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SUMS OF C-FRAMES, C-RIESZ BASES AND ORTHONORMAL MAPPINGS

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In this paper we introduce and prove some new concepts and results on c-frames for Hilbert spaces. We define c-Riesz bases for a Hilbert space H and state some results to characterize them. Then, we give necessary and sufficient conditions on c-Bessel mappings f and g and operators L_1, L_2 on H so that $L_1 f + L_2 g$ is a c-frame for H. This allows us to construct a large number of new c-frames by existing c-frames. Also, we define orthonormal mappings for H and we specify a necessary and sufficient condition to represent a c-frame as a linear combination of two orthonormal mappings. Moreover, we show that every c-frame can be written as a (multiple of a) sum of two Parseval c-frames; it can also be written as a (multiple of a) sum of an orthonormal mapping and a c-Riesz basis.

Keywords: Hilbert space, frame, c-frame, Parseval c-frame, c-Riesz basis, orthonormal mapping.

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1. Introduction

The concept of frames (discrete frames) in Hilbert spaces has been introduced by Duffin and Schaeffer [8] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [6] by Daubechies, Grossmann and Meyer, frame theory popularized greatly.

Generally, frames have been used in signal processing, image processing, data compression and sampling theory. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements. Later, motivated by the theory of coherent states, this concept was generalized by Antoine *et al.* to families indexed by some locally compact space endowed with a Radon measure. Their approach leads to the notion of continuous frames [1, 2, 13, 18]. Prominent examples are connected to the continuous wavelet transform [1, 16] and the short time Fourier transform [14]. In mathematical physics, these frames are referred to as coherent states [1, 15]. Some results about continues frames were discussed in [3, 9, 10, 11, 12, 20].

In this paper we generalize some results in [19] and [4] from frame theory to c-frames.

The paper is organized as follows. In Section 2, we introduce the concept of c-Riesz bases for Hilbert spaces and discuss about their characteristics and their

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relations by continuous frames. Our aim in Section 3 is producing new continuous frames by old ones and some operators, especially their associated frame operators. Indeed, we construct new continuous frames by sum of some existing continuous frames or local adding. Section 4 is devoted to introducing of orthonormal mappings for a Hilbert space. Via these mappings, we describe every continuous frame as a multiple of a sum of two Parseval continuous frames. We conclude the section by showing that every continuous frame can be written as a linear combination of an orthonormal mapping and a c-Riesz basis.

Throughout this paper, H will be a separable Hilbert space.

We first recall the definition of continuous frame [20].

Definition 1.1. ([20]) *Suppose that (Ω,μ) is a measure space with positive measure μ. A mapping $f:\Omega\longrightarrow H$ is called a* continuous frame*, or simply* c-frame*, with respect to (Ω,μ) for H, if:*
(i) For each $h\in H$, $\omega\longmapsto\langle h,f(\omega)\rangle$ is a measurable function,
(ii) there exist positive constants A and B such that

$$A\|h\|^2\leq\int_\Omega|\langle h,f(\omega)\rangle|^2d\mu(\omega)\leq B\|h\|^2,\quad h\in H. \tag{1}$$

The constants A,B are called c-frame bounds*. If A,B can be chosen such that $A=B$, then f is called a* tight c-frame *and if $A=B=1$, it is called a Parseval c-frame. A mapping f is called* c-Bessel mapping *if the second inequality in (1) holds. In this case, B is called the* Bessel constant.

We can define some operators associated to a c-Bessel mapping. The following proposition is a useful tool in the rest of our discussion.

