EXTENSION OF BESSEL SEQUENCES TO DUAL FRAMES IN HILBERT SPACES

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In this paper, first we consider a wave packet system of the form $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$, and we present sufficient conditions for the extension of a pair of wave packet Bessel sequence to wave packet dual frames for $L^2(\mathbb{R})$. In the second part, we provide necessary and sufficient conditions for the finite extension of a pair of Hilbert Bessel sequences to a pair of Hilbert dual frames. Examples and counter-examples are given to illustrate the results.

Keywords: Frames, dual frame, Bessel sequence, wave packet system.

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1. Introduction and Preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle .,. \rangle$. Let Λ be a countable set. A sequence $\{f_k\}_{k\in\Lambda} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in\Lambda} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1}$$

The numbers A and B are called *lower* and *upper frame bounds*, respectively. If the upper inequality in (1) is satisfied, then $\{f_k\}_{k\in\Lambda}$ is a *Bessel sequence* (or *Hilbert Bessel sequence*) with *Bessel bound B*. A frame $\{f_k\}_{k\in\Lambda}$ is *tight* if it is possible to choose $A = B$.

Associated with a frame $\{f_k\}_{k\in\Lambda}$ for $\mathcal{H}$, there are three bounded linear operators:

$$\textit{pre-frame operator } \Theta : \ell^2(\Lambda) \to \mathcal{H}, \quad \Theta\{c_k\}_{k\in\Lambda} = \sum_{k\in\Lambda} c_k f_k, \ \{c_k\}_{k\in\Lambda} \in \ell^2(\Lambda),$$

$$\textit{analysis operator} \quad \Theta^* : \mathcal{H} \to \ell^2(\Lambda), \quad \Theta^* f = \{\langle f, f_k\rangle\}_{k\in\Lambda}, \quad f \in \mathcal{H},$$

$$\textit{frame operator} \quad S = \Theta\Theta^* : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k\in\Lambda} \langle f, f_k\rangle f_k, \quad f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k\in\Lambda} \langle S^{-1}f, f_k\rangle f_k \quad \left(= \sum_{k\in\Lambda} \langle f, S^{-1}f_k\rangle f_k \right).$$

The scalars $\{\langle S^{-1}f, f_k\rangle\}_{k\in\Lambda}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$. The representation of f in the reconstruction formula need not be unique. Thus, frames are redundant building blocks which allow each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required.

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Duffin and Schaeffer [14] introduced frames for Hilbert spaces, while addressing some deep problems in nonharmonic Fourier series. Later, in 1986, Daubechies, Grossmann and Meyer revived frames in [12]. For application of frames in applied mathematics in different directions, we refer books by Casazza and Kutyniok [3], Christensen [4, 5] and Daubechies [11].

Let $\{f_k\}_{k\in\Lambda}$ be a frame for $\mathcal{H}$. A frame $\{g_k\}_{k\in\Lambda}$ for $\mathcal{H}$ satisfying

$$f=\sum_{k\in\Lambda}\langle f,g_k\rangle f_k \text{ for all } f\in\mathcal{H}$$

is called a *dual frame* of $\{f_k\}_{k\in\Lambda}$. For any frame $\{f_k\}_{k\in\Lambda}$ there exist at least one dual frame $\{S^{-1}f_k\}_{k\in\Lambda}$ which is called the *canonical dual* frame of $\{f_k\}_{k\in\Lambda}$. If $\{f_k\}_{k\in\Lambda}$ is a tight frame, then $\{f_k\}_{k\in\Lambda}$ has a dual of the form $g_k = Cf_k$ for some constant $C > 0$. If $\{f_k\}_{k\in\Lambda}$ is a tight frame with frame bounds $A = B = 1$, then we can take $g_k = f_k$ and the elements of $\mathcal{H}$ have representation of the form

$$f=\sum_{k\in\Lambda}\langle f,f_k\rangle f_k \text{ for all } f\in\mathcal{H}. \tag{2}$$

1.1. Background on wave packets

The wave packet system is a system of functions generated by combined action of translation, dilation and modulation operators on $L^2(\mathbb{R})$. More precisely, a system of the form

$$\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}},$$

where $\psi\in L^2(\mathbb{R})$, $\{a_j\}_{j\in\mathbb{Z}}\subset\mathbb{R}^+$, $b\neq 0$ and $\{c_m\}_{m\in\mathbb{Z}}\subset\mathbb{R}$ is called *irregular Weyl-Heisenberg wave packet system* (or simply IWH *wave packet system*) in $L^2(\mathbb{R})$. In more general setting the wave packet system is a system of the form $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j\in\Lambda_1,k\in\Lambda_2,m\in\Lambda_3}$, where Λ_i $(i = 1, 2, 3)$ are countable sets. In general, we take $\Lambda_i = \mathbb{Z}$. The wave packet system was introduced by Cordoba and Fefferman [9] by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Later, Labate et al. [17] adopted the same expression to describe, more generally, any collection of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R})$. More precisely, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Lacey and Thiele [18, 19] gave applications of wave packet systems in boundedness of the Hilbert transforms. For further development in wave packet systems and their frame like properties in different directions, see [6, 7, 10, 13, 15, 16].

