HYBRID ITERATIVE TECHNIQUES APPROACH TO A MINIMIZATION PROBLEM

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This paper is to investigate iterative techniques for solving a constrained convex minimization problem in Hilbert spaces. We propose a hybrid gradient projection method for solving this constrained convex minimization problem. Strong convergence result is obtained under some additional conditions.

Keywords: minimization problem, gradient projection, hybrid method, strong convergence.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $\mathcal{C}$ a nonempty closed and convex subset of $\mathcal{H}$. Recall that the (nearest point or metric) projection from $\mathcal{H}$ onto $\mathcal{C}$, is denoted by $P_{\mathcal{C}}$ which assigns, to each $q^\dagger \in \mathcal{H}$, the unique point $P_{\mathcal{C}}(q^\dagger) \in \mathcal{C}$ fulfilling the following inequality

$$\|q^\dagger - P_{\mathcal{C}}(q^\dagger)\| \le \|p - q^\dagger\|, \ \forall p \in \mathcal{C}.$$

It is well known that $P_{\mathcal{C}}$ satisfies the following basic result: for all $q^\dagger \in \mathcal{H}$,

$$\langle q^\dagger - P_{\mathcal{C}}(q^\dagger), p - P_{\mathcal{C}}(q^\dagger)\rangle \le 0, \ \forall p \in \mathcal{C}. \tag{1}$$

In this paper, our purpose aims to solve the following constrained convex minimization problem:

$$\min_{z^\dagger \in \mathcal{C}} \varphi(z^\dagger), \tag{2}$$

where $\varphi : \mathcal{H} \to \mathcal{R}$ is a real-valued convex function.

Throughout, we assume that the constrained convex minimization problem (2) is consistent, i.e., its solution set is nonempty. Denote the solution set of (2) by $\mathrm{Sol}(\mathcal{C}, \varphi)$.

Assume that the convex function $\varphi : \mathcal{H} \to \mathcal{R}$ is Fréchet differentiable. Use $\nabla\varphi$ to denote the gradient of φ. It is well known that $q^\dagger \in \mathrm{Sol}(\mathcal{C}, \varphi)$ is equivalent to solving the following variational inequality problem

$$\langle \nabla\varphi(q^\dagger), p - q^\dagger\rangle \ge 0, \ \forall p \in \mathcal{C}. \tag{3}$$

Note that the above optimality condition (3) can be converted into the following inequality

$$\langle q^\dagger - (q^\dagger - \nabla\varphi(q^\dagger)), p - q^\dagger\rangle \ge 0, \ \forall p \in \mathcal{C}. \tag{4}$$

With the help of the characteristic inequality (1) of the projection $P_{\mathcal{C}}$, inequality (4) is equivalent to the following fixed point equation

$$q^\dagger = P_{\mathcal{C}}(q^\dagger - \varpi\nabla\varphi(q^\dagger)), \tag{5}$$

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where ϖ is an any positive constant.

Based on the fixed point equation (5), we can apply the well-known gradient projection method to solve the minimization problem (2). The gradient projection method defines an iterative sequence $\{x_n\}$ by the following form

$$x_0 \in \mathcal{C}, x_{n+1} = P_{\mathcal{C}}(x_n - \varpi \nabla \varphi(x_n)),\ n \geq 0, \tag{6}$$

In general, if $\nabla\varphi$ is only assumed to be Lipschitz continuous, then the sequence $\{x_n\}$ generated by (6) is weak convergent in an infinite dimensional Hilbert spaces. If $\nabla\varphi$ is Lipschitz and strongly monotone, then the sequence $\{x_n\}$ generated by (6) strongly converges to a minimizer of φ in $\mathcal{C}$. The gradient projection algorithm (6) is a powerful tool for solving the constrained convex optimization problems ([1, 6, 9–11, 13–15, 17, 24, 47]), fixed point problems ([3, 8, 12, 16, 19–23, 40]), variational inequality problems ([2, 28, 35–38, 41–43, 46, 48, 49]), equilibrium problems ([30, 45, 50]), and split feasibility problems ([4, 5, 27, 29, 31–34, 39, 44]). Many scholars constructed and modified various projection iterative algorithms for solving (2). Especially, Xu [26] suggested a viscosity-type gradient projective algorithm and proved that the proposed algorithm converges strongly to a minimizer of (2).

