g-dual) of Λ if and only if $\{\Gamma_i^*\Gamma_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate g-dual (resp. (a,m)-g-dual) of $\{\Gamma_i^*\Lambda_i\}_{i\in\mathbb{I}}$.*
- (ii) *Let T be an isometric operator on X. If Γ is an (a,m)-approximate g-dual (resp. (a,m)-g-dual) of Λ, then $\{\Gamma_iT\}_{i\in\mathbb{I}}$ is an (a,m)-approximate g-dual (resp. (a,m)-g-dual) of $\{\Lambda_iT\}_{i\in\mathbb{I}}$.*
- (iii) *Let T be a co-isometric operator on $\mathcal{X}$. If Γ is an (a,m)-approximate g-dual (resp. an (a,m)-g-dual) of Λ, then $\{\Gamma_iT^*\}_{i\in\mathbb{I}}$ is an (a,m)-approximate g-dual (resp. an (a,m)-g-dual) of $\{\Lambda_iT^*\}_{i\in\mathbb{I}}$.*

Proof. (i) It is easy to see that $\Psi = \{\Gamma_i^*\Gamma_i\}_{i\in\mathbb{I}}$ and $\Phi = \{\Gamma_i^*\Lambda_i\}_{i\in\mathbb{I}}$ are standard g-Bessel sequences. Since Γ_i's are partial isometry, using the explanation stated after [17, Proposition 3.8], we have $\Gamma_i^*\Gamma_i\Gamma_i^* = \Gamma_i^*$, so

$$aM_{m\Psi\Phi}f = a\sum_{i\in\mathbb{I}} m_i\Gamma_i^*\Gamma_i\Gamma_i^*\Lambda_if = a\sum_{i\in\mathbb{I}} m_i\Gamma_i^*\Lambda_if = aM_{m\Gamma\Lambda}f.$$

Hence Γ is an (a,m)-approximate g-dual (resp. (a,m)-g-dual) of Λ if and only if $\{\Gamma_i^*\Gamma_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate g-dual (resp. (a,m)-g-dual) of $\{\Gamma_i^*\Lambda_i\}_{i\in\mathbb{I}}$.
(ii) It is easy to see that $\Phi = \{\Lambda_iT\}_{i\in\mathbb{I}}$ and $\Psi = \{\Gamma_iT\}_{i\in\mathbb{I}}$ are standard g-Bessel sequences with upper bounds $B_\Lambda\|T\|^2$ and $B_\Gamma\|T\|^2$, respectively. Then for each $f \in \mathcal{X}$, we have

$$\begin{aligned} \|aM_{m,\Psi,\Phi}f - f\| &= \left\| T^*\Big[(a\sum_{i\in\mathbb{I}} m_i\Gamma_i^*\Lambda_iT)f - Tf\Big]\right\| \\ &= \|T^*(aM_{m,\Gamma,\Lambda} - \mathrm{Id}_{\mathcal{X}})Tf\| \le \|aM_{m,\Gamma,\Lambda} - \mathrm{Id}_{\mathcal{X}}\|\|f\|. \end{aligned}$$

Since $\|aM_{m,\Gamma,\Lambda} - \mathrm{Id}_{\mathcal{X}}\| < 1$ (resp. $aM_{m,\Gamma,\Lambda} = \mathrm{Id}_{\mathcal{X}}$), we get $\|aM_{m,\Psi,\Phi} - \mathrm{Id}_{\mathcal{X}}\| < 1$ (resp. $aM_{m,\Psi,\Phi} = \mathrm{Id}_{\mathcal{X}}$) and the result follows.
(iii) The result follows from part (ii) by considering T^* instead of T. □

Corollary 2.2. (i) *Let T be an isometric operator on $\mathcal{X}$. If $\{g_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate dual (resp. an (a,m)-dual) of $\{f_i\}_{i\in\mathbb{I}}$, then $\{T^*g_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate dual (resp. (a,m)-dual) of $\{T^*f_i\}_{i\in\mathbb{I}}$.*

(ii) *Let T be a co-isometric operator on $\mathcal{X}$. If $\{g_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate dual (resp. (a,m)-dual) of $\{f_i\}_{i\in\mathbb{I}}$, then $\{Tg_i\}_{i\in\mathbb{I}}$ is an (a,m)-approximate dual (resp. (a,m)-dual) of $\{Tf_i\}_{i\in\mathbb{I}}$.*

Note that if Λ is an (A,B) standard g-frame, then $\widetilde{\Lambda}=\{\widetilde{\Lambda_i}\}_{i\in\mathbb{I}}$ is an $(\frac{1}{B},\frac{1}{A})$ standard g-frame, where $\widetilde{\Lambda_i}=\Lambda_i S_\Lambda^{-1}$ and we have $\sum_{i\in\mathbb{I}}\Lambda_i^*\widetilde{\Lambda_i}x=x=\sum_{i\in\mathbb{I}}\widetilde{\Lambda_i}^*\Lambda_i x$, for each $x\in\mathcal{X}$. $\widetilde{\Lambda}$ is called the *canonical g-dual* of Λ.
The following proposition is a generalization of [24, Theorem 4.1] to Hilbert C^*-modules which is also stated in Theorem 3.4 in [23] and we recall it since it is used in the sequel.