In the rest of this section, we recall basic notations, definitions and results to make the paper self-contained. By χ_E, we denote the characteristic function of a set E. The *range* and *kernel* of a bounded linear operator T from a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is denoted by $\mathrm{Im}(T)$ and $\mathrm{Ker}T$, respectively. A bounded linear operator T from $\mathcal{X}$ into $\mathcal{Y}$ is said to be of *finite rank*, if $\dim(\mathrm{Im}T)$ is finite. We consider three classes of unitary operators $T_a,\ E_b,\ D_c$ on $L^2(\mathbb{R})$ which play central role in Gabor frame and wavelets:

$$\begin{aligned}
&T_af(t)=f(t-a) \quad \text{(Translation by } a\in\mathbb{R})\\
&E_bf(t)=e^{2\pi ibt}f(t) \quad \text{(Modulation by } b\in\mathbb{R})\\
&D_cf(t)=|c|^{\frac{1}{2}}f(ct) \quad \text{(Dilation by } c\in\mathbb{R}\setminus\{0\}).
\end{aligned}$$

For $\phi\in L^1(\mathbb{R})$, the Fourier transform $\widehat{\phi}$ is defined by

$$\widehat{\phi}(\gamma)=\int_{\mathbb{R}}\phi(x)e^{-2\pi\iota x\gamma}dx,\quad \gamma\in\mathbb{R}.$$

As usual, we can extend the Fourier transform to a unitary mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Proposition 1.1. [7] *Let* $\{a_j\} \subset \mathbb{R}^+$, $b > 0$, $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ *and* $\phi \in L^2(\mathbb{R})$. *If*

$$B = \frac{1}{b}\sup_{\gamma\in\mathbb{R}} \sum_{j,k,m\in\mathbb{Z}} |\widehat{\phi}(a_j^{-1}\gamma - c_m)\widehat{\phi}(a_j^{-1}\gamma - c_m - \frac{ka_j}{b})| < \infty,$$

then $\{D_{a_j}T_{bk}E_{c_m}\phi\}_{j,k,m\in\mathbb{Z}}$ *is a Bessel sequence with bound* B. *Also, if*

$$A = \frac{1}{b}\inf_{\gamma\in\mathbb{R}}(\sum_{j,m\in\mathbb{Z}} |\widehat{\phi}(a_j^{-1}\gamma - c_m)|^2 - \sum_{0\neq k\in\mathbb{Z}}\sum_{j,m\in\mathbb{Z}} |\widehat{\phi}(a_j^{-1}\gamma - c_m)\widehat{\phi}(a_j^{-1}\gamma - c_m - \frac{ka_j}{b})|)$$
$$> 0,$$

then $\{D_{a_j}T_{bk}E_{c_m}\phi\}_{j,k,m\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with frame bounds* A *and* B.

1.2. Our contribution

The outcomes in the paper are organised as follows: In Section 2, we provide sufficient conditions for the extension of a pair of irregular Weyl-Heisenberg wave packet Bessel sequences to wave packet type dual frames for $L^2(\mathbb{R})$. For this we use certain techniques given in [8]. The result given in this section is supported by examples and counter-examples. We consider the finite extension of Hilbert Bessel sequences to dual Hilbert frames in Section 3. A characterization for the finite extension of a pair of Hilbert Bessel sequences to a pair of dual Hilbert frames is proved. An application of the characterization is given. Finally, necessary conditions in the direction of the finite extension are proved.

2. Extension of Wave Packet Bessel Sequences to Frames in $L^2(\mathbb{R})$

Let $\psi \in L^2(\mathbb{R})$, $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, $b \neq 0$ and $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$. A system $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is called an *irregular Weyl-Heisenberg wave packet frame* (in short, IWH *wave packet frame*) for $L^2(\mathbb{R})$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in L^2(\mathbb{R}). \tag{3}$$

If the upper condition in (3) is satisfied, then $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is called an *irregular Weyl-Heisenberg wave packet Bessel sequence* with *Bessel bound* B (in short, IWH *wave packet Bessel sequence*) in $L^2(\mathbb{R})$. Regarding the existence of IWH wave packet Bessel sequence, we have the following example.

Example 2.1. *Let* $a_j = 2^{-j}$ *for* $j < 0$, $a_j = 2^j$ *for* $j \geq 0$, $b = 1$, $c_m = m$ *for* $m \in \{1, 2, .., p\} = \mathbb{N}_p$ *and* $\psi \in L^2(\mathbb{R})$ *such that* $\widehat{\psi} = \chi_{[\frac{1}{4},\frac{3}{4})}$. *Then,* $\{D_{a_j}T_kE_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\mathbb{N}_p}$ *is a Bessel sequence in* $L^2(\mathbb{R})$. *Indeed, for arbitrary* $\gamma \in \mathbb{R}$, *we compute*

$$\sum_{j,k\in\mathbb{Z},m\in\mathbb{N}_p} |\widehat{\psi}(a_j^{-1}\gamma - c_m)\widehat{\psi}(a_j^{-1}\gamma - c_m - ka_j)|$$
$$= \sum_{j\geq 0,k\in\mathbb{Z},m\in\mathbb{N}_p} |\widehat{\psi}(2^{-j}\gamma - m)\widehat{\psi}(2^{-j}\gamma - m - k2^j)|$$
$$+ \sum_{j<0,k\in\mathbb{Z},m\in\mathbb{N}_p} |\widehat{\psi}(2^{j}\gamma - m)\widehat{\psi}(2^{j}\gamma - m - k2^{-j})|$$
$$= \sum_{j\geq 0,k\in\mathbb{Z},m\in\mathbb{N}_p} |\chi_{[2^j(\frac{1}{4}+m),2^j(\frac{3}{4}+m))}(\gamma)\chi_{[2^j(\frac{1}{4}+m+k2^j),2^j(\frac{3}{4}+m+k2^j))}(\gamma)|$$
$$+ \sum_{j<0,k\in\mathbb{Z},m\in\mathbb{N}_p} |\chi_{[2^{-j}(\frac{1}{4}+m),2^{-j}(\frac{3}{4}+m))}(\gamma)\chi_{[2^{-j}(\frac{1}{4}+m+k2^{-j}),2^{-j}(\frac{3}{4}+m+k2^{-j}))}(\gamma)|$$