In this paper, we continue to study iterative algorithms for solving the constrained convex minimization problem (2). We propose a hybrid gradient projection method for solving this constrained convex minimization problem. Strong convergence result is obtained under some additional conditions.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Use $\to$ and $\rightharpoonup$ to stand for strong convergence and weak convergence, respectively. Use $\omega_w(x_n) := \{x \in \mathcal{H} :$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup x\}$ to mean the weak ω-limit set of the sequence $\{x_n\}$.

Definition 2.1. *An operator $U : \mathcal{C} \to \mathcal{H}$ is said to be Lipschitz continuous if*

$$\|U(x) - U(y)\| \leq \varsigma \|x - y\|,\ \forall x, y \in \mathcal{C},$$

where $\varsigma > 0$ is a constant.

We call U nonexpansive when $\varsigma = 1$. It is well-known that the projection $P_{\mathcal{C}}$ is nonexpansive. If $L < 1$, then U is said to be contractive.

Definition 2.2. *An operator $U : \mathcal{C} \to \mathcal{H}$ is said to be averaged, if and only if U can be written as the average of the identity I and a nonexpansive operator; namely,*

$$U = (1 - \gamma)I + \gamma S \tag{7}$$

where $\gamma \in (0, 1)$ is a constant and S is a nonexapnsive operator.

In general, we call U γ-averaged if (7) holds. In the sequel, we use $Fix(U)$ to mean the fixed point set of U.

Definition 2.3. *Recall that an operator $U : \mathcal{C} \to \mathcal{H}$ is said to be firmly nonexpansive, if*

$$\|U(p^\dagger) - U(q^\dagger)\|^2 \leq \langle U(p^\dagger) - U(q^\dagger), p^\dagger - q^\dagger \rangle$$

for all $p^\dagger \in \mathcal{C}$ and $q^\dagger \in \mathcal{C}$.

It is well known that the metric projection $P_{\mathcal{C}}$ is firmly nonexpansive.

Definition 2.4. *Recall that an operator $\phi : \mathcal{H} \to \mathcal{H}$ is said to be strongly positive if there exists a constant $\sigma > 0$ such that $\langle \phi(x), x \rangle \geq \sigma \|x\|^2$, $\forall x \in \mathcal{H}$.*

Lemma 2.1 ([18]). *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in $\mathcal{H}$. Suppose that the following conditions are satisfied:*

- $x_{n+1} = (1-\eta_n)y_n + \eta_n x_n, \forall n \geq 0$;
- $\eta_n \in (0,1)$ *and* $0 < \liminf_{n\to\infty} \eta_n \leq \limsup_{n\to\infty} \eta_n < 1$;
- $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$.

Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.2 ([7])**.** *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $S : \mathcal{C} \to \mathcal{H}$ be a nonexpansive operator. If $Fix(S) \neq \emptyset$, then S is demiclosed, namely, $x_n \rightharpoonup p^\dagger$ and $x_n - Sx_n \to 0$ imply that $p^\dagger \in Fix(S)$.*

Lemma 2.3 ([25])**.** *Suppose the following conditions hold:*

- $\alpha_n \in (0,+\infty)$, $\varpi_n \in (0,1)$ *and* $\beta_n \in \mathcal{R}$;
- $\alpha_{n+1} \leq (1-\varpi_n)\alpha_n + \beta_n$;
- $\sum_{n=1}^{\infty} \varpi_n = \infty$ *and* $\limsup_{n\to\infty} \beta_n/\varpi_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. **Main results**

In this section, we will state and prove our main results.

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Assume that $\varphi : \mathcal{C} \to \mathcal{R}$ is a Fréchet differentiable convex function with the gradient $\nabla\varphi$ being ς-Lipschitz continuous.

Next, we propose a hybrid iterative algorithm for solving the minimization problem (2).

