EXACT ANALYTICAL SOLUTIONS FOR NONLINEAR SYSTEMS OF CONFORMABLE PARTIAL DIFFERENTIAL EQUATIONS VIA AN ANALYTICAL APPROACH

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Many numerical and analytical methods have been developed for solving Partial Differential Equations (PDEs) and conformable PDEs, most of which provide approximate solutions. Exact solutions, however, are vitally important in the proper understanding of the qualitative features of the concerned phenomena and processes. This paper introduces an effective analytical approach for solving nonlinear systems of conformable space-time PDEs. Moreover, the convergence theorem and error analysis of the proposed method are also shown. An essential benefit of this paper is that it yields exact analytical solutions for some nonlinear dynamical systems of conformable space-time PDEs. The Graphical representations of solutions are shown to confirm the accuracy and efficiency of the suggested method.

Keywords: Analytical Approach, Exact analytical solutions, Nonlinear systems of conformable space-time PDEs

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1. Introduction

During the last decades, it has been realized that modeling physical phenomena with fractional derivatives provides a better fit due to their non-local nature. Fractional derivatives are effective while formulating processes having memory effects. In the field of mathematical modeling of life-science applications and the analysis, there are many problems modeled in terms of PDEs and fractional PDEs. However, fractional PDEs are the generalization of PDEs

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with fractional partial derivatives of arbitrary orders that have been considered in many ways as a novel topic, and they have been the subjects of several conferences due to their important applications discovered in various areas of science, engineering, and finance. Fractional PDEs are useful tools for modeling the memory properties of various materials and processes with a nonlinear relationship to time, such as anomalous diffusion. Moreover, fractional PDEs are more applicable in the study of fluids flow, chemical physics, dynamics of protein molecules, population dynamics, biological systems, and intelligent systems [22, 7, 24, 23, 26, 27, 16, 4, 19, 17]. The conformable PDEs are simply PDEs with conformable partial derivatives. Several definitions of fractional derivatives are available in the literature. The researchers in fractional calculus realize that some nonlinear phenomena cannot be described, and some properties of fractional derivatives are not satisfied. Therefore, a rather definition called "the conformable derivative" was introduced [14].

Throughout the new definition of the conformable derivative, several works were devoted [5, 11, 12, 21, 18, 2, 9, 3, 13, 28]. Further, with the conformable derivatives, it has been proved the product rule, the mean value theorem with fractional order, and solved some differential equations of conformable derivatives. However, some functions could not be represented, or its integral transforms could not be calculated, but it is possible to do so with the help of conformable calculus theory. Therefore, the conformable calculus theory is still an active area of research.

Recently, Yücel Çenesiz et al. [6] studied PDEs with conformable derivative using the first integral method, K. Hosseini et al. [11] applied the Kudryashov method for Klein–Gordon equations with conformable derivatives, and Farid Samsami Khodadad et al. [15] proposed the method of Riccati sub equation for solving Zakharov-Kuznetsov equation with conformable derivatives. This paper aims to demonstrate that an effective analytical approach is introduced for solving nonlinear systems of conformable PDEs. The rest of the sections are organized a follows. In Section 2, we introduce some properties of the conformable calculus theory that are used in this paper. Section 3 introduces an efficient method to solve systems of conformable space-time nonlinear PDEs. In Section 4, we obtain exact solutions for some nonlinear dynamical systems of PDEs with conformable space-time derivatives. In Section 5, we introduce a discussion and graphical representations.

2. Basic Results and Definitions

Various results and definitions of conformable calculus theory are available in the literature. This section presents some modified properties which can be found in [8, 10, 20, 29] and among other references.