Proposition 2.4. *Let Λ and Γ be (A_1,B_1) and (A_2,B_2) standard g-frames, respectively. If $\Lambda-\Gamma=\{\Lambda_i-\Gamma_i\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence with upper bound C, then $\widetilde{\Lambda}-\widetilde{\Gamma}=\{\widetilde{\Lambda_i}-\widetilde{\Gamma}_i\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence with upper bound $\left(\frac{\sqrt{B_1C}}{A_1A_2}(\sqrt{B_1}+\sqrt{B_2})+\frac{\sqrt{C}}{A_2}\right)^2$.*

If $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ is a standard frame for $\mathcal{X}$, then the operator $S_\mathcal{F}\colon\mathcal{X}\longrightarrow\mathcal{X}$ defined by $S_\mathcal{F}(x)=\sum_{i\in\mathbb{I}}\langle x,f_i\rangle f_i$ is an invertible operator. It is easy to see that if $\mathcal{F}$ is an (A,B) standard frame, then $\widetilde{\mathcal{F}}=\{S_\mathcal{F}^{-1}f_i\}_{i\in\mathbb{I}}$ is an $(\frac{1}{B},\frac{1}{A})$ standard frame with $x=\sum_{i\in\mathbb{I}}\langle x,S_\mathcal{F}^{-1}f_i\rangle f_i=\sum_{i\in\mathbb{I}}\langle x,f_i\rangle S_\mathcal{F}^{-1}f_i$, for each $x\in\mathcal{X}$. $\widetilde{\mathcal{F}}$ is called the *canonical dual* of $\mathcal{F}$.

Corollary 2.3. *Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ and $\mathcal{G}=\{g_i\}_{i\in\mathbb{I}}$ be (A_1,B_1) and (A_2,B_2) standard frames for E, respectively. If $\mathcal{F}-\mathcal{G}=\{f_i-g_i\}_{i\in\mathbb{I}}$ is a standard Bessel sequence with upper bound C, then $\widetilde{\mathcal{F}}-\widetilde{\mathcal{G}}=\{\widetilde{f_i}-\widetilde{g_i}\}_{i\in\mathbb{I}}$ is a standard Bessel sequence with upper bound $\left(\frac{\sqrt{B_1C}}{A_1A_2}(\sqrt{B_1}+\sqrt{B_2})+\frac{\sqrt{C}}{A_2}\right)^2$.*

Proof. Let $\Lambda=\{\Lambda_i\in\mathcal{L}(\mathcal{X},\mathfrak{A}):i\in\mathbb{I}\}$ and $\Gamma=\{\Gamma_i\in\mathcal{L}(\mathcal{X},\mathfrak{A}):i\in\mathbb{I}\}$, where $\Lambda_i(x)=\langle x,f_i\rangle$ and $\Gamma_i(x)=\langle x,g_i\rangle$. Since $\mathcal{F}$ and $\mathcal{G}$ are (A_1,B_1) and (A_2,B_2) standard frames, it is easy to see that Λ and Γ are (A_1,B_1) and (A_2,B_2) standard g-frames, respectively. Also, it is easy to obtain that $\mathcal{F}-\mathcal{G}$ and $\widetilde{\mathcal{F}}-\widetilde{\mathcal{G}}$ are standard Bessel sequences if and only if $\Lambda-\Gamma$ and $\widetilde{\Lambda}-\widetilde{\Gamma}$ are standard g-Bessel sequences with the same upper bounds, respectively. Now the result follows from Proposition 2.4. □

We recall the following definition from [19].

Definition 2.8. *Let $\Lambda=\{\Lambda_i\in\mathcal{L}(\mathcal{X},\mathcal{X}_i):i\in\mathbb{I}\}$, $a_1,a_2\in\mathfrak{A}$ with $\|a_1\|,\|a_2\|<1$ and $\{a_i\}_{i\in\mathbb{I}}\in\ell^2(\mathbb{I},\mathfrak{A})$. We say that $\Gamma=\{\Gamma_i\in\mathcal{L}(\mathcal{X},\mathcal{X}_i):i\in\mathbb{I}\}$ is an $(a_1,a_2,\{a_i\}_{i\in\mathbb{I}})$-perturbation of Λ if*

$$\left|(\Lambda_i-\Gamma_i)x\right|^2\le\left|a_1|\Lambda_ix|+a_2|\Gamma_ix|+|x|a_i\right|^2,\tag{2}$$

for each $x\in\mathcal{X}$ and $i\in\mathbb{I}$.