Proposition 1.1. ([20]) *Let (Ω,μ) be a measure space and $f:\Omega\longrightarrow H$ be a c-Bessel mapping for H. Then the operator $T_f:L^2(\Omega,\mu)\longrightarrow H$, weakly defined by*

$$\langle T_f\varphi,h\rangle=\int_\Omega\varphi(\omega)\langle f(\omega),h\rangle d\mu(\omega),\quad h\in H \tag{2}$$

is well defined, linear, bounded, and its adjoint is given by

$$T_f^*:H\longrightarrow L^2(\Omega,\mu),\quad T_f^*h(\omega)=\langle h,f(\omega)\rangle,\ \ \omega\in\Omega. \tag{3}$$

The operator T_f is called *synthesis operator* and T_f^* is called *analysis operator* of f.

If f is a c-Bessel mapping with respect to (Ω,μ) for H, then the operator $S_f:H\longrightarrow H$ defined by $S_f=T_fT_f^*$, is called *frame operator* of f. Thus

$$\langle S_fh,k\rangle=\int_\Omega\langle h,f(\omega)\rangle\langle f(\omega),k\rangle d\mu(\omega),\quad h,k\in H.$$

If f is a c-frame for H, then S is invertible.

The converse of Proposition 1.1 holds when in the measure space (Ω,μ), μ is σ-finite.

Proposition 1.2. ([20]) *Let (Ω,μ) be a measure space where μ is σ-finite. Let $f:\Omega\longrightarrow H$ be a mapping such that for each $h\in H$, $\omega\longmapsto\langle h,f(\omega)\rangle$ is measurable. If the mapping $T_f:L^2(\Omega,\mu)\longrightarrow H$ defined by (2), is a bounded operator, then f is a c-Bessel mapping.*

The next theorem gives an equivalent characterization of a continuous frame.

Theorem 1.1. ([20]) *Suppose that (Ω, μ) is a measure space where μ is σ-finite. Let $f : \Omega \longrightarrow H$ be a mapping such that for each $h \in H$, $\omega \longmapsto \langle h, f(\omega)\rangle$ is measurable. The mapping f is a c-frame with respect to (Ω, μ) for H if and only if the operator $T_f : L^2(\Omega, \mu) \longrightarrow H$ defined by (2), is a bounded and onto operator.*

Now, we state a known result that will be employed to prove some results.

Proposition 1.3. ([4]) *Let $K : H \longrightarrow H$ be a bounded linear operator. Then the following hold.*
(i) $K = \alpha(U_1 + U_2 + U_3)$, where each U_j, $j = 1, 2, 3$, is a unitary operator and α is a constant.
(ii) If K is onto, then it can be written as a linear combination of two unitary operators if and only if K is invertible.

The following Lemma shows that if an operator has closed range, there exists a *right-inverse* operator in the following sense:

Lemma 1.1. ([5]) *Let H and K be Hilbert spaces, and suppose that $U : K \longrightarrow H$ is a bounded operator with closed range R_U. Then there exists a bounded operator $U^\dagger : H \longrightarrow K$ for which*

$$UU^\dagger h = h, \quad h \in R_U.$$

The operator $U^\dagger$ is called the *pseudo-inverse* of U.

2. c-Riesz bases

In this section, we define c-Riesz bases which are generalization of (discrete) Riesz bases. Then we characterize c-Riesz bases and verify relations between c-Riesz bases and c-frames.

Now, we define c-Riesz bases as follows:

Definition 2.1. *Let (Ω, μ) be a measure space. A mapping $f : \Omega \longrightarrow H$ is called a* c-Riesz basis *with respect to (Ω, μ) for H, if:*
(i) $\big\{h : \langle h, f(\omega)\rangle = 0$, a.e. $[\mu]\big\} = \{0\}$,
(ii) for each $h \in H$, $\omega \longmapsto \langle h, f(\omega)\rangle$ is measurable and the operator $T_f : L^2(\Omega, \mu) \longrightarrow H$ defined by (2), is well-defined and there are positive constants A and B such that

$$A\|\varphi\|_2 \leq \|T_f\varphi\| \leq B\|\varphi\|_2, \quad \varphi \in L^2(\Omega, \mu).$$

If in the definition of c-Riesz basis, the measure space $\Omega = \mathbb{N}$ and μ is the counting measure, then our c-Riesz basis will be a (discrete) Riesz basis and so we expect that some properties of Riesz bases can be satisfied in c-Riesz bases.