Algorithm 3.1. *Assume that $\psi : \mathcal{C} \to \mathcal{H}$ is a contractive operator with coefficient $\delta \in (0,1)$. Assume that $\phi : \mathcal{H} \to \mathcal{H}$ is a strongly positive bounded linear operator with coefficient σ. Assume that $\{\varpi_n\} \subset (0, \frac{2}{\varsigma})$ and $\{\tau_n\} \subset (0,1)$ are two real number sequences. Assume that μ is a positive constant. For a given initial point $x_0 \in \mathcal{C}$, define a sequence $\{x_n\}$ iteratively by the following pattern*

$$x_{n+1} = P_{\mathcal{C}}(I - \varpi_n \nabla\varphi)P_{\mathcal{C}}(\tau_n \mu \psi(x_n) + (I - \tau_n \phi)x_n),\ n \geq 0. \tag{8}$$

Now, we demonstrate the convergence of Algorithm 3.1.

Theorem 3.1. *Suppose that* $\mathrm{Sol}(\mathcal{C},\varphi) \neq \emptyset$. *Suppose that the following conditions hold:*

(C1): $\lim_{n\to+\infty} \tau_n = 0$, $\sum_{n=0}^{+\infty} \tau_n = +\infty$ *and* $0 < \mu < \frac{\sigma}{\delta}$;
(C2): $0 < \liminf_{n\to+\infty} \varpi_n \leq \limsup_{n\to+\infty} \varpi_n < \frac{2}{\varsigma}$ *and* $\lim_{n\to+\infty}(\varpi_{n+1} - \varpi_n) = 0$.

Then the sequence $\{x_n\}$ generated by (8) converges to a minimizer $\hat{x} \in \mathrm{Sol}(\mathcal{C},\varphi)$ which is the unique solution of the following VI

$$\langle \phi(\hat{x}) - \mu\psi(\hat{x}), x - \hat{x} \rangle \geq 0,\ \forall x \in \mathrm{Sol}(\mathcal{C},\varphi). \tag{9}$$

Proof. Let $x^* \in \mathrm{Sol}(\mathcal{C},\varphi)$. Note that $x^* \in \mathrm{Sol}(\mathcal{C},\varphi) \Leftrightarrow x^* = P_C(x^* - \varpi\nabla\varphi(x^*)), \forall \varpi > 0$. Thanks to condition (C2), we obtain $x^* = P_C(x^* - \varpi_n \nabla\varphi(x^*))$ for all $n \geq 0$. Since $P_{\mathcal{C}}$ and

$I - \varpi_n \nabla \varphi$ are nonexpansive ([26]), from (8), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P_{\mathcal{C}}(I - \varpi_n \nabla\varphi)P_{\mathcal{C}}[\tau_n \mu\psi(x_n) + (I - \tau_n\phi)x_n] - P_{\mathcal{C}}(I - \varpi_n\nabla\varphi)x^*\| \\
&\leq \|P_{\mathcal{C}}[\tau_n\mu\psi(x_n) + (I - \tau_n\phi)x_n] - P_{\mathcal{C}}[x^*]\| \\
&\leq \|\tau_n\mu\psi(x_n) + (I - \tau_n\phi)x_n - x^*\| \\
&= \|\tau_n\mu(\psi(x_n) - \psi(x^*)) + (I - \tau_n\phi)(x_n - x^*) + \mu\tau_n\psi(x^*) - \tau_n\phi(x^*)\| \\
&\leq \tau_n\mu\|\psi(x_n) - \psi(x^*)\| + |I - \tau_n\phi|\|x_n - x^*\| + \tau_n\|\mu\psi(x^*) - \phi(x^*)\| \\
&\leq \mu\delta\tau_n\|x_n - x^*\| + (1 - \sigma\tau_n)|\|x_n - x^*\| + \tau_n\|\mu\psi(x^*) - \phi(x^*)\| \\
&= [1 - (\sigma - \mu\delta)\tau_n]|\|x_n - x^*\| + \tau_n\|\mu\psi(x^*) - \phi(x^*)\| \\
&\leq \max\left\{\|x_n - x^*\|, \frac{\|\mu\psi(x^*) - \phi(x^*)\|}{\sigma - \mu\delta}\right\}.
\end{aligned}$$

By induction, we have

$$\|x_{n+1} - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|\mu\psi(x^*) - \phi(x^*)\|}{\sigma - \mu\delta}\right\}.$$