$$\leq 2 \sum_{j\geq 0, k\in\mathbb{Z}, m\in\mathbb{N}_p} |\chi_{[2^j(\frac{1}{4}+m), 2^j(\frac{3}{4}+m))}(\gamma)\chi_{[2^j(\frac{1}{4}+m+k2^j), 2^j(\frac{3}{4}+m+k2^j))}(\gamma)|$$
$$= 2 \sum_{j\geq 0, m\in\mathbb{N}_p} |\chi_{[2^j(\frac{1}{4}+m), 2^j(\frac{3}{4}+m))}(\gamma)|^2$$
$$\leq 2p.$$

Therefore, $\frac{1}{b}\sup_{\gamma\in\mathbb{R}} \sum_{j,k\in\mathbb{Z}, m\in\mathbb{N}_p} |\widehat{\psi}(a_j^{-1}\gamma - c_m)\widehat{\psi}(a_j^{-1}\gamma - c_m - \frac{ka_j}{b})| \leq 2p < \infty.$
Hence by Proposition 1.1, the system $\{D_{a_j}T_kE_{c_m}\psi\}_{j,k\in\mathbb{Z}, m\in\mathbb{N}_p}$ *is an* IWH *wave packet Bessel sequence in* $L^2(\mathbb{R})$.

The wave packet Bessel sequence, in general, is not a frame for $L^2(\mathbb{R})$. It would be interesting to extend a Bessel wave packet system to a frame for $L^2(\mathbb{R})$. By using a technique in [8], the following theorem gives sufficient conditions for the extension of IWH wave packet Bessel sequences to dual IWH wave packet frames in $L^2(\mathbb{R})$. This generalizes [13, Theorem 1].

Theorem 2.1. *Let* $\{D_{a_j}T_{bk}E_{c_m}\psi_1\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}}$ *be* IWH *wave packet Bessel sequences in* $L^2(\mathbb{R})$ *with pre-frame operator* T *and* U*, respectively. Assume there exists* $\phi \in L^2(\mathbb{R})$ *such that*

(i) $\{D_{a_j}T_{bk}E_{c_m}\phi\}_{j,k,m\in\mathbb{Z}}$ *is an* IWH *wave packet frame with a dual* $\{D_{a_j}T_{bk}E_{c_m}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$.
(ii) $TU^*D_{a_j}T_{bk}E_{c_m}\phi = D_{a_j}T_{bk}E_{c_m}TU^*\phi$.

Then, for $\psi_2 = \Xi^*\phi$ *and* $\widetilde{\psi_2} = \widetilde{\phi}$ *(where* $\Xi = I - UT^*$*,* I *is the identity operator on* $L^2(\mathbb{R})$ *), the systems* $\{D_{a_j}T_{bk}E_{c_m}\psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a_j}T_{bk}E_{c_m}\psi_2\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ *constitute a dual* IWH *wave packet frame for* $L^2(\mathbb{R})$.

Proof. By definition of U and T^*, we have

$$UT^*f = U(\{\langle f, D_{a_j}T_{bk}E_{c_m}\psi_1\rangle\}_{j,k,m\in\mathbb{Z}})$$
$$= \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi_1\rangle D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_1}, \ f \in L^2(\mathbb{R}).$$

Define a bounded linear operator $\Xi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as $\Xi = I - UT^*$. Then, by using the fact that the systems $\{D_{a_j}T_{bk}E_{c_m}\phi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j}T_{bk}E_{c_m}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$ form a pair of dual IWH wave packet frame for $L^2(\mathbb{R})$, we have

$$\Xi f = \sum_{j,k,m\in\mathbb{Z}} \langle \Xi(f), D_{a_j}T_{bk}E_{c_m}\phi\rangle D_{a_j}T_{bk}E_{c_m}\widetilde{\phi}$$
$$= \sum_{j,k,m\in\mathbb{Z}} \langle f, \Xi^*(D_{a_j}T_{bk}E_{c_m}\phi)\rangle D_{a_j}T_{bk}E_{c_m}\widetilde{\phi}, \ f \in L^2(\mathbb{R}). \quad (4)$$

By using condition (ii), we compute

$$\Xi^*D_{a_j}T_{bk}E_{c_m}\phi = (I - TU^*)D_{a_j}T_{bk}E_{c_m}\phi$$
$$= D_{a_j}T_{bk}E_{c_m}(I - TU^*)\phi$$
$$= D_{a_j}T_{bk}E_{c_m}\Xi^*\phi. \quad (5)$$

The definition of Ξ gives

$$\Xi f = f - \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi_1\rangle D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_1}, \ f \in L^2(\mathbb{R}). \quad (6)$$

Choose $\psi_2 = \Xi^*\phi$ and $\widetilde{\psi_2} = \widetilde{\phi}$.
By using (4), (5) and (6), each $f \in L^2(\mathbb{R})$ can be expanded as

$$f = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi_1\rangle D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_1} + \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a_j}T_{bk}E_{c_m}\psi_2\rangle D_{a_j}T_{bk}E_{c_m}\widetilde{\psi}_2.$$

Thus, the systems $\{D_{a_j}T_{bk}E_{c_m}\psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a_j}T_{bk}E_{c_m}\psi_2\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a_j}T_{bk}E_{c_m}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ form a dual IWH wave packet frame for $L^2(\mathbb{R})$. The theorem is proved. □

The conditions given in Theorem 2.1 are sufficient but not necessary. This is justified in the following example.