Definition 2.1. *For $\varphi : \mathbb{R} \times [a, \infty) \to \mathbb{R}$, the conformable partial derivative of order γ with respect to the time t for φ is defined as follows:*

$$ {}_a\mathfrak{T}_t^{\gamma}\varphi(x,t) = \lim_{\xi \to 0} \frac{\varphi(x, t + \xi(t-b)^{1-\gamma}) - \varphi(x,t)}{\xi}, \ \textit{for} \ \forall\ t > b, 0 < \gamma \leq 1. \quad (1) $$

Theorem 2.1 ([1, 25]). *For a function $\varphi(x,t) = \sum_{j=0}^{\infty} \lambda^j \varphi_j(x,t)$, the operator $N(\varphi(x,t))$ satisfies the following property:*

$$ N(\varphi(x,t)) = N(\sum_{j=0}^{\infty} \lambda^j \varphi_j(x,t)) = \sum_{j=0}^{\infty} \big(\frac{1}{j!}\frac{\partial^j}{\partial \lambda^j}\big(N(\sum_{k=0}^{j} \lambda^k \varphi_k(x,t))\big)_{\lambda=0}\big)\lambda^j. \quad (2) $$

Definition 2.2. *For a function $u : [x_0, \infty) \times \mathbb{R} \to \mathbb{R}$, $0 < \alpha \leq 1$. The left partial integral of conformable order α with respect to the space x for u is given by*

$$ {}_{x_0}\mathfrak{J}_x^{\alpha} u(x,t) = \int_{x_0}^{x} (\tau - x_0)^{\alpha-1} u(\tau, t) d\tau. \quad (3) $$

Definition 2.3. *Let $u : \mathbb{R} \times [t_0, \infty) \to \mathbb{R}$, $0 < \beta \leq 1$. The left integral of conformable order β with respect to the time t for u is given by*

$$ {}_{t_0}\mathfrak{J}_t^{\beta} u(x,t) = \int_{t_0}^{t} (\tau - t_0)^{\beta-1} u(x, \tau) d\tau. \quad (4) $$

3. Description of the Proposed Method

This section introduces an efficient extended analytical method that is called "a homotopy conformable integral method" to solve a nonlinear system of conformable PDEs of the following form:

$$ \begin{cases} {}_{t_0}\mathfrak{T}_t^{h_i\beta_i}(u_i(x,t)) + L_i(\bar{u}(x,t)) + N_i(\bar{u}(x,t)) = f_i(x,t), \\ \dfrac{\partial^{k_i} u_i(x,0)}{\partial t^{k_i}} = f_{ik_i}(x), \ k_i = 0, 1, 2, \ldots, h_i - 1, \ i = 1, 2, \ldots, m, \end{cases} \quad (5) $$

for $h_i - 1 < \beta_i < h_i \in \mathbb{N}$ and $\bar{u}(x,t) = (u_1, u_2, \ldots, u_m), t > 0$, where $L_i(\bar{u}(x,t))$, $N_i(\bar{u}(x,t))$ are linear and nonlinear operators respectively of functions $u_i(x,t)$ and their conformable partial derivatives which might include other conformable partial derivatives, and $f_i(x,t)$, $f_{ik}(x)$ are known analytic functions and ${}_{t_0}\mathfrak{T}_t^{h_i\beta_i}$ are the h_i-times conformable partial derivatives of orders β_i.

Definition 3.1. *Let $u(x,t)$ be defined on $[x_0, \infty) \times \mathbb{R}$ and $0 < \alpha \leq 1$. The partial integral transform of conformable order α with respect to the space x for u is given by*

$$ \widehat{U}_{\alpha,0}(k,t) = {}_{x_0}\mathfrak{J}_x^{k\alpha} u(x,t) = {}_{x_0}\mathfrak{J}_x^{k}\big((x-x_0)^{k\alpha-k} u(x,t)\big)_{(x_0,t)}, \ {}_{x_0}\mathfrak{J}_x^{k} = \underbrace{\int_{x_0}^{x} \cdots \int_{x_0}^{x}}_{k-times}. $$

Definition 3.2. *For $u : \mathbb{R} \times [t_0, \infty) \to \mathbb{R}$ such that $\frac{\partial^h u(x,t)}{\partial t^h}$ is continuous and $\beta \in (h-1, h)$, the partial integral transform of conformable order β with respect to the time t for u is given by*

$$\widehat{U}_{0,\beta}(x,h) = {}_{t_0}\mathfrak{J}_t^{h\beta} u(x,h) = {}_{t_0}\mathfrak{J}_t^{h}\big((t-t_0)^{h\beta-h}u(x,t)\big)_{(x,t_0)}, \quad {}_{t_0}\mathfrak{J}_t^{h} = \underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{t}}_{h-times}.$$