The following proposition shows that, under some conditions, a c-Riesz basis is a special case of a c-frame.

Proposition 2.1. *Suppose that (Ω, μ) is a measure space where μ is σ-finite and consider the mapping $f : \Omega \longrightarrow H$.*
(i) Assume that for each $h \in H$, $\omega \longmapsto \langle h, f(\omega)\rangle$ is measurable. Then f is a c-Riesz basis with respect to (Ω, μ) for H if and only if T_f defined by (2) is a well defined, bounded and invertible operator from $L^2(\Omega, \mu)$ onto H.
(ii) If f is a c-Riesz basis with respect to (Ω, μ) for H, then f is a c-frame with respect to (Ω, μ) for H.

Proof. (i) By Proposition 1.2 and Proposition 1.1 and Theorem 4.12 in [21], it is clear.
(ii) By assumption and (i), the operator T_f defined by (2) is an invertible bounded operator. So by Theorem 1.1, f is a c-frame with respect to (Ω, μ) for H. $\square$

We will now present some equivalent conditions for a c-frame being a c-Riesz basis.

Theorem 2.1. *Suppose (Ω, μ) is a measure space where μ is σ-finite. Let f be a c-frame with respect to (Ω, μ) for H with synthesis operator T_f. Then the following statements are equivalent:*
(i) f is a c-Riesz basis with respect to (Ω, μ) for H.
(ii) T_f is one-to-one.
(iii) $R_{T_f^*} = L^2(\Omega, \mu)$.

Proof. $(i) \to (ii)$: It is obvious.
$(ii) \to (i)$: By Theorem 1.1, the operator T_f defined by (2) is bounded and onto. By (ii), T_f is also one-to-one. Therefore T_f has a bounded inverse $T_f^{-1} : H \longrightarrow L^2(\Omega, \mu)$ and hence f is a c-Riesz basis with respect to (Ω, μ) for H by Proposition 2.1.
$(i) \to (iii)$: By Proposition 2.1, T_f has a bounded inverse on $R_{T_f} = H$. Consequently, $R_{T_f^*} = L^2(\Omega, \mu)$.
$(iii) \to (i)$: Since the operator T_f^* is invertible, so is T_f. $\square$

3. Sums of c-frames

We want to verify some situations to produce new c-frames by adding some known existing c-frames. Also, we use some operators, especially frame operators of assumed c-frames, to construct new c-frames.

Proposition 3.1. *Let f be a c-frame with respect to (Ω, μ) for H with frame operator S and frame bounds A, B and let $L : H \longrightarrow H$ be a bounded operator. Then Lf is a c-frame for H if and only if L is onto. Moreover, in this case the frame operator of Lf is LSL^* and its bounds are $\|L^\dagger\|^{-2}A$ and $\|L\|^2 B$, where $L^\dagger$ is pseudo-inverse of L.*

Proof. If L is onto, then by Lemma 1.1, the operator $L^\dagger$ is well defined and bounded. For each $h \in H$

$$\begin{aligned}\int_\Omega |\langle h, Lf(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega |\langle L^* h, f(\omega)\rangle|^2 d\mu(\omega) \\ &\geq A\|L^* h\|^2 \geq (\|L^\dagger\|^{-2} A)\|h\|^2,\end{aligned}$$

also

$$\begin{aligned}\int_\Omega |\langle h, Lf(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega |\langle L^* h, f(\omega)\rangle|^2 d\mu(\omega) \\ &\leq B\|L^* h\|^2 \leq (\|L\|^2 B)\|h\|^2.\end{aligned}$$

Hence Lf is a c-frame with respect to (Ω, μ) for H with frame bounds $\|L^\dagger\|^{-2}A$ and $\|L\|^2 B$.