Example 2.2. *Let $b = \frac{1}{4}$, $c_m = m$ for all $m \in \mathbb{Z}$ and $\psi_1 = \chi_{[0,\frac{1}{2})}$. Then, the wave packet system $\{D_1T_{bk}E_{c_m}\psi_1\}_{k,m\in\mathbb{Z}} = \{T_{\frac{1}{4}k}E_m\chi_{[0,\frac{1}{2})}\}_{k,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, see* [5, p. 206]. *Let S be the frame operator of $\{D_1T_{bk}E_{c_m}\psi_1\}_{k,m\in\mathbb{Z}}$.*
Then, for any $k', m' \in \mathbb{Z}$, we have

(i) $ST_{\frac{1}{4}k'}f = T_{\frac{1}{4}k'}Sf$ *for all* $f \in L^2(\mathbb{R})$.

(ii) $SE_{m'}f = E_{m'}Sf$ *for all* $f \in L^2(\mathbb{R})$.

Indeed, for any $f \in L^2(\mathbb{R})$, we compute

$$\begin{aligned}
ST_{\frac{1}{4}k'}f &= T_{\frac{1}{4}k'}(T_{\frac{1}{4}k'})^{-1}ST_{\frac{1}{4}k'}f \\
&= T_{\frac{1}{4}k'}T_{-\frac{1}{4}k'} \sum_{1,k,m\in\mathbb{Z}} \langle T_{\frac{1}{4}k'}f, D_1T_{\frac{1}{4}k}E_m\psi_1\rangle D_1T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'}T_{-\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle T_{\frac{1}{4}k'}f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle f, T_{-\frac{1}{4}k'}T_{\frac{1}{4}k}E_m\psi_1\rangle T_{-\frac{1}{4}k'}T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}(k-k')}E_m\psi_1\rangle T_{\frac{1}{4}(k-k')}E_m\psi_1 \\
&= T_{\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'} \sum_{1,k,m\in\mathbb{Z}} \langle f, D_1T_{\frac{1}{4}k}E_m\psi_1\rangle D_1T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'}Sf,
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
SE_{m'}f &= E_{m'}(E_{m'})^{-1}SE_{m'}f \\
&= E_{m'}E_{-m'} \sum_{1,k,m\in\mathbb{Z}} \langle E_{m'}f, D_1T_{\frac{1}{4}k}E_m\psi_1\rangle D_1T_{\frac{1}{4}k}E_m\psi_1 \\
&= E_{m'}E_{-m'} \sum_{k,m\in\mathbb{Z}} \langle E_{m'}f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\psi_1 \\
&= E_{m'} \sum_{k,m\in\mathbb{Z}} \langle f, E_{-m'}T_{\frac{1}{4}k}E_m\psi_1\rangle E_{-m'}T_{\frac{1}{4}k}E_m\psi_1 \\
&= E_{m'} \sum_{k,m\in\mathbb{Z}} \langle f, \exp^{-2\pi i\frac{1}{4}km} E_{-m'}E_mT_{\frac{1}{4}k}\psi_1\rangle \exp^{-2\pi i\frac{1}{4}km} E_{-m'}E_mT_{\frac{1}{4}k}\psi_1
\end{aligned}$$

$$= E_{m'} \sum_{k,m\in\mathbb{Z}} \langle f, E_{(m-m')}T_{\frac{1}{4}k}\psi_1\rangle E_{(m-m')}T_{\frac{1}{4}k}\psi_1$$
$$= E_{m'} \sum_{k,m\in\mathbb{Z}} \langle f, E_m T_{\frac{1}{4}k}\psi_1\rangle E_m T_{\frac{1}{4}k}\psi_1$$
$$= E_{m'} \sum_{k,m\in\mathbb{Z}} \langle f, \exp^{2\pi i\frac{1}{4}km} T_{\frac{1}{4}k}E_m\psi_1\rangle \exp^{2\pi i\frac{1}{4}km} T_{\frac{1}{4}k}E_m\psi_1$$
$$= E_{m'} \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\psi_1$$
$$= E_{m'} \sum_{1,k,m\in\mathbb{Z}} \langle f, D_1T_{\frac{1}{4}k}E_m\psi_1\rangle D_1T_{\frac{1}{4}k}E_m\psi_1$$
$$= E_{m'}Sf. \tag{8}$$

By using (7) *and* (8), *we have*

$$\begin{aligned} S^{-1}D_1T_{bk}E_{c_m}\psi_1 &= S^{-1}T_{\frac{1}{4}k}E_m\psi_1 \\ &= (E_{-m}T_{-\frac{1}{4}k}S)^{-1}\psi_1 \\ &= (SE_{-m}T_{-\frac{1}{4}k})^{-1}\psi_1 \\ &= T_{\frac{1}{4}k}E_mS^{-1}\psi_1 \\ &= D_1T_{bk}E_{c_m}S^{-1}\psi_1. \end{aligned}$$