It follows that the sequence $\{x_n\}$ is bounded. It is obviously that the sequences $\{\psi(x_n)\}$, $\{\phi(x_n)\}$ and $\{\nabla\varphi(x_n)\}$ are all bounded. Observe that $P_{\mathcal{C}}(I - \varpi_n\nabla\varphi)$ is $\frac{2+\varsigma\varpi_n}{4}$-averaged for each $n \geq 0$. Then, according to the definition of the averaged operator, we have

$$P_{\mathcal{C}}(I - \varpi_n\nabla\varphi) = \frac{2 - \varsigma\varpi_n}{4}I + \frac{2 + \varsigma\varpi_n}{4}U_n, \tag{10}$$

where U_n is a nonexpansive operator.

Set $v_n = \tau_n\mu\psi(x_n) + (I - \tau_n\phi)x_n$ for all $n \geq 0$. Taking into account of (8), we get

$$\begin{aligned}
x_{n+1} &= P_{\mathcal{C}}(I - \varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n] \\
&= \frac{2 - \varsigma\varpi_n}{4}x_n + \frac{2 + \varsigma\varpi_n}{4}U_nP_{\mathcal{C}}[v_n] + \frac{2 - \varsigma\varpi_n}{4}(P_{\mathcal{C}}[v_n] - x_n) \\
&= \frac{2 - \varsigma\varpi_n}{4}x_n + \frac{2 + \varsigma\varpi_n}{4}\big(U_nP_{\mathcal{C}}[v_n] + \frac{2 - \varsigma\varpi_n}{2 + \varsigma\varpi_n}(P_{\mathcal{C}}[v_n] - x_n)\big).
\end{aligned} \tag{11}$$

Using the definition of v_n, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|\tau_{n+1}\mu\psi(x_{n+1}) + (I - \tau_{n+1}\phi)x_{n+1} - \tau_n\mu\psi(x_n) - (I - \tau_n\phi)x_n\| \\
&\leq \tau_{n+1}\mu\|\psi(x_{n+1}) - \psi(x_n)\| + \mu|\tau_{n+1} - \tau_n|\|\psi(x_n)\| \\
&\quad + |I - \tau_{n+1}\phi|\|x_{n+1} - x_n\| + |\tau_{n+1} - \tau_n|\|\phi(x_n)\| \\
&\leq [1 - (\sigma - \mu\delta)\tau_{n+1}]\|x_{n+1} - x_n\| + \mu|\tau_{n+1} - \tau_n|\|\psi(x_n)\| \\
&\quad + |\tau_{n+1} - \tau_n|\|\phi(x_n)\|,
\end{aligned} \tag{12}$$

and

$$\|v_n - x_n\| = \|\tau_n\mu\psi(x_n) + (I - \tau_n\phi)x_n - x_n\| \leq \tau_n(\mu\|\psi(x_n)\| + \|\phi(x_n)\|). \tag{13}$$

From (10), we receive

$$\begin{aligned} U_{n+1}P_{\mathcal{C}}[v_n]-U_nP_{\mathcal{C}}[v_n] &= \frac{4}{2+\varsigma\varpi_{n+1}}(P_{\mathcal{C}}(I-\varpi_{n+1}\nabla\varphi)P_{\mathcal{C}}[v_n]-\frac{2-\varsigma\varpi_{n+1}}{4}P_{\mathcal{C}}[v_n]) \\ &\quad -\frac{4}{2+\varsigma\varpi_n}(P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n]-\frac{2-\varsigma\varpi_n}{4}P_{\mathcal{C}}[v_n]) \\ &= \frac{4}{2+\varsigma\varpi_{n+1}}(P_{\mathcal{C}}(I-\varpi_{n+1}\nabla\varphi)P_{\mathcal{C}}[v_n]-\frac{2-\varsigma\varpi_{n+1}}{4}P_{\mathcal{C}}[v_n]) \\ &\quad -\frac{4}{2+\varsigma\varpi_{n+1}}(P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n]-\frac{2-\varsigma\varpi_n}{4}P_{\mathcal{C}}[v_n]) \\ &\quad +(\frac{4}{2+\varsigma\varpi_{n+1}}-\frac{4}{2+\varsigma\varpi_n})(P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n] \\ &\quad -\frac{2-\varsigma\varpi_n}{4}P_{\mathcal{C}}[v_n]). \end{aligned}$$