Definition 3.3. *Let a function $u(x,t) : [x_0, \infty) \times [t_0, \infty) \to \mathbb{R}$. The partial integral transform of conformable orders α, β with respect to the space and time for $u(x,t)$ is given by*

$$\widehat{U}_{\alpha,\beta}(k,h) = {}_{x_0}\mathfrak{J}_x^{k\alpha}\,{}_{t_0}\mathfrak{J}_t^{h\beta} u(x,t) = {}_{x_0}\mathfrak{J}_x^{k}\,{}_{t_0}\mathfrak{J}_t^{h}\big((x-x_0)^{k\alpha-k}(t-t_0)^{h\beta-h}u(x,t)\big)_{x_0,t_0},$$

where ${}_{x_0}\mathfrak{J}_x^{k} = \underbrace{\int_{x_0}^{x}\cdots\int_{x_0}^{x}}_{k-times}$, ${}_{t_0}\mathfrak{J}_t^{h} = \underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{t}}_{h-times}$, *for all* $x \geq x_0, t \geq t_0,\ k-1 < \alpha < k,\ h-1 < \beta < h$.

Theorem 3.1. *Let $\varphi(x,t) : [x_0, \infty) \times \mathbb{R} \to \mathbb{R}$, $k < \alpha \leq k$. Then, we have*

$$ {}_{x_0}\mathfrak{T}_x^{k\alpha}\,{}_{x_0}\mathfrak{J}_x^{k\alpha}\varphi(x,t) = \varphi(x,t). \tag{6}$$

Proof. By using Theorem **??** and Definition 3.1, we have

$$\begin{aligned} {}_{x_0}\mathfrak{T}_x^{k\alpha}\,{}_{x_0}\mathfrak{J}_x^{k\alpha}u(x,t) &= {}_{x_0}\mathfrak{T}_x^{k\alpha}\left({}_{x_0}\mathfrak{J}_x^{k}(x-x_0)^{k\alpha-k}u(x,t)\right) \\ &= (x-x_0)^{k-k\alpha}\frac{\partial^k}{\partial x^k}\left({}_{x_0}\mathfrak{J}_x^{k}(x-x_0)^{k\alpha-k}u(x,t)\right) \\ &= (x-x_0)^{k-k\alpha}\big((x-x_0)^{k\alpha-k}u(x,t)\big) = u(x,t). \end{aligned}$$

□

Theorem 3.2. *For $\varphi : \mathbb{R} \times [t_0, \infty) \to \mathbb{R}$ and $\beta \in (h-1, h)$, we have*

$$ {}_{t_0}\mathfrak{T}_t^{h\beta}\,{}_{t_0}\mathfrak{J}_t^{h\beta}\varphi(x,t) = \varphi(x,t). \tag{7}$$

Theorem 3.3. *Let $u(x,t) : (x_0, \infty) \times (t_0, \infty) \to \mathbb{R}$ be k, h-differentiable. Then, for all $x \geq x_0, t \geq t_0$, we have*

$$\begin{cases} {}_{x_0}\mathfrak{J}_x^{k\alpha}\,{}_{x_0}\mathfrak{T}_x^{k\alpha}u(x,t) = u(x,t) - \displaystyle\sum_{i'=0}^{k-1}\frac{x^{i'}}{i'!}\frac{\partial^{i'}u(x,t)}{\partial x^{i'}}|_{(x_0,t)}, \\ {}_{t_0}\mathfrak{J}_t^{h\beta}\,{}_{t_0}\mathfrak{T}_t^{h\beta}u(x,t) = u(x,t) - \displaystyle\sum_{j'=0}^{h-1}\frac{t^{j'}}{j'!}\frac{\partial^{j'}u(x,t)}{\partial t^{j'}}|_{(x,t_0)}. \end{cases} \tag{8}$$