Conversly, let Lf be a c-frame with respect to (Ω,μ) for H. Then for each $\varphi \in L^2(\Omega,\mu)$ and $h \in H$

$$\begin{aligned}\langle T_{Lf}\varphi, h\rangle &= \int_\Omega \varphi(\omega)\langle Lf(\omega), h\rangle d\mu(\omega)\\ &= \int_\Omega \varphi(\omega)\langle f(\omega), L^*h\rangle d\mu(\omega)\\ &= \langle LT_f\varphi, h\rangle.\end{aligned}$$

So $T_{Lf} = LT_f$. Since Lf is a c-frame, so T_{Lf} is onto and therefore L is onto. Furthermore, for each $h, k \in H$

$$\begin{aligned}&\int_\Omega \langle h, Lf(\omega)\rangle\langle Lf(\omega), k\rangle d\mu(\omega)\\ =&\int_\Omega \langle L^*h, f(\omega)\rangle\langle f(\omega), L^*k\rangle d\mu(\omega)\\ =&\langle SL^*h, L^*k\rangle = \langle LSL^*h, k\rangle.\end{aligned}$$

Thus the frame operator of Lf is LSL^*. □

The following corollary states that when we can construct a c-frame by adding a c-frame f to Lf, where L is a bounded operator on H.

Corollary 3.1. *If f is a c-frame with respect to (Ω,μ) for H and $L : H \longrightarrow H$ is a bounded operator, then $f + Lf$ is a c-frame for H if and only if $I + L$ is onto. In this case, the frame operator of $f + Lf$ is $(I + L)S(I + L^*)$ and the frame bounds are*

$$\|(I + L)^\dagger\|^{-2}A \quad and \quad \|I + L\|^2 B.$$

In particular, if L is a positive operator (or just $I + L \geq \epsilon$, for some $\epsilon > 0$) then $f + Lf$ is a c-frame with frame operator

$$S + LS + SL^* + LSL^*.$$

Proof. By Proposition 3.1, it is obvious. □

The above corollary can be used for the orthogonal projections.

Corollary 3.2. *If f is a c-frame with respect to (Ω,μ) for H and P is an orthogonal projection on H, then for all $a \neq -1$, $f + aPf$ is a c-frame for H.*

Proof. Let $L = aP$ in Corollary 3.1. □

Now, we give some necessary and sufficient conditions on c-Bessel mappings f and g and operators L_1, L_2 on H such that $L_1f + L_2g$ is a c-frame.

Proposition 3.2. *Let (Ω,μ) be a measure space where μ is σ-finite. Let f and g be c-Bessel mappings with respect to (Ω,μ) for H with synthesis operators T_f, T_g and frame operators S_f, S_g, respectively. For the given bounded operators $L_1, L_2 : H \longrightarrow H$, the following are equivalent:*
(i) $L_1f + L_2g$ is a c-frame for H.
(ii) $T_f^*L_1^* + T_g^*L_2^*$ is a bounded operator on H, which is bounded below.
(iii) $S = L_1S_fL_1^* + L_2S_gL_2^* + L_1T_fT_g^*L_2^* + L_2T_gT_f^*L_1^* \geq \epsilon$, for some $\epsilon > 0$. Moreover, in this case, S is the frame operator for $L_1f + L_2g$.