It follows that

$$\begin{aligned} \|U_{n+1}P_{\mathcal{C}}[v_n]-U_nP_{\mathcal{C}}[v_n]\| &\le \frac{4}{2+\varsigma\varpi_{n+1}}\|P_{\mathcal{C}}(I-\varpi_{n+1}\nabla\varphi)P_{\mathcal{C}}[v_n] \\ &\quad -P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n]\|+\frac{\varsigma|\varpi_{n+1}-\varpi_n|}{2+\varsigma\varpi_{n+1}}\|P_{\mathcal{C}}[v_n]\| \\ &\quad +|\frac{4}{2+\varsigma\varpi_{n+1}}-\frac{4}{2+\varsigma\varpi_n}|\|P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n] \\ &\quad -\frac{2-\varsigma\varpi_n}{4}P_{\mathcal{C}}[v_n]\| \end{aligned}$$

Then,

$$\begin{aligned} \|U_{n+1}P_{\mathcal{C}}[v_n]-U_nP_{\mathcal{C}}[v_n]\| &\le \frac{4|\varpi_{n+1}-\varpi_n|}{2+\varsigma\varpi_{n+1}}\|\nabla\varphi(P_{\mathcal{C}}[v_n])\|+\frac{\varsigma|\varpi_{n+1}-\varpi_n|}{2+\varsigma\varpi_{n+1}}\|P_{\mathcal{C}}[v_n]\| \\ &\quad +\frac{4\varsigma|\varpi_{n+1}-\varpi_n|}{(2+\varsigma\varpi_{n+1})(2+\varsigma\varpi_n)}\|P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n] \qquad (14) \\ &\quad -\frac{2-\varsigma\varpi_n}{4}P_{\mathcal{C}}[v_n]\|. \end{aligned}$$

Set $y_n=U_nP_{\mathcal{C}}[v_n]+\frac{2-\varsigma\varpi_n}{2+\varsigma\varpi_n}(P_{\mathcal{C}}[v_n]-x_n)$ for all $n\ge 0$. By virtue of (12), (13) and (14), we acquire

$$\begin{aligned} \|y_{n+1}-y_n\| &= \|U_{n+1}P_{\mathcal{C}}[v_{n+1}]-U_nP_{\mathcal{C}}[v_n]+\tfrac{2-\varsigma\varpi_{n+1}}{2+\varsigma\varpi_{n+1}}(P_{\mathcal{C}}[v_{n+1}]-x_{n+1}) \qquad (15) \\ &\quad -\tfrac{2-\varsigma\varpi_n}{2+\varsigma\varpi_n}(P_{\mathcal{C}}[v_n]-x_n)\| \\ &\le \|U_{n+1}P_{\mathcal{C}}[v_{n+1}]-U_{n+1}P_{\mathcal{C}}[v_n]\|+\|U_{n+1}P_{\mathcal{C}}[v_n]-U_nP_{\mathcal{C}}[v_n]\| \\ &\quad +\tfrac{2-\varsigma\varpi_{n+1}}{2+\varsigma\varpi_{n+1}}\|P_{\mathcal{C}}[v_{n+1}]-x_{n+1}\|+\tfrac{2-\varsigma\varpi_n}{2+\varsigma\varpi_n}\|P_{\mathcal{C}}[v_n]-x_n\| \\ &\le \|U_{n+1}P_{\mathcal{C}}[v_n]-U_nP_{\mathcal{C}}[v_n]\|+\tfrac{2-\varsigma\varpi_{n+1}}{2+\varsigma\varpi_{n+1}}\|v_{n+1}-x_{n+1}\| \\ &\quad +\|v_{n+1}-v_n\|+\tfrac{2-\varsigma\varpi_n}{2+\varsigma\varpi_n}\|v_n-x_n\| \\ &\le [1-(\sigma-\mu\delta)\tau_{n+1}]\|x_{n+1}-x_n\|+\mu|\tau_{n+1}-\tau_n|\|\psi(x_n)\| \\ &\quad +|\tau_{n+1}-\tau_n|\|\phi(x_n)\|+\tau_n(\mu\|\psi(x_n)\|+\|\phi(x_n)\|)+\tau_{n+1}(\mu\|\psi(x_{n+1})\| \\ &\quad +\|\phi(x_{n+1})\|)+\tfrac{4|\varpi_{n+1}-\varpi_n|}{2+\varsigma\varpi_{n+1}}\|\nabla\varphi(P_{\mathcal{C}}[v_n])\|+\tfrac{\varsigma|\varpi_{n+1}-\varpi_n|}{2+\varsigma\varpi_{n+1}}\|P_{\mathcal{C}}[v_n]\| \\ &\quad +\tfrac{4\varsigma|\varpi_{n+1}-\varpi_n|}{(2+\varsigma\varpi_{n+1})(2+\varsigma\varpi_n)}\|P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}[v_n]-\tfrac{2-\varsigma\varpi_n}{4}P_{\mathcal{C}}[v_n]\|. \end{aligned}$$