Proposition 2.5. *Let A_1 and B_1 be positive real numbers and let $\|\{a_i\}_{i\in\mathbb{I}}\|_2<(1-\|a_1\|)\sqrt{A_1}$, $A_2=\left(\frac{(1-\|a_1\|)\sqrt{A_1}-\|\{a_i\}_{i\in\mathbb{I}}\|_2}{1+\|a_2\|}\right)^2$, $B_2=\left(\frac{(1+\|a_1\|)\sqrt{B_1}+\|\{a_i\}_{i\in\mathbb{I}}\|_2}{1-\|a_2\|}\right)^2$, $C=(\|a_1\|\sqrt{B_1}+\|a_2\|\sqrt{B_2}+\|\{a_i\}_{i\in\mathbb{I}}\|_2)^2$ and $K=\left(\frac{\sqrt{B_1C}}{A_1A_2}(\sqrt{B_1}+\sqrt{B_2})+\frac{\sqrt{C}}{A_2}\right)^2$.*

(i) *Let Λ be an (A_1, B_1) standard g-frame. If Γ is an $(a_1, a_2, \{a_i\}_{i\in\mathbb{I}})$-perturbation of Λ, then Γ is an (A_2, B_2) standard g-frame and $\widetilde{\Lambda}-\widetilde{\Gamma}$ is a standard g-Bessel sequence with upper bound K.*

(ii) *Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{X}$ be an (A_1, B_1) standard frame. If $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ is a sequence in $\mathcal{X}$ such that $|\langle x, f_i - g_i\rangle|^2 \leq \Big|a_1|\langle x, f_i\rangle| + a_2|\langle x, g_i\rangle| + a_i|x|\Big|^2$, for each $x \in \mathcal{X}$, then $\mathcal{G}$ is an (A_2, B_2) standard frame and $\widetilde{\mathcal{F}} - \widetilde{\mathcal{G}}$ is a standard Bessel sequence with upper bound K.*

Proof. (i) It follows from the proof of Theorem 4.1 in [19] that Γ is an

$$\left(\Big(\frac{(1-\|a_1\|)\sqrt{A_1}-\|\{a_i\}_{i\in\mathbb{I}}\|_2}{1+\|a_2\|}\Big)^2, \Big(\frac{(1+\|a_1\|)\sqrt{B_1}+\|\{a_i\}_{i\in\mathbb{I}}\|_2}{1-\|a_2\|}\Big)^2\right)$$

standard g-frame. Also the inequality

$$\begin{aligned}\|\{(\Lambda_i-\Gamma_i)x\}_{i\in\mathbb{F}}\|_2 &\leq \|\{a_1|\Lambda_i x| + a_2|\Gamma_i x| + |x|a_i\}_{i\in\mathbb{F}}\|_2\\ &\leq \|a_1\|\|\{\Lambda_i x\}_{i\in\mathbb{F}}\|_2 + \|a_2\|\|\{\Gamma_i x\}_{i\in\mathbb{F}}\|_2 + \|x\|\|\{a_i\}_{i\in\mathbb{F}}\|_2,\end{aligned}$$

which is obtained from (2) for each finite subset $\mathbb{F}$ of $\mathbb{I}$ and $x \in \mathcal{X}$ implies that $\Lambda - \Gamma$ is a standard g-Bessel sequence with upper bound $(\|a_1\|\sqrt{B_1} + \|a_2\|\sqrt{B_2} + \|\{a_i\}_{i\in\mathbb{I}}\|_2)^2$. The remainder is obtained from Proposition 2.4.
(ii) Let $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{X}, \mathfrak{A}) : i \in \mathbb{I}\}$ and $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{X}, \mathfrak{A}) : i \in \mathbb{I}\}$, where $\Lambda_i(x) = \langle x, f_i\rangle$ and $\Gamma_i(x) = \langle x, g_i\rangle$. It is easy to see that Λ is an (A_1, B_1) standard g-frame and Γ is an $(a_1, a_2, \{a_i\}_{i\in\mathbb{I}})$-perturbation of Λ. Now the result follows from part (i), Corollary 2.3 and using the fact that $\mathcal{F}$ (resp. $\mathcal{G}$) is a standard frame if and only if Λ (resp. Γ) is a standard g-frame. □