In order to solve the system (5), let the solutions $u_i(x,t)$ for (5) be given by

$$u_i(x,t) = \sum_{r=0}^{\infty} u_{ij}(x,t). \tag{9}$$

Next, let the functions $u_i(x,t),\ (i=1,\cdots,m)$ be h_i-differentiable functions, and we consider the following homotopy:

$$ {}_{t_0}\mathfrak{T}_t^{h_i\beta_i}\varphi_i(x,t,p) = p[f_i(x,t) - L_i[\bar{\varphi}] - N_i[\bar{\varphi}]], \tag{10}$$

where $p \in [0,1]$, $\bar{\varphi}(x,t,p) = (\varphi_1, \varphi_2, \ldots, \varphi_m)$, and $\varphi_i(x,t,p)$ are unknown functions defined by

$$\varphi_i(x,t,p) = u_{i0}(x,t) + \sum_{j\geq 1} p^r u_{ij}(x,t). \tag{11}$$

Therefore, when $p=0$, we have $\varphi_i(x,t,0) = u_{i0}$. For $p=1$, we have $\varphi_i = u_i$. Then, when p changes (0 to 1), we have solution functions $\varphi_i(x,t,p)$ vary from $u_{i0}(x,t)$ to solution functions $u_i(x,t)$. The solution $u_{i0}(x,t)$ can be evaluated by

$$\begin{cases} {}_{t_0}\mathfrak{T}_t^{h_i\beta_i} u_{0i}(x,t) - f_i(x,t) = 0, \\ \dfrac{\partial^{k_i} u_0(x,0)}{\partial t^{k_i}} = f_{k_i}(x),\ k_i = 0,1,2,\ldots,\ h_i - 1. \end{cases} \tag{12}$$

By substituting (11) in the equation (10), we have

$$ {}_{t_0}\mathfrak{T}_t^{h_i\beta_i}(\sum_{r=0}^{\infty} p^r u_{ij}(x,t)) = p[f_i(x,t) - L_i[\sum_{r=0}^{\infty} p^r \bar{u}_r(x,t)] - N_i[\sum_{r=0}^{\infty} p^r \bar{u}_r(x,t)]]. \tag{13}$$

By using Theorem 2.1 into (13), we obtain

$$ {}_{t_0}\mathfrak{T}_t^{h_i\beta_i}(\sum_{r=0}^{\infty} p^r u_{ij} = p[f_i - L_i[\sum_{r=0}^{\infty} p^r \bar{u}_r] - \sum_{r=0}^{\infty}\big(\frac{1}{r!}\frac{\partial^r}{\partial p^r}\big(N_i \sum_{s=0}^{r} p^s \bar{u}_s\big)_{p=0}\big)p^r. \tag{14}$$

Equating the terms in (14) with identical powers of p, we get

$$\begin{cases} {}_{t_0}\mathfrak{T}_t^{h_i\beta_i} u_{i0} = 0,\ {}_{t_0}\mathfrak{T}_t^{h_i\beta_i} u_{i1} = f_i - L_i(\bar{u}_0) - N_i[\bar{u}_0], \\ {}_{t_0}\mathfrak{T}_t^{h_i\beta_i} u_{ir} = -L_i(\bar{u}_{(r-1)}) - \dfrac{1}{(r-1)!}\dfrac{\partial^{r-1}}{\partial p^{r-1}}\big(N_i[\sum_{s=0}^{r-1} p^s \bar{u}_s]\big)_{p=0}, \end{cases} \tag{15}$$

for $r = 2,3,\ldots$. Next, we apply the conformable partial integral to both sides of the system (15) and using the initial conditions given by (5), we obtain

$$\begin{cases} u_{i0}(x,t) = \sum_{j'=0}^{k-1} \dfrac{t^{j'}}{j'!} f_{ij'},\ u_{i1}(x,t) = {}_{t_0}\mathfrak{J}_t^{\beta_i}\big(f_i(x,t) - L_i(\bar{u}_0(x,t)) - N_i[\bar{u}_0(x,t)]\big], \\ u_{ij}(x,t) = {}_{t_0}\mathfrak{J}_t^{\beta_i}\big(- L_i(\bar{u}_{(r-1)}(x,t)) - \dfrac{1}{(r-1)!}\dfrac{\partial^{r-1}}{\partial p^{r-1}}\big(N_i[\sum_{s=0}^{r-1} p^s \bar{u}_s(x,t)]\big)_{p=0}\big), \end{cases} \tag{16}$$

for $r = 2,3,\ldots$. By substituting the components from (16) into (9), we obtain the analytical solutions for (5).