Proof. $(i) \leftrightarrow (ii)$ $L_1 f + L_2 g$ is a c-frame if and only its analysis operator T^* is a bounded and bounded below operator; in this case for each $\omega \in \Omega$

$$\begin{aligned} T^* h(\omega) =&\langle h, L_1 f(\omega) + L_2 g(\omega)\rangle \\ =&\langle L_1^* h, f(\omega)\rangle + \langle L_2^* h, g(\omega)\rangle \\ =&T_f^* L_1^* h(\omega) + T_g^* L_2^* h(\omega). \end{aligned}$$

$(ii) \leftrightarrow (iii)$ Let T be the synthesis operator of c-Bessel mapping $L_1 f + L_2 g$. The frame operator of $L_1 f + L_2 g$ is

$$\begin{aligned} S = TT^* &= (T_f^* L_1^* + T_g^* L_2^*)^* (T_f^* L_1^* + T_g^* L_2^*) \\ &= L_1 S_f L_1^* + L_2 S_g L_2^* + L_1 T_f T_g^* L_2^* + L_2 T_g T_f^* L_1^*. \end{aligned}$$

So T^* is bounded below if and only if $S \geq \epsilon$, for some $\epsilon > 0$. □

The following theorem gives us a c-frame by sum of a c-frame and a c-Bessel mapping.

Theorem 3.1. *Let f be a c-frame with respect to (Ω, μ) for H with synthesis operator T_f and frame operator S_f and g be a c-Bessel mapping with respect to (Ω, μ) for H with synthesis operator T_g and frame operator S_g. Suppose that range$(T_g^*) \subseteq$ range(T_f^*). If the operator $R = T_f T_g^*$ is a positive operator, then $f + g$ is a c-frame for H with frame operator $S_f + R + R^* + S_g$.*

Proof. Let $L_1 = I = L_2$ and $S = S_f + R + R^* + S_g$. Then $S \geq A$, where A is a below frame bound of f. By Proposition 3.2, $f + g$ is a c-frame for H with frame operator $S_f + R + R^* + S_g$. □

If L is a normal operator on the Hilbert space H, then for a $\psi \in C(\sigma(L))$, $\psi(L)$ is defined by $\psi(L) = \Gamma^{-1}\psi$, where Γ is the Gelfand map. So if L is a positive operator, then for all $a \in \mathbb{R}$, L^a is well defined. For more details, refer to the definition of functional calculus in [7, p.93].

Now, as an application of above theorem we have:

Corollary 3.3. *If f is a c-frame with respect to (Ω, μ) for H with frame operator S and g is a c-Bessel mapping with respect to (Ω, μ) for H such that*

$$\langle h, k\rangle = \int_\Omega \langle h, f(\omega)\rangle \langle g(\omega), k\rangle d\mu(\omega), \quad h, k \in H,$$

then for all real numbers a and b, $S^a f + S^b g$ is a c-frame for H.

Proof. For each $h, k \in H$,

$$\begin{aligned} \langle S^{a+b} h, k\rangle =&\langle S^a h, S^b k\rangle = \int_\Omega \langle S^a h, f(\omega)\rangle \langle g(\omega), S^b k\rangle d\mu(\omega) \\ &= \int_\Omega \langle h, S^a f(\omega)\rangle \langle S^b g(\omega), k\rangle d\mu(\omega) = \langle \tilde{T_1}^* h, \tilde{T_2}^* k\rangle, \end{aligned}$$

where $\tilde{T_1}$ and $\tilde{T_2}$ are the synthesis operators of c-frames $S^a f$ and $S^b g$, respectively. So $S^{a+b} = \tilde{T_1}\tilde{T_2}^* = R$. Therefore $S^a f + S^b g$ is a c-frame by Theorem 3.1. □

Especially, if g is a dual c-frame of f, then we have the same result.

Corollary 3.4. *If f is a c-frame with respect to (Ω, μ) for H with frame operator S and g is a dual c-frame of f, then for all real numbers a and b, $S^a f + S^b g$ is a c-frame for H.*

Proof. By definition of a dual c-frame and previous corollary, the conclusion follows. □

Moreover, we can produce new c-frames by local addition on an existing c-frame.

Proposition 3.3. *Let f be a c-frame with respect to (Ω, μ) for H with frame operator S and frame bounds A, B. Let $\{\Omega_1, \Omega_2\}$ be a partition of Ω such that Ω_1 