Let $\Phi_j = \{\Lambda_{ij} \in \mathcal{L}(\mathcal{X}_j, \mathcal{X}_{ij}) : i \in \mathbb{I}\}$ be an (A_j, B_j) standard g-frame for $\mathcal{X}_j$, $j \in \mathbb{J}$, such that $A = \inf\{A_j : j \in \mathbb{J}\} > 0$ and $B = \sup\{B_j : j \in \mathbb{J}\} < \infty$. Then we say that $\{\Phi_j\}_{j\in\mathbb{J}}$ is an (A, B)-*bounded family of standard g-frames* or shortly (A, B)-*BFSGF.*
It was proved in [19, Theorem 4.2] that $\{\Phi_j\}_{j\in\mathbb{J}}$ is an (A, B)-BFSGF if and only if $\oplus_{j\in\mathbb{J}}\Phi_j = \{\oplus_{j\in\mathbb{J}}\Lambda_{ij}\}_{i\in\mathbb{I}}$ is an (A, B) standard g-frame. Now we have the following result:

Corollary 2.4. *Let A_1 and B_1 be positive numbers and let $\|\{a_i\}_{i\in\mathbb{I}}\|_2 < (1 - \|a_1\|)\sqrt{A_1}$, $A_2 = \left(\frac{(1-\|a_1\|)\sqrt{A_1}-\|\{a_i\}_{i\in\mathbb{I}}\|_2}{1+\|a_2\|}\right)^2$, $B_2 = \left(\frac{(1+\|a_1\|)\sqrt{B_1}+\|\{a_i\}_{i\in\mathbb{I}}\|_2}{1-\|a_2\|}\right)^2$, $C = (\|a_1\|\sqrt{B_1}+\|a_2\|\sqrt{B_2}+\|\{a_i\}_{i\in\mathbb{I}}\|_2)^2$ and $K = \left(\frac{\sqrt{B_1 C}}{A_1 A_2}(\sqrt{B_1} + \sqrt{B_2}) + \frac{\sqrt{C}}{A_2}\right)^2$. Let $\{\Phi_j\}_{j\in\mathbb{J}}$ be an (A_1, B_1)-BFSGF. If $\Psi_j = \{\Gamma_{ij} \in \mathcal{L}(\mathcal{X}_j, \mathcal{X}_{ij}) : i \in \mathbb{I}\}$ is an $(a_1, a_2, \{a_i\}_{i\in\mathbb{I}})$-perturbation of Φ_j, for each $j \in \mathbb{J}$, then $\oplus_{j\in\mathbb{J}}\Psi_j$ is an (A_2, B_2) standard g-frame. In this case $\widetilde{\oplus_{j\in\mathbb{J}}\Phi_j} - \widetilde{\oplus_{j\in\mathbb{J}}\Psi_j}$ is a standard g-Bessel sequence with upper bound K.*

Proof. The result follows from Theorem 4.2 in [19] and the above proposition. □

Proposition 2.6. [18] *Let Λ be a standard g-frame with upper bound B and let Φ be a g-dual of Λ with upper bound D. If Γ is a sequence such that $\Gamma - \Lambda = \{\Gamma_i - \Lambda_i\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence with upper bound C and $CD < 1$, then Γ is a standard g-frame. Moreover, Γ and Φ are approximately g-duals.*

Corollary 2.5. *Suppose that Λ is an (A_1, B_1) standard g-frame and Γ is a sequence such that $\Lambda - \Gamma$ is a standard g-Bessel sequence with upper bound C. If one of the followings holds, then Γ is a standard g-frame and $\widetilde{\Lambda} - \widetilde{\Gamma}$ is a standard g-Bessel sequence with upper* *bound* $\left(\frac{\sqrt{B_1 C}}{A_1 A_2}(\sqrt{B_1}+\sqrt{B_2})+\frac{\sqrt{C}}{A_2}\right)^2$, *where* A_2 *and* B_2 *are lower and upper bounds of* Γ, *respectively:*

(i) Λ *admits a dual with upper bound* D *and* $CD < 1$.

(ii) $C < A_1$.

Proof. If (i) holds, then using Proposition 2.6 and the condition $CD < 1$, we obtain that Γ is a standard g-frame. Now the result follows from Proposition 2.4.
Now let $C < A_1$. Since $\widetilde{\Lambda}$ is a g-dual of Λ with upper bound $D := \frac{1}{A_1}$ and $CD = \frac{C}{A_1} < 1$, condition (i) is satisfied and the result is obtained. □

3. **Stability of fusion frames and resolutions of the identity**

In this section, we study perturbations of fusion frames and resolutions of the identity in Hilbert C^*-modules. The following definition is a generalization of Definitions 3.24 and 3.29 in [6] to Hilbert C^*-modules (see also [13]).

Definition 3.1. *Let* T_i *be an adjointable operator on* $\mathcal{X}$, *for each* $i \in \mathbb{I}$.