Theorem 3.4. *Assume that the space B is a Banach space. The solutions series in* (9) *converges to $S_i \in B$ if $\exists\ \gamma_i,\ 0 \le \gamma_i < 1$ such that, $\|u_{in}\| \le \gamma_i \|u_{i(n-1)}\|$ for $\forall n \in \mathbb{N}$.*

Proof. Let S_{in} be a series of partial sums defined as follows:

$$\begin{cases} S_{i0} = u_{i0}(x,t),\ S_{i1} = u_{i0}(x,t) + u_{i1}(x,t), \\ S_{i2} = u_{i0}(x,t) + u_{i1}(x,t) + u_{i2}(x,t), \\ \ \vdots \\ S_{in} = u_{i0}(x,t) + u_{i1}(x,t) + u_{i2}(x,t) + \cdots + u_{in}(x,t), \end{cases} \tag{17}$$

and we want to claim that $\{S_{in}\}$ is a family of Cauchy sequences in the Banach space B. In this regard, we introduce the following inequality

$$\begin{aligned} \|S_{i(n+1)} - S_{in}\| &= \|u_{i(n+1)}(x,t)\| \le \gamma_i \|u_{in}(x,t)\| \le \gamma_i^2 \|u_{i(n-1)}(x,t)\| \\ &\le \cdots \le \gamma_i^{n+1} \|u_{i0}(x,t)\|, \end{aligned} \tag{18}$$

For all $r, r' \in \mathbb{N},\ r \ge r'$ and by using (18) with help of triangle inequality, we obtain

$$\begin{aligned} \|S_{ir} - S_{ir'}\| &= \|S_{i(r'+1)} - S_{ir'} + S_{i(r'+2)} - S_{i(r'+1)} + \cdots + S_{ir} - S_{i(r-1)}\| \\ &\le \|S_{i(r'+1)} - S_{ir'}\| + \|S_{i(r'+2)} - S_{i(r'+1)}\| + \cdots + \|S_{ir} - S_{i(r-1)}\| \\ &\le \gamma_i^{r'+1} \|u_{i0}(x,t)\| + \gamma_i^{r'+2} \|u_{i0}\| + \cdots + \gamma_i^{r} \|u_{i0}\| \\ &= \gamma_i^{r'+1} (1 + \gamma_i + \cdots + \gamma_i^{r-r'-1}) \|u_{i0}\| \le \gamma_i^{r'+1} \Big(\frac{1 - \gamma^{r-r'}}{1 - \gamma_i} \Big) \|u_{i0}\|. \end{aligned}$$

Since $0 < \gamma_i < 1$, so $1 - \gamma_i^{r-r'} \le 1$. Then $\|S_{ir} - S_{ir'}\| \le \frac{\gamma_i^{r'+1}}{1-\gamma}{}_i \|u_{i0}\|$. Since $u_{i0}(x,t)$ is bounded, then $\lim_{r,r' \to \infty} \|S_{ir} - S_{ir'}\| = 0,\ i = 1, 2, \ldots, m$. Thus the family of $\{S_{ir}\}$ are Cauchy sequences in B. Therefore, solutions series given by (17) converges. □

Theorem 3.5. *For the solutions series* (9) *of* (5), *the error is*

$$\sup_{(x,t)\in\Omega} \Big| u_i(x,t) - \sum_{k=0}^{m} u_{ik}(x,t) \Big| \le \frac{\gamma_i^{m+1}}{1 - \gamma_i} \sup_{(x,t)\in\Omega} |u_{i0}(x,t)|,\ \Omega \subset \mathbb{R}^2. \tag{19}$$

Proof. By using Theorem 3.4, we get $\|S_{ir} - S_{ir'}\| \le \frac{\gamma_i^{r'+1}}{1-\gamma_i} \sup_{(x,t)\in\Omega} |u_{i0}(x,t)|$. But we assume $S_{ir} = \sum_{k=0}^{r} u_{ik}$. Since $r \to \infty$, we have $S_{in} \to u_i(x,t)$, we have

$$\|u_i(x,t) - S_{ir'}\| = \|u_i(x,t) - \sum_{k=0}^{r'} u_{ik}(x,t)\| \le \frac{\gamma_i^{r'+1}}{1 - \gamma_i} \sup_{(x,t)\in\Omega} |u_{i0}(x,t)|. \tag{20}$$

Therefore, $\sup_{(x,t)\in\Omega} \big| u_i - \sum_{k=0}^{r'} u_{ik} \big| \le \frac{\gamma_i^{r'+1}}{1-\gamma_i} \sup_{(x,t)\in\Omega} |u_{i0}|$. □

4. Applications to Dynamical Systems of Conformable PDEs

In this section, we introduce exact solutions for some examples of the conformable dynamical systems. These examples are chosen as they have not been considered before in the current forms or their exact solutions are not available in the literature.