Set $\widetilde{\psi}_1 = S^{-1}\psi_1$.
Then

$$\{D_1T_{bk}E_{c_m}\psi_1\}_{k,m\in\mathbb{Z}} = \{T_{\frac{1}{4}k}E_m\chi_{[0,\frac{1}{2})}\}_{k,m\in\mathbb{Z}}$$

and

$$\{D_1T_{bk}E_{c_m}\widetilde{\psi_1}\}_{k,m\in\mathbb{Z}} = \{D_1T_{bk}E_{c_m}S^{-1}\psi_1\}_{k,m\in\mathbb{Z}} = \{S^{-1}D_1T_{bk}E_{c_m}\psi_1\}_{k,m\in\mathbb{Z}}$$

is a pair of dual IWH *wave packet frames in* $L^2(\mathbb{R})$ *and hence a pair of* IWH *wave packet Bessel sequences in* $L^2(\mathbb{R})$ *with pre-frame operator* T *and* U *(say), respectively.*
Choose $\psi_2 = \chi_{[0,1]}$ *and* $\widetilde{\psi}_2 = 0$.
We compute

$$\sum_{k,m\in\mathbb{Z}} \langle f, D_1T_{\frac{1}{4}k}E_m\psi_1\rangle D_1T_{\frac{1}{4}k}E_m\widetilde{\psi_1} + \sum_{k,m\in\mathbb{Z}} \langle f, D_1T_{\frac{1}{4}k}E_m\psi_2\rangle D_1T_{\frac{1}{4}k}E_m\widetilde{\psi}_2$$
$$= \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\widetilde{\psi_1} + \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_2\rangle T_{\frac{1}{4}k}E_m\widetilde{\psi}_2$$
$$= \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\widetilde{\psi_1} + \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_2\rangle T_{\frac{1}{4}k}E_m0$$
$$= \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\widetilde{\psi_1} = f.$$

Therefore, the systems $\{D_1T_{bk}E_{c_m}\psi_1\}_{k,m\in\mathbb{Z}} \bigcup \{D_1T_{bk}E_{c_m}\psi_2\}_{k,m\in\mathbb{Z}}$ *and* $\{D_1T_{bk}E_{c_m}\widetilde{\psi_1}\}_{k,m\in\mathbb{Z}} \bigcup \{D_1T_{bk}E_{c_m}\widetilde{\psi_2}\}_{k,m\in\mathbb{Z}}$ *constitute a dual* IWH *wave packet frame for* $L^2(\mathbb{R})$.
Furthermore, for any $f \in L^2(\mathbb{R})$, *we have*

$$TU^*(f) = T(\{\langle f, D_1T_{bk}E_{c_m}\widetilde{\psi_1}\rangle\}_{k,m\in\mathbb{Z}})$$

$$= \sum_{k,m\in\mathbb{Z}} \langle f, D_1 T_{\frac{1}{4}k} E_m \widetilde{\psi_1}\rangle D_1 T_{\frac{1}{4}k} E_m \psi_1$$
$$= f.$$

This gives $\Xi = I - UT^* = 0$. *Therefore, for any* $\phi \in L^2(\mathbb{R})$, $\Xi^*(\phi) = 0$. *So, there does not exist* $\phi \in L^2(\mathbb{R})$ *such that* $\psi_2 = \chi_{[0,1]} = \Xi^*(\phi)$.
To sum up, there is no $\phi \in L^2(\mathbb{R})$ *such that*

(*i*) $\{D_1 T_{bk} E_{c_m} \phi\}_{k,m\in\mathbb{Z}}$ *is an* IWH *wave packet frame with a dual* $\{D_1 T_{bk} E_{c_m} \widetilde{\phi}\}_{k,m\in\mathbb{Z}}$.

(*ii*) $TU^* D_1 T_{bk} E_{c_m} \phi = D_1 T_{bk} E_{c_m} TU^* \phi$.

with $\psi_2 = \Xi^*\phi$ *and* $\widetilde{\psi_2} = \widetilde{\phi}$.

Application of Theorem 2.1: Let $b = \frac{1}{4}$, $c_m = m$ for all $m \in \mathbb{Z}$, $\psi_1 = \chi_{[0,3]}$ and $\widetilde{\psi}_1 = \chi_{[0,2]}$.
Then

$$\{D_1 T_{bk} E_{c_m} \psi_1\}_{k,m\in\mathbb{Z}} = \{T_{\frac{1}{4}k} E_m \chi_{[0,3]}\}_{k,m\in\mathbb{Z}}$$

and

$$\{D_1 T_{bk} E_{c_m} \widetilde{\psi_1}\}_{k,m\in\mathbb{Z}} = \{T_{\frac{1}{4}k} E_m \chi_{[0,2]}\}_{k,m\in\mathbb{Z}}$$

is a pair of IWH wave packet Bessel sequences in $L^2(\mathbb{R})$, see [5, p. 204]. Let T and U be pre-frame operators associated with $\{D_1 T_{bk} E_{c_m} \psi_1\}_{k,m\in\mathbb{Z}}$ and $\{D_1 T_{bk} E_{c_m} \widetilde{\psi_1}\}_{k,m\in\mathbb{Z}}$, respectively.
Choose $\phi = \chi_{[0,\frac{1}{2})}$. Then, as done in Example 2.2, the following system

$$\{D_1 T_{bk} E_{c_m} \phi\}_{k,m\in\mathbb{Z}} = \{T_{\frac{1}{4}k} E_m \chi_{[0,\frac{1}{2})}\}_{k,m\in\mathbb{Z}}$$

is an IWH wave packet frame for $L^2(\mathbb{R})$ with dual IWH wave packet frame

$$\{D_1 T_{bk} E_{c_m} \widetilde{\phi}\}_{k,m\in\mathbb{Z}} = \{D_1 T_{\frac{1}{4}k} E_m S^{-1}\phi\}_{k,m\in\mathbb{Z}}.$$