(i) $\{T_i\}_{i\in\mathbb{I}}$ *is called a resolution of the identity on* $\mathcal{X}$ *if* $x = \sum_{i\in\mathbb{I}} T_i x$ *and the series converges in norm for each* $x \in \mathcal{X}$.

(ii) *Assume that* $\{\upsilon_i : i \in \mathbb{I}\} \subseteq \mathfrak{A}$ *is a family of weights, i.e., each* υ_i *is a positive, invertible element from the center of* $\mathfrak{A}$. *Then* $\{(T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ *is called an* ℓ^2*-resolution of the identity if* $\{T_i\}_{i\in\mathbb{I}}$ *is a resolution of the identity and there exists a real number* $B > 0$ *such that*
$$\sum_{i\in\mathbb{I}} \upsilon_i^{-2}|T_i x|^2 \leq B|x|^2,$$
and the series converges in norm for each $x \in \mathcal{X}$.

A closed submodule $\mathcal{M}$ of $\mathcal{X}$ is *orthogonally complemented* if $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$. In this case, $\pi_{\mathcal{M}} \in \mathcal{L}(\mathcal{X}, \mathcal{M})$, where $\pi_{\mathcal{M}} : \mathcal{X} \longrightarrow \mathcal{M}$ is the projection onto $\mathcal{M}$.
Now we recall the definition of fusion frames in Hilbert C^*-modules from [13] (see also [1]). Suppose that $\{\omega_i : i \in \mathbb{I}\} \subseteq \mathfrak{A}$ is a family of weights and $\{\mathcal{W}_i : i \in I\}$ is a family of orthogonally complemented submodules of $\mathcal{X}$. Then $\{(\mathcal{W}_i, \omega_i)\}_{i\in\mathbb{I}}$ is a *fusion frame* if there exist real constants $0 < A \leq B < \infty$ such that
$$A\langle x, x\rangle \leq \sum_{i\in\mathbb{I}} \omega_i^2 \langle \pi_{\mathcal{W}_i}(x), \pi_{\mathcal{W}_i}(x)\rangle \leq B\langle x, x\rangle \quad (x \in \mathcal{X}).$$
We call A and B the lower and upper bounds of the fusion frame, respectively. In this case, we call it an (A, B) fusion frame. If the sum converges in norm, it is called a *standard* fusion frame and if we only require to have the upper bound, then $\{(\mathcal{W}_i, \omega_i)\}_{i\in\mathbb{I}}$ is called a *Bessel fusion sequence* with upper bound B.
If $\mathcal{W} = \{(\mathcal{W}_i, \omega_i)\}_{i\in\mathbb{I}}$ is a standard Bessel fusion sequence, then the operator $S_{\mathcal{W}} : \mathcal{X} \longrightarrow \mathcal{X}$ defined by $S_{\mathcal{W}}(x) = \sum_{i\in\mathbb{I}} \omega_i^2 \pi_{\mathcal{W}_i} x$ is adjointable. $S_{\mathcal{W}}$ is called the *operator* of $\mathcal{W}$ and if $\mathcal{W}$ is a standard fusion frame, then $S_{\mathcal{W}}$ is invertible.
Note that if $a \in \mathcal{Z}(\mathfrak{A})$ and $T \in \mathcal{L}(\mathcal{X})$, then the operator $aT : \mathcal{X} \longrightarrow \mathcal{X}$ which is defined by $(aT)(x) = aT(x)$ is adjointable with $(aT)^* = a^*T^*$. Hence $\upsilon_i^2 T_i$ is an adjointable operator on $\mathcal{X}$.
It was proved (in Hilbert spaces) in Lemma 3.27, Proposition 3.28 and Theorem 3.30 in [6] that ℓ^2-resolutions of the identity and resolutions of the identity of the form $\{\upsilon_i^2 T_i\}_{i\in\mathbb{I}}$ are useful in applications. Now we have the following two results for this kind of operator sequences:

Proposition 3.1. *Suppose that T_i is an adjointable operator on $\mathcal{X}$ and $\mathcal{W}_i = \overline{T_i(\mathcal{X})}$ is orthogonally complemented, for each $i \in \mathbb{I}$.*

(i) *Suppose that the sequence $\{(\upsilon_i^2 T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ satisfies the second condition of ℓ^2-resolution of the identity (the inequality of ℓ^2-resolution of the identity) and $\Gamma = \{\Gamma_i \in \mathcal{L}(\mathcal{X})\colon i \in \mathbb{I}\}$ is a standard g-Bessel sequence with upper bound C such that $\sum_{i\in\mathbb{I}} \upsilon_i \Gamma_i^* T_i x = x$. If R is a positive number with $\sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i}(x) - T_i(x)|^2 \leq R|x|^2$ and $CR < 1$, then $\{(\mathcal{W}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is a standard fusion frame and $\{\upsilon_i \pi_{\mathcal{W}_i}\}_{i\in\mathbb{I}}$ and Γ are approximately g-duals.*