Example 4.1. *For $0 < \alpha, \beta < 1$, we consider the following system:*

$$\begin{cases} {}_0\mathcal{T}_t^{\beta} u(x,t) + u(x,t) + v(x,t){}_0\mathcal{T}_x^{\alpha} u(x,t) = 1, \ u(x,0) = e^{x^{\alpha}/\alpha}, \\ {}_0\mathcal{T}_t^{\beta} v(x,t) - v(x,t) - u(x,t){}_0\mathcal{T}_x^{\alpha} v(x,t) = 1, \ v(x,0) = e^{-x^{\alpha}/\alpha}. \end{cases} \tag{21}$$

For $\alpha = \beta = 1$, the exact solutions for system (21) are [30] $u(x,t) = e^{\frac{x-t}{2}}$, $v(x,t) = e^{\frac{-x+t}{2}}$. Next, assume the solutions u and v for (5) are

$$u(x,t) = \sum_{r=0}^{\infty} u_r(x,t), \ v(x,t) = \sum_{r=0}^{\infty} v_r(x,t), \tag{22}$$

From the system (23), we have

$$\begin{cases} {}_0\mathcal{T}_t^{\beta} u_0 = 0, \ {}_0\mathcal{T}_t^{\beta} v_0 = 0, \ {}_0\mathcal{T}_t^{\beta} u_1 = f_1 - L_1(u_0, v_0) - N_1(u_0, v_0), \\ {}_0\mathcal{T}_t^{\beta} v_1 = f_2 - L_2(u_0, v_0) - N_2(u_0, v_0), \\ {}_0\mathcal{T}_t^{\beta} u_k = -L_1(u_{k-1}, v_{k-1}) - \dfrac{1}{(k-1)!} \dfrac{\partial^{k-1}}{\partial p^{k-1}} \big(N_1[\sum_{s=0}^{k-1} p^s (u_s, v_s)]\big)_{p=0}, \\ {}_0\mathcal{T}_t^{\beta} v_k = -L_2(u_{k-1}, v_{k-1}) - \dfrac{1}{(k-1)!} \dfrac{\partial^{k-1}}{\partial p^{k-1}} \big(N_2[\sum_{s=0}^{k-1} p^s (u_s, v_s)]\big)_{p=0}, \end{cases} \tag{23}$$

for $k = 2, 3, \ldots$. Next, we apply the conformable space-time partial integral transform given by Definition 3.3 to (21), we obtain

$$u_0 = e^{x^{\alpha}/\alpha}, \ v_0 = e^{\frac{-x^{\alpha}}{\alpha}}, \ u_k = (-1)^k \frac{t^{k\beta} e^{x^{\alpha}/\alpha}}{k!\beta^k}, \ v_k = \frac{t^{k\beta} e^{-x^{\alpha}/\alpha}}{k!\beta^k}, \tag{24}$$

for $k = 1, 2, \ldots$ By substituting the components from (24) into the system (22), the solutions u and v for (21) are

$$u = e^{x^{\alpha}/\alpha} - \frac{t^{\beta} e^{x^{\alpha}/\alpha}}{\beta} + \frac{t^{2\beta} e^{x^{\alpha}/\alpha}}{2!\beta^2} - \frac{t^{3\beta} e^{x^{\alpha}/\alpha}}{3!\beta^3} + \cdots = e^{x^{\alpha}/\alpha - t^{\beta}/\beta}, \tag{25}$$

$$v = e^{-x^{\alpha}/\alpha} + \frac{t^{\beta} e^{-x^{\alpha}/\alpha}}{\beta} + \frac{t^{2\beta} e^{-x^{\alpha}/\alpha}}{2!\beta^2} + \frac{t^{3\beta} e^{-x^{\alpha}/\alpha}}{3!\beta^3} + \cdots = e^{-x^{\alpha}/\alpha + t^{\beta}/\beta}. \tag{26}$$

Which is same to the exact solutions in case of $\alpha = \beta = 1$.