Since the sequences $\{x_n\}$, $\{\psi(x_n)\}$, $\{\phi(x_n)\}$ and $\{\nabla\varphi(x_n)\}$ are bounded, with the help of (15), we can deduce

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{16}$$

Owing to (11), we have $x_{n+1}=\frac{2-\varsigma\varpi_n}{4}x_n+\frac{2+\varsigma\varpi_n}{4}y_n$. By condition (C2), we get $0<\liminf_{n\to\infty}\frac{2-\varsigma\varpi_n}{4}\le\limsup_{n\to\infty}\frac{2-\varsigma\varpi_n}{4}<1$. In the light of (16) and Lemma 2.1, we conclude

$$\lim_{n\to\infty}\|y_n-x_n\|=0.$$

Hence,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\frac{2+\varsigma\varpi_n}{4}\|y_n-x_n\|=0. \tag{17}$$

In terms of (8), we attain

$$\begin{aligned}\|x_n-P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)x_n\|&\le\|x_n-x_{n+1}\|+\|x_{n+1}-P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)x_n\|\\&=\|x_n-x_{n+1}\|+\|P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}(\tau_n\mu\psi(x_n)+(I-\tau_n\phi)x_n)\\&\quad-P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)x_n\|\\&\le\|x_n-x_{n+1}\|+\tau_n(\mu\|\psi(x_n)\|+\|\phi(x_n)\|).\end{aligned}$$

This together with condition (C1) and (17) implies that

$$\lim_{n\to+\infty}\|x_n-P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)x_n\|=0. \tag{18}$$

Next we show that $\omega_w(x_n)\subset \mathrm{Sol}(\mathcal{C},\varphi)$. Select any $\tilde{x}\in\omega_w(x_n)$. Since $\{x_n\}$ and $\{\varpi_n\}$ are bounded, we can choose a common subsequence $\{n_i\}\subset\{n\}$ such that $x_{n_i}\rightharpoonup\tilde{x}$ and $\varpi_{n_i}\to\varpi\in(0,\frac{2}{\varsigma})$ as $i\to+\infty$.

Observe that

$$\begin{aligned}\|x_{n_i}-P_{\mathcal{C}}(I-\varpi\nabla\varphi)x_{n_i}\|&\le\|x_{n_i}-P_{\mathcal{C}}(I-\varpi_{n_i}\nabla\varphi)x_{n_i}\|+\|P_{\mathcal{C}}(I-\varpi_{n_i}\nabla\varphi)x_{n_i}\\&\quad-P_{\mathcal{C}}(I-\varpi\nabla\varphi)x_{n_i}\|\\&\le\|x_{n_i}-P_{\mathcal{C}}(I-\varpi_{n_i}\nabla\varphi)x_{n_i}\|+|\varpi_{n_i}-\varpi|\|\nabla\varphi(x_{n_i})\|,\end{aligned}$$

which together with (18) implies that

$$\lim_{i\to+\infty}\|x_{n_i}-P_{\mathcal{C}}(I-\varpi\nabla\varphi)x_{n_i}\|=0. \tag{19}$$

Since $\varpi\in(0,\frac{2}{\varsigma})$, $P_{\mathcal{C}}(I-\varpi\nabla\varphi)$ is nonexpansive. Noting that $x_{n_i}\rightharpoonup\tilde{x}$, applying Lemma 2.2 to (19), we conclude that $\tilde{x}\in Fix(P_{\mathcal{C}}(I-\varpi\nabla\varphi))=\mathrm{Sol}(\mathcal{C},\varphi)$. Therefore, $\omega_w(x_n)\subset\mathrm{Sol}(\mathcal{C},\varphi)$.