(ii) *If $\{(\upsilon_i^2 T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is an ℓ^2-resolution of the identity and there exists a positive number R such that $\sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i}(x) - T_i(x)|^2 \leq R|x|^2$, for each $x \in \mathcal{X}$, then $\{(\mathcal{W}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is a standard fusion frame for $\mathcal{X}$.*

Proof. (i) Suppose that $\{(\upsilon_i^2 T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ satisfies the inequality of ℓ^2-resolution of the identity. We have

$$\sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 = \sum_{i\in\mathbb{I}} \upsilon_i^{-2} |\upsilon_i^2 T_i x|^2 \leq B|x|^2, \tag{3}$$

so $\{\upsilon_i T_i\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence. The equality $\sum_{i\in\mathbb{I}} \upsilon_i \Gamma_i^* T_i x = x$ implies that Γ is a g-dual of $\{\upsilon_i T_i\}_{i\in\mathbb{I}}$. Now the result follows from Proposition 2.6.

(ii) For each $x \in \mathcal{X}$, we have

$$\|\{\upsilon_i \pi_{\mathcal{W}_i}(x)\}_{i\in\mathbb{I}}\|_2 \leq \left\|\{\upsilon_i(\pi_{\mathcal{W}_i}(x) - T_i(x))\}_{i\in\mathbb{I}}\right\|_2 + \|\{\upsilon_i T_i x\}_{i\in\mathbb{I}}\|_2 \leq (\sqrt{R} + \sqrt{B})\|x\|,$$

so $\left\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i}(x)|^2 \right\| \leq (\sqrt{R} + \sqrt{B})^2 \|x\|^2$ and Theorem 3.1 in [26] yields that $\{\upsilon_i \pi_{\mathcal{W}_i}\}_{i\in\mathbb{I}}$ is a standard g-Bessel sequence and consequently $\{(\mathcal{W}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is a standard Bessel fusion sequence. The lower bound for $\{(\mathcal{W}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is obtained similar to the proof of Proposition 4.1 in [19] using Cauchy-Schwarz inequality in Hilbert C^*-modules, the inequalities

$$\begin{aligned} \|x\|^4 = \left\|\langle \sum_{i\in\mathbb{I}} \upsilon_i^2 T_i x, x\rangle\right\|^2 &= \left\| \sum_{i\in\mathbb{I}} \langle \upsilon_i T_i(x), \upsilon_i \pi_{\mathcal{W}_i}(x)\rangle \right\|^2 \\ &\leq \left\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 \right\| \left\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i}(x)|^2 \right\| \\ &\leq B\|x\|^2 \left\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i}(x)|^2 \right\|, \end{aligned}$$

$\left\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i}(x)|^2 \right\| \geq \frac{1}{B}\|x\|^2$ and Theorem 3.1 in [26]. □

In the next theorem, we show that for each $x \in \mathcal{X}$ the family $\{\upsilon_i \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1}(x)\}_{i\in\mathbb{I}}$ has a certain minimum property. This result is a generalization of Theorem 2.2 and Corollary 2.1 in [12] to Hilbert C^*-modules.

Theorem 3.1. *Let $\mathcal{W} = \{(\mathcal{W}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ be a standard fusion frame for $\mathcal{X}$.*

(i) *If $\{(\upsilon_i^2 T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is an ℓ^2-resolution of the identity, where $T_i\colon \mathcal{X} \longrightarrow \mathcal{W}_i$, then*

$$\sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1}(x)|^2 \leq \sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2,$$

for each $x \in \mathcal{X}$ and $\{\upsilon_i T_i\}_{i\in\mathbb{I}}$ is a standard g-frame.

(ii) *If $k \in \mathbb{I}$ with $\mathcal{W}_k = \mathcal{X}$, then $S_{\mathcal{W}}^{-1} \leq \upsilon_k^{-2} \mathrm{Id}_{\mathcal{X}}$.*

Proof. (i) For each $x \in \mathcal{X}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1}(x)|^2 &= \Big\langle \sum_{i\in\mathbb{I}} \upsilon_i^2 \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x, S_{\mathcal{W}}^{-1} x \Big\rangle = \langle x, S_{\mathcal{W}}^{-1} x\rangle \\ &= \Big\langle \sum_{i\in\mathbb{I}} \upsilon_i^2 T_i x, S_{\mathcal{W}}^{-1} x \Big\rangle = \sum_{i\in\mathbb{I}} \upsilon_i^2 \langle T_i x, \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x\rangle.\end{aligned}$$