Example 4.2. *We consider nonlinear dynamical system of conformable space-time PDEs with initial values of the following form:*

$$\begin{cases} {}_0\mathfrak{T}_t^{\beta} u + (v + \frac{t^\beta}{\beta}) {}_0\mathfrak{T}_x^{\alpha} u + \frac{t^\beta {}_0\mathfrak{T}_y^{\alpha} u {}_0\mathfrak{T}_t^{\beta} v}{\beta} + \frac{(x^\alpha - y^\alpha)}{\alpha} + \frac{t^\beta}{\beta} - 1 = 0, \\ {}_0\mathfrak{T}_t^{\beta} v - (u - \frac{t^\beta}{\beta}) {}_0\mathfrak{T}_x^{\alpha} v - \frac{t^\beta {}_0\mathfrak{T}_t^{\beta} u {}_0\mathfrak{T}_y^{\alpha} u}{\beta} - \frac{(x^\alpha - y^\alpha)}{\alpha} - \frac{t^\beta}{\beta} + 1 = 0, \end{cases} \tag{27}$$

subject to the following initial conditions:

$$u(x,y,0) = -\frac{x^\alpha}{\alpha} + \frac{y^\alpha}{\alpha}, v(x,y,0) = \frac{x^\alpha}{\alpha} - \frac{y^\alpha}{\alpha}, \ 0 < \alpha \le \beta < 1 \tag{28}$$

Next, we assume that the solutions of (5) have the analytic expansions

$$u(x,y,t) = \sum_{r=0}^{\infty} u_r(x,y,t), \ v(x,y,t) = \sum_{r=0}^{\infty} v_r(x,y,t). \tag{29}$$

From the system (23), we have

$$\begin{cases} {}_0\mathfrak{T}_t^{\beta} u_0 = 0, \ {}_0\mathfrak{T}_t^{\beta} v_0 = 0, \\ {}_0\mathfrak{T}_t^{\beta} u_1 = f_1 - L_1(u_0, v_0) - N_1(u_0, v_0), \ {}_0\mathfrak{T}_t^{\beta} v_1 = f_1 - L_2(u_0, v_0) - N_2(u_0, v_0), \\ {}_0\mathfrak{T}_t^{\beta} u_k = -L_2(u_{k-1}) - \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial p^{k-1}} \big(N_1[\sum_{s=0}^{k-1} p^s(u_s, v_s)]\big)_{p=0}, \\ {}_0\mathfrak{T}_t^{\beta} v_k = -L_2(u_{k-1}) - \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial p^{k-1}} \big(N_2[\sum_{s=0}^{k-1} p^s(u_s, v_s)]\big)_{p=0}, \end{cases} \tag{30}$$

for $k = 2, 3, \ldots$. Next, we apply the conformable space-time partial integral transform given by Definition 3.3 to both sides of the system (30) and using the initial conditions, we obtain

$$u_0 = \frac{-x^\alpha + y^\alpha}{\alpha}, v_0 = \frac{x^\alpha - y^\alpha}{\alpha}, u_1 = \frac{t^\beta}{\beta}, v_1 = -\frac{t^\beta}{\beta}, u_k = 0, v_k = 0, k = 2, , 3 \ldots.$$

And finally we obtain th exact analytical solutions for the system 27 as follows:

$$u(x,y,t) = -\frac{x^\alpha}{\alpha} + \frac{y^\alpha}{\alpha} + \frac{t^\beta}{\beta}, v(x,y,t) = \frac{x^\alpha}{\alpha} - \frac{y^\alpha}{\alpha} - \frac{t^\beta}{\beta}, \tag{31}$$

which is same to the exact solutions in case of $\alpha = \beta = 1$.