Next, we compute

$$\begin{aligned}
&TU^* D_1 T_{bk'} E_{c_{m'}} \phi \\
&= TU^* T_{\frac{1}{4}k'} E_{m'} \phi \\
&= T_{\frac{1}{4}k'} E_{m'} (T_{\frac{1}{4}k'} E_{m'})^{-1} TU^* T_{\frac{1}{4}k'} E_{m'} \phi \\
&= T_{\frac{1}{4}k'} E_{m'} E_{-m'} T_{-\frac{1}{4}k'} \sum_{1,k,m\in\mathbb{Z}} \langle T_{\frac{1}{4}k'} E_{m'} \phi, D_1 T_{\frac{1}{4}k} E_m \widetilde{\psi}_1\rangle D_1 T_{\frac{1}{4}k} E_m \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} E_{-m'} T_{-\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle T_{\frac{1}{4}k'} E_{m'} \phi, T_{\frac{1}{4}k} E_m \widetilde{\psi}_1\rangle T_{\frac{1}{4}k} E_m \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} E_{-m'} \sum_{k,m\in\mathbb{Z}} \langle E_{m'} \phi, T_{-\frac{1}{4}k'} T_{\frac{1}{4}k} E_m \widetilde{\psi}_1\rangle T_{-\frac{1}{4}k'} T_{\frac{1}{4}k} E_m \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} E_{-m'} \sum_{k,m\in\mathbb{Z}} \langle E_{m'} \phi, T_{\frac{1}{4}(k-k')} E_m \widetilde{\psi}_1\rangle T_{\frac{1}{4}(k-k')} E_m \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} E_{-m'} \sum_{k,m\in\mathbb{Z}} \langle E_{m'} \phi, T_{\frac{1}{4}k} E_m \widetilde{\psi}_1\rangle T_{\frac{1}{4}k} E_m \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} \sum_{k,m\in\mathbb{Z}} \langle \phi, E_{-m'} T_{\frac{1}{4}k} E_m \widetilde{\psi}_1\rangle E_{-m'} T_{\frac{1}{4}k} E_m \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} \sum_{k,m\in\mathbb{Z}} \langle \phi, \exp^{-2\pi i \frac{1}{4}km} E_{-m'} E_m T_{\frac{1}{4}k} \widetilde{\psi}_1\rangle \exp^{-2\pi i \frac{1}{4}km} E_{-m'} E_m T_{\frac{1}{4}k} \psi_1
\end{aligned}$$

$$
\begin{aligned}
&= T_{\frac{1}{4}k'} E_{m'} \sum_{k,m \in \mathbb{Z}} \langle \phi, E_{(m-m')} T_{\frac{1}{4}k} \widetilde{\psi_1} \rangle E_{(m-m')} T_{\frac{1}{4}k} \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} \sum_{k,m \in \mathbb{Z}} \langle \phi, E_m T_{\frac{1}{4}k} \widetilde{\psi_1} \rangle E_m T_{\frac{1}{4}k} \psi_1 \\
&= T_{\frac{1}{4}k'} E_{m'} \sum_{k,m \in \mathbb{Z}} \langle \phi, \exp^{2\pi i \frac{1}{4} km} T_{\frac{1}{4}k} E_m \widetilde{\psi_1} \rangle \exp^{2\pi i \frac{1}{4} km} T_{\frac{1}{4}k} E_m \psi_1 \\
&= D_1 T_{\frac{1}{4}k'} E_{m'} \sum_{1,k,m \in \mathbb{Z}} \langle \phi, D_1 T_{\frac{1}{4}k} E_m \widetilde{\psi_1} \rangle D_1 T_{\frac{1}{4}k} E_m \psi_1 \\
&= D_1 T_{\frac{1}{4}k'} E_{m'} T U^* \phi \\
&= D_1 T_{bk'} E_{c_{m'}} T U^* \phi, \text{ for } k', m' \in \mathbb{Z}.
\end{aligned}
$$

Therefore, both conditions of Theorem 2.1 are satisfied.
Hence for $\psi_2 = \Xi^* \phi$ and $\widetilde{\psi_2} = \widetilde{\phi}$, where $\Xi = I - UT^*$, the systems $\{D_1 T_{bk} E_{c_m} \psi_1\}_{k,m \in \mathbb{Z}} \bigcup \{D_1 T_{bk} E_{c_m} \psi_2\}_{k,m \in \mathbb{Z}}$ and $\{D_1 T_{bk} E_{c_m} \widetilde{\psi_1}\}_{k,m \in \mathbb{Z}} \bigcup \{D_1 T_{bk} E_{c_m} \widetilde{\psi_2}\}_{k,m \in \mathbb{Z}}$ form a dual IWH wave packet frame for $L^2(\mathbb{R})$.

3. Finite Extension of Bessel Sequences to Dual Hilbert Frames

This section studies the finite extension of Hilbert Bessel sequences to Hilbert dual frames . Suppose $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be a pair of Bessel sequences in a Hilbert space $\mathcal{H}$. Our aim is to find a pair of finite collection of vectors $\{p_n\}_{n=1}^{k}$ and $\{q_n\}_{n=1}^{k}$ such that $\{f_n\}_{n=1}^{\infty} \cup \{p_n\}_{n=1}^{k}$ and $\{g_n\}_{n=1}^{\infty} \cup \{q_n\}_{n=1}^{k}$ form a pair of dual frames in $\mathcal{H}$. That is

$$
f = \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n + \sum_{n=1}^{k} \langle f, p_n \rangle q_n \quad \text{for all } f \in \mathcal{H}.
$$

Bakić and Berić studied finite extensions of Bessel sequences in infinite dimensional Hilbert spaces in [1]. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be Bessel sequences in $\mathcal{H}$ with analysis operators U and V, respectively. Let I be the identity operator on $\mathcal{H}$. Bakić and Berić proved a characterization of Bessel sequences that can be extended to frames by adding finitely many vectors, where it is assumed that the operator $I - V^*U$ is compact. We present a characterization of Bessel sequences that can be extended to dual frames by adding a finite family of vectors in infinite dimensional separable Hilbert spaces.