Now we have

$$\begin{aligned}&\sum_{i\in\mathbb{I}} \upsilon_i^2 |(\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} - T_i)x|^2 \\ = \ &\sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x|^2 + \sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 - \sum_{i\in\mathbb{I}} \upsilon_i^2 \langle \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x, T_i x\rangle - \sum_{i\in\mathbb{I}} \upsilon_i^2 \langle T_i x, \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x\rangle \\ = \ &\sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 - \sum_{i\in\mathbb{I}} \upsilon_i^2 \langle \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x, T_i x\rangle.\end{aligned}$$

Also, we get

$$\begin{aligned}&\sum_{i\in\mathbb{I}} \upsilon_i^2 \langle \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x, T_i x\rangle = \langle S_{\mathcal{W}}^{-1} x, \sum_{i\in\mathbb{I}} \upsilon_i^2 \pi_{\mathcal{W}_i} T_i(x)\rangle \\ = \ &\Big\langle S_{\mathcal{W}}^{-1} x, \sum_{i\in\mathbb{I}} \upsilon_i^2 T_i(x) \Big\rangle = \langle S_{\mathcal{W}}^{-1} x, x\rangle = \langle x, S_{\mathcal{W}}^{-1} x\rangle = \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1}(x)|^2\end{aligned}$$

This yields that

$$0 \le \sum_{i\in\mathbb{I}} \upsilon_i^2 |(\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} - T_i)x|^2 = \Big(\sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 - \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1}(x)|^2 \Big),$$

so $\sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1}(x)|^2 \le \sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2$. If A is a lower bound of $\{(W_i, \upsilon_i)\}_{i\in\mathbb{I}}$, then

$$\begin{aligned}A\|x\|^2 &\le A\|S_{\mathcal{W}}\|^2 \|S_{\mathcal{W}}^{-1} x\|^2 \\ &\le \|S_{\mathcal{W}}\|^2 \Big\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x|^2 \Big\| \le \|S_{\mathcal{W}}\|^2 \Big\| \sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 \Big\|.\end{aligned}$$

Hence $\frac{A}{\|S_{\mathcal{W}}\|^2}$ is a lower bound for $\{\upsilon_i T_i\}_{i\in\mathbb{I}}$ and the upper bound is obtained from (3).
(ii) Define $T_i\colon \mathcal{X} \longrightarrow \mathcal{W}_i$ by $T_k = \upsilon_k^{-2} \pi_{\mathcal{W}_k}$ and $T_i = 0$ for each $i \neq k$. Since $\mathcal{W}_k = \mathcal{X}$, we have $T_k = \upsilon_k^{-2} \mathrm{Id}_{\mathcal{X}}$, so

$$\sum_{i\in\mathbb{I}} \upsilon_i^2 T_i x = \upsilon_k^2 \upsilon_k^{-2} \mathrm{Id}_{\mathcal{X}} x = x,$$

also

$$\sum_{i\in\mathbb{I}} \upsilon_i^{-2} |\upsilon_i^2 T_i x|^2 = \upsilon_k^2 |x|^2 \le \|\upsilon_k^2\| |x|^2.$$

Hence $\{(\upsilon_i^2 T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is an ℓ^2-resolution of the identity. Then using part (i), we have

$$\begin{aligned}\langle x, S_{\mathcal{W}}^{-1} x\rangle &= \Big\langle \sum_{i\in\mathbb{I}} \upsilon_i^2 \pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x, S_{\mathcal{W}}^{-1} x \Big\rangle \\ &= \sum_{i\in\mathbb{I}} \upsilon_i^2 |\pi_{\mathcal{W}_i} S_{\mathcal{W}}^{-1} x|^2 \le \sum_{i\in\mathbb{I}} \upsilon_i^2 |T_i x|^2 = \upsilon_k^2 \upsilon_k^{-4} |x|^2 = \langle \upsilon_k^{-2} \mathrm{Id}_{\mathcal{X}} x, x\rangle,\end{aligned}$$

so $\langle (\upsilon_k^{-2} \mathrm{Id}_{\mathcal{X}} - S_{\mathcal{W}}^{-1})x, x\rangle \ge 0$. Now Lemma 4.1 in [17] implies that $S_{\mathcal{W}}^{-1} \le {\upsilon_k}^{-2} \mathrm{Id}_{\mathcal{X}}$. □

φ-morphisms are important operators on Hilbert C^*-modules with great applications in operator theory. It recently has been shown in [21] that they are also nice operators in frame theory, especially it was shown that they preserve most of the properties of a frame. Now we state the definition of φ-morphisms and refer to [3] for more details about the properties of these operators.