It is clear that the VI (9) has a unique solution which is denoted by $\hat{x}$. Next, we show $\limsup_{n\to\infty}\langle\mu\psi(\hat{x})-\phi(\hat{x}),x_n-\hat{x}\rangle\le 0$. In fact, we have

$$\limsup_{n\to\infty}\langle\mu\psi(\hat{x})-\phi(\hat{x}),x_n-\hat{x}\rangle=\lim_{k\to\infty}\langle\mu\psi(\hat{x})-\phi(\hat{x}),x_{n_k}-\hat{x}\rangle \tag{20}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup x^\dagger\in\mathrm{Sol}(\mathcal{C},\varphi)$. Note that $\hat{x}$ solves (9). Hence,

$$\lim_{j\to\infty}\langle\mu\psi(\hat{x})-\phi(\hat{x}),x_{n_{k_j}}-\hat{x}\rangle=\langle\mu\psi(\hat{x})-\phi(\hat{x}),x^\dagger-\hat{x}\rangle\le 0. \tag{21}$$

Combining (20) and (21), we deduce $\limsup_{n\to\infty}\langle\mu\psi(\hat{x})-\phi(\hat{x}),x_n-\hat{x}\rangle\le 0$. This together with $\tau_n\to 0$ implies that

$$\limsup_{n\to\infty}\langle\mu\psi(\hat{x})-\phi(\hat{x}),\tau_n\mu(\psi(x_n)-\psi(\hat{x}))+(I-\tau_n\phi)(x_n-\hat{x})\rangle\le 0. \tag{22}$$

Finally, we show $x_n \to \hat{x}$. From (8), we have

$$\begin{aligned}\|x_{n+1}-\hat{x}\|^2 &= \|P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)P_{\mathcal{C}}(\tau_n\mu\psi(x_n)+(I-\tau_n\phi)x_n)-P_{\mathcal{C}}(I-\varpi_n\nabla\varphi)\hat{x}\|^2\\ &\le \|\tau_n\mu\psi(x_n)+(I-\tau_n\phi)x_n-\hat{x}\|^2\\ &= \|\tau_n\mu(\psi(x_n)-\psi(\hat{x}))+(I-\tau_n\phi)(x_n-\hat{x})+\tau_n(\mu\psi(\hat{x})-\phi(\hat{x}))\|^2\\ &\le \|\tau_n\mu(\psi(x_n)-\psi(\hat{x}))+(I-\tau_n\phi)(x_n-\hat{x})\|^2\\ &\quad +2\tau_n\langle\mu\psi(\hat{x})-\phi(\hat{x}),\tau_n\mu(\psi(x_n)-\psi(\hat{x}))+(I-\tau_n\phi)(x_n-\hat{x})\rangle\\ &\le [1-(\sigma-\mu\delta)\tau_n]\|x_n-\hat{x}\|^2+2\tau_n\langle\mu\psi(\hat{x})-\phi(\hat{x}),\tau_n\mu(\psi(x_n)-\psi(\hat{x}))\\ &\quad +(I-\tau_n\phi)(x_n-\hat{x})\rangle.\end{aligned} \tag{23}$$

According to Lemma 2.3, (22) and (23), we conclude that $x_n \to \hat{x}$. The proof is completed. □

4. **Conclusions**

This paper, we investigate iterative algorithms for solving a constrained convex minimization problem (2) in Hilbert spaces. A popular way for finding a minimizer of (2) is to apply the well-known gradient projection algorithm (6). In this paper, we propose a hybrid gradient projection algorithm [Algorithm 3.1] for solving the constrained convex minimization problem (2). We prove a strong convergence result [Theorem 3.1] under some assumptions. Our result improves and extends some existing results in the literature.

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