Theorem 3.1. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be a pair of Bessel sequences in $\mathcal{H}$ with pre-frame operators U and V, respectively. Then, $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ can be extended to a pair of dual frames by adding a pair of finite collection of vectors if and only if $I - VU^*$ has finite rank.*

Proof. Suppose first that there exist $\{p_n\}_{n=1}^{k}$ and $\{q_n\}_{n=1}^{k}$ such that $\{f_n\}_{n=1}^{\infty} \cup \{p_n\}_{n=1}^{k}$ and $\{g_n\}_{n=1}^{\infty} \cup \{q_n\}_{n=1}^{k}$ form a pair of dual frames in $\mathcal{H}$. Then, for any $f \in \mathcal{H}$, we have

$$
\begin{aligned}
f &= \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n + \sum_{n=1}^{k} \langle f, p_n \rangle q_n \\
&= VU^*(f) + \sum_{n=1}^{k} \langle f, p_n \rangle q_n.
\end{aligned}
$$

Therefore, $(I - VU^*)(f) = \sum\limits_{n=1}^{k} \langle f, p_n \rangle q_n$ for all $f \in \mathcal{H}$.
This gives $\mathrm{Im}(I - VU^*) \subseteq \mathrm{span}\{q_1, ..., q_k\}$. Hence $I - VU^*$ has finite rank.

For the reverse part, assume that $I - VU^*$ has finite rank.
Let $\dim(\mathrm{Im}(I - VU^*)) = d$. We can find a frame $\{p_n\}_{n=1}^k$ for $\mathrm{Im}(I - VU^*)$, where $k \geq d$.
Let $\{q_n\}_{n=1}^k$ be a dual frame of $\{p_n\}_{n=1}^k$ in $\mathrm{Im}(I - VU^*)$.
For $f \in \mathcal{H}$, we compute

$$\begin{aligned}(I - VU^*)(f) &= \sum_{n=1}^{k} \langle (I - VU^*)(f), p_n \rangle q_n \\ &= \sum_{n=1}^{k} \langle f, (I - VU^*)^* p_n \rangle q_n \\ &= \sum_{n=1}^{k} \langle f, (I - UV^*) p_n \rangle q_n. \end{aligned} \tag{9}$$

Choose $x_n = (I - UV^*)p_n$, for $1 \leq n \leq k$. Then, $\{f_n\}_{n=1}^\infty \cup \{x_n\}_{n=1}^k$ and $\{g_n\}_{n=1}^\infty \cup \{q_n\}_{n=1}^k$ is a pair of Bessel sequences in $\mathcal{H}$.
For any $f \in \mathcal{H}$, by using (9), we have

$$f = \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n + \sum_{n=1}^{k} \langle f, x_n \rangle q_n.$$

Hence the systems $\{f_n\}_{n=1}^\infty \cup \{x_n\}_{n=1}^k$ and $\{g_n\}_{n=1}^\infty \cup \{q_n\}_{n=1}^k$ form a pair of dual frames in $\mathcal{H}$. □

Application of Theorem 3.1: Consider the discrete signal space $\mathcal{H}_o = L^2(\mathbb{N}, \mu)$, where μ is the counting measure. Let $\{\chi_j\}_{j \in \mathbb{N}}$ be the sequence of canonical unit vectors in $\mathcal{H}_o$.

(1) Define $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty \subset \mathcal{H}_o$ by

$$g_n = f_n = 0,\ 1 \leq n \leq k \text{ and } g_n = f_n = \chi_n,\ n \geq k+1\ (k \in \mathbb{N}).$$

Then, $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ is a pair of Bessel sequences in $\mathcal{H}_o$ with pre-frame operators U and V(say), respectively. Let I be the identity operator on $\mathcal{H}_o$. Then, for arbitrary $f = \{\alpha_1, \alpha_2, ...\} \in \mathcal{H}_o$, we compute

$$\begin{aligned}(I - VU^*)(f) &= f - VU^*(f) \\ &= f - V(\{\langle f, f_n \rangle\}_{n=1}^\infty) \\ &= f - \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n \\ &= \{\alpha_1, \alpha_2, ...\alpha_k, \alpha_{k+1}, \alpha_{k+2}, ...\} - \{0, ...0, \alpha_{k+1}, \alpha_{k+2}, ...\} \\ &= \{\alpha_1, \alpha_2, ...\alpha_k, 0, 0, ...\}.\end{aligned}$$

This gives $\dim(\mathrm{Im}(I - VU^*)) < \infty$, that is $I - VU^*$ has finite rank. Therefore, by Theorem 3.1, the pair of Bessel sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ can be extended to a pair of dual frames by adding a pair of finite collections of vectors.