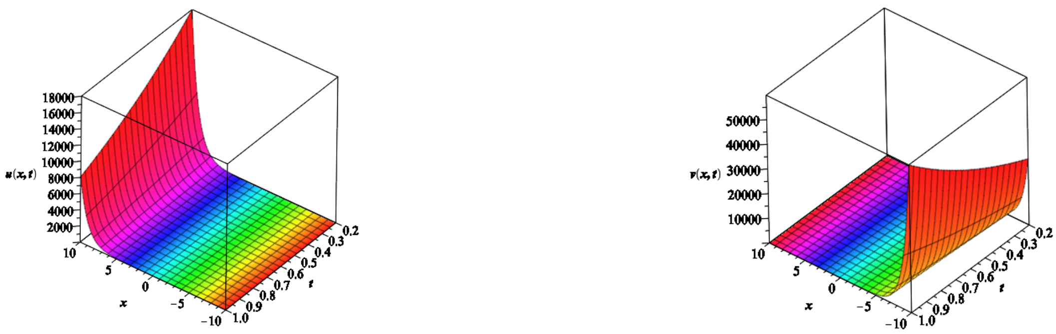
5. Discussion and Graphical Representations

The solutions for Example 4.1 are graphically described in Figure 1a and Figure 1b when $\alpha = \beta = 0.5$ and in Figure2a and Figure 2b when $\alpha = 1$ and $\beta = 1$ through several points of x and t. We represent u, v for Example 4.2 when y is fixed at $y = 1$ in Figure 3a and Figure 3b for $\alpha = \beta = 0.5$ and in Figure 4a and Figure 4b for $\alpha = \beta = 1$ through several values of x and t.

(a) Graphical representation of u for Example 4.1 as $\alpha = \beta = 0.5$.

(b) Graphical representation of $v(x,t)$ for Example 4.1 $\alpha = \beta = 0.5$.

Fig. 1. Graphical representations of solutions when $\alpha = 0.5,\ \beta = 0.5$ for Example 4.1.



(a) Graphical representation of u for Example 4.1 as $\alpha = 1$ and $\beta = 1$.

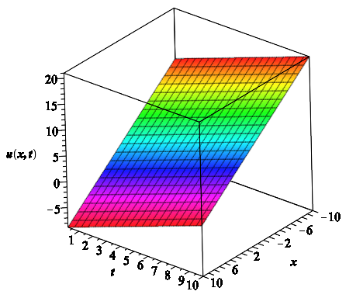
(b) Graphical representation of $v(x,t)$ for Example 4.1 as $\alpha = 1$ and $\beta = 1$.

Fig. 2. Graphical representations of solutions for Example 4.1 when $\alpha = \beta = 1$.

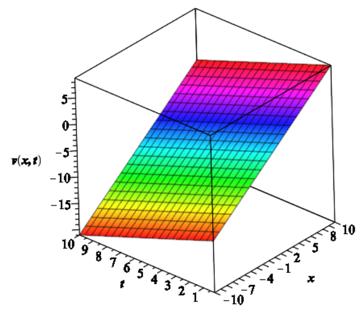
(a) Graphical representation of $u(x,y,t)$ for Example 4.2 at $y = 1$ and $\alpha = \beta = 0.5$.

(b) Graphical representation of $v(x,y,t)$ for Example 4.2 at $y = 1$ and $\alpha = \beta = 0.5$.

Fig. 3. Graphical representations of solutions for Example 4.2 at $y = 1$ as $\alpha = \beta = 0.5$.



(a) Graphical representation of u for Example 4.2 at $y = 1$ as $\alpha = \beta = 1$.



(b) Graphical representation of $v(x, t)$ at $y = 1$ when $\alpha = \beta = 1$ for Example 4.2.

Fig. 4. Graphical representations of solutions for Example 4.2 at $y = 1$ as $\alpha = \beta = 1$.

6. **Conclusion**

In this paper, we introduced an effective analytical approach for solving nonlinear systems of conformable space-time PDEs. Further, the convergence and error analysis of the solution are also shown. The real benefit of the introduced method is that it provides exact solutions of nonlinear dynamical systems of conformable space-time PDEs, which are in excellent agreement to prove the efficiency of the introduced method. It is valued to state that putting the proposed method into practice is very dependable, well-organized, and applicable to solve other nonlinear physical systems.

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Hayman Thabet and Subhash Kendre contributed substantially to this paper. Hayman Thabet wrote this paper, Subhash Kendre supervised the development of the paper, Dumitru Baleanu, and James Peters helped in evaluating and editing the paper.

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