Definition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert C^*-modules over C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a morphism of C^*-algebras. A map $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be a φ-morphism of Hilbert C^*-modules if*

$$\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle),$$

for each $x, y \in \mathcal{X}$.

It is easy to see that each φ-morphism is a linear operator and $\Phi(ax) = \varphi(a)\Phi(x)$, for each $a \in \mathfrak{A}$ and $x \in \mathcal{X}$. Also, since φ is a morphism of C^*-algebras, we have $\|\varphi\| \leq 1$, so the relation

$$\|\Phi(x)\| = \|\langle \Phi(x), \Phi(x)\rangle\|^{\frac{1}{2}} = \|\varphi(\langle x, x\rangle)\|^{\frac{1}{2}} \leq \|x\|$$

yields that Φ is bounded with $\|\Phi\| \leq 1$.

We recall the following definition from [3].

Definition 3.3. *A map $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ is called a unitary if there exists an injective morphism of C^*-algebras $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that Φ is a surjective φ-morphism.*

Note that if Φ is a unitary φ-morphism, then it is surjective and since φ is an injection, then Φ is isometric and so it is invertible (for more details, see [3]).

Proposition 3.2. *Assume that $\{(T_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is an ℓ^2-resolution of the identity on $\mathcal{X}$ with upper bound B.*

(i) *If Φ is a φ-morphism, then for each $x \in \mathcal{X}$, $\sum_{i\in\mathbb{I}} \varphi(\upsilon_i)^{-2}|\Phi(T_i x)|^2$ converges in norm and*

$$\left\|\sum_{i\in\mathbb{I}} \varphi(\upsilon_i)^{-2}|\Phi(T_i x)|^2\right\| \leq B\|x\|^2.$$

(ii) *If φ is an isomorphism and Φ is a unitary, adjointable φ-morphism, then the sequence $\{(\Phi \circ T_i \circ \Phi^{-1}, \varphi(\upsilon_i))\}_{i\in\mathbb{I}}$ is an ℓ^2-resolution of the identity on $\mathcal{Y}$.*

Proof. (i) For each $x \in \mathcal{X}$, we have

$$\begin{aligned} \sum_{i\in\mathbb{I}} \varphi(\upsilon_i)^{-2}|\Phi(T_i x)|^2 &= \varphi\left(\sum_{i\in\mathbb{I}} \upsilon_i^{-2}|T_i x|^2\right) \\ &\leq B\varphi(|x|^2) = B|\Phi(x)|^2. \end{aligned}$$

The above relation shows that $\sum_{i\in\mathbb{I}} \varphi(\upsilon_i)^{-2}|\Phi(T_i x)|^2$ converges in norm and

$$\left\|\sum_{i\in\mathbb{I}} \varphi(\upsilon_i)^{-2}|\Phi(T_i x)|^2\right\| \leq B\||\Phi(x)|^2\| \leq B\|x\|^2.$$

(ii) Since each υ_i is an invertible and positive element in the center of $\mathfrak{A}$ and φ is a surjective morphism of C^*-algebras, it is easy to see that $\varphi(\upsilon_i)$ is also an invertible and positive element in the center of $\mathfrak{B}$. Let $y_1, y_2 \in \mathcal{Y}$ and $x_1, x_2 \in \mathcal{X}$ with $x_1 = \Phi^{-1}(y_1)$ and $x_2 = \Phi^{-1}(y_2)$. Then $\varphi(\langle x_1, x_2\rangle) = \langle \Phi(x_1), \Phi(x_2)\rangle$, so $\varphi(\langle \Phi^{-1}(y_1), \Phi^{-1}(y_2)\rangle) = \langle y_1, y_2\rangle$ and consequently

$\varphi^{-1}(\langle y_1, y_2\rangle) = \langle \Phi^{-1}(y_1), \Phi^{-1}(y_2)\rangle$. Thus Φ^{-1} is a unitary φ^{-1}-morphism. Now, for each $y \in \mathcal{Y}$, we have

$$\sum_{i\in\mathbb{I}}(\Phi \circ T_i \circ \Phi^{-1})y = \Phi\left(\sum_{i\in\mathbb{I}} T_i(\Phi^{-1}y)\right) = \Phi(\Phi^{-1}y) = y.$$

Also,

$$\begin{aligned}\sum_{i\in\mathbb{I}}\varphi(\upsilon_i)^{-2}|\Phi \circ T_i \circ \Phi^{-1}(y)|^2 &= \varphi\left(\sum_{i\in\mathbb{I}}\upsilon_i^{-2}|T_i\Phi^{-1}(y)|^2\right)\\ &\leq B\varphi(\langle \Phi^{-1}y, \Phi^{-1}y\rangle)\\ &= B\varphi(\varphi^{-1}|y|^2) = B|y|^2.\end{aligned}$$

□

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