(2) Define sequences $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty$ in $\mathcal{H}_o$ by

$$f_n = \chi_n \text{ if } 1 \leq n \leq k;\ f_n = 0, \quad \text{for } n \geq k+1$$

and

$$g_n = 0 \text{ if } 1 \leq n \leq k;\ g_n = \chi_n, \quad \text{for } n \geq k+1.$$

Then, $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ is a pair of Bessel sequences in $\mathcal{H}_o$. Let U and V be pre-frame operators for $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$, respectively. For arbitrary $f \in \mathcal{H}_o$, we compute

$$\begin{aligned}(I - VU^*)(f) &= f - VU^*(f)\\ &= f - V(\{\langle f, f_n\rangle\}_{n=1}^{\infty})\\ &= f - \sum_{n=1}^{\infty}\langle f, f_n\rangle g_n\\ &= f - 0\\ &= f.\end{aligned}$$

Therefore, $I - VU^* = I$. That is, $\dim(\mathrm{Im}(I - VU^*))$ is not finite. Hence by Theorem 3.1, the pair of Bessel sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ can not be extended to a pair of dual frames by adding a pair of finite collections of vectors.

Next, we give necessary conditions for the finite extension of a pair of Bessel sequences to a pair of dual frames.

Theorem 3.2. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be a pair of Bessel sequences in $\mathcal{H}$ with pre-frame operators U and V, respectively. Suppose $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ can be extended to a pair of dual frame by adding a pair of finite collection of vectors. Then, $KerU^*$, $KerV^*$ and $KerVU^*$ are finite dimensional.*

Proof. Suppose there exist $\{p_n\}_{n=1}^{k}$ and $\{q_n\}_{n=1}^{k}$ in $\mathcal{H}$ such that $\{f_n\}_{n=1}^{\infty} \cup \{p_n\}_{n=1}^{k}$ and $\{g_n\}_{n=1}^{\infty} \cup \{q_n\}_{n=1}^{k}$ form a pair of dual frames in $\mathcal{H}$.
Let $f \in \mathrm{Ker}U^*$. Then, $0 = U^*(f) = \{\langle f, f_n\rangle\}_{n=1}^{\infty}$. That is, $\langle f, f_n\rangle = 0$ for all $n \in \mathbb{N}$.
We compute

$$\begin{aligned}f &= \sum_{n=1}^{\infty}\langle f, f_n\rangle g_n + \sum_{n=1}^{k}\langle f, p_n\rangle q_n\\ &= \sum_{n=1}^{k}\langle f, p_n\rangle q_n.\end{aligned}$$

Therefore $\mathrm{Ker}U^* \subseteq \mathrm{span}\{q_1, q_2, .., q_k\}$. Hence $\dim(\mathrm{Ker}U^*) < \infty$.
Similarly, $\dim(\mathrm{Ker}V^*) < \infty$.
To show $\mathrm{Ker}VU^* < \infty$, let $g \in \mathrm{Ker}VU^*$.
We compute

$$\begin{aligned}g &= \sum_{n=1}^{\infty}\langle g, f_n\rangle g_n + \sum_{n=1}^{k}\langle g, p_n\rangle q_n\\ &= VU^*(g) + \sum_{n=1}^{k}\langle g, p_n\rangle q_n\\ &= \sum_{n=1}^{k}\langle g, p_n\rangle q_n.\end{aligned}$$

This gives, $\mathrm{Ker}VU^* \subseteq \mathrm{span}\{q_1, q_2, .., q_k\}$. That is, $\dim(\mathrm{Ker}VU^*) < \infty$. The theorem is proved. □

The conditions given in Theorem 3.2 are only necessary but not sufficient. This is justified in the following example.

Example 3.1. *Define sequences* $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ *in the discrete signal space* $\mathcal{H}_o$ *by*

$$f_n = \frac{\chi_n}{\sqrt{n}},\ n \in \mathbb{N},$$

and

$$g_n = \chi_n,\ n \in \mathbb{N}.$$

Then, $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *is a pair of Bessel sequences in* $\mathcal{H}_o$*. Let* U *and* V *be pre-frame operators for* $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$*, respectively.*
For $g \in KerU^*$*, we have*

$$\begin{aligned} 0 = U^*(g) &= \{\langle g, f_n\rangle\}_{n=1}^{\infty} \\ &= \left\{\left\langle g, \frac{\chi_n}{\sqrt{n}}\right\rangle\right\}_{n=1}^{\infty}. \end{aligned}$$

This gives $\langle g, \chi_n\rangle = 0$ *for all* $n \in \mathbb{N}$*. That is,* $KerU^* = \{0\}$*.*
Similarly, $KerV^* = 0$*. To compute* $KerVU^*$*, let* $h = \{\alpha_1, \alpha_2, ...\} \in KerVU^*$ *be arbitrary. Then*

$$\begin{aligned} 0 = VU^*(h) &= \sum_{n=1}^{\infty}\langle h, f_n\rangle g_n \\ &= \sum_{n=1}^{\infty}\left\langle \{\alpha_1, \alpha_2, ...\}, \frac{\chi_n}{\sqrt{n}}\right\rangle \chi_n \\ &= \left\{\frac{\alpha_1}{\sqrt{1}}, \frac{\alpha_2}{\sqrt{2}}, ...\right\}. \end{aligned}$$

This gives $KerVU^* = \{0\}$*. Thus, we have shown that* $KerU^*$*,* $KerV^*$ *and* $KerVU^*$ *are finite dimensional. On the other hand, one may observe that* $\{\chi_2, \chi_3, \chi_4, ...\} \subseteq Im(I - VU^*)$*. That is,* $Im(I - VU^*)$ *is infinite dimensional. Therefore, by Theorem 3.2 the systems* $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *can not be extended to a pair of dual frames by adding a pair of finite collection of vectors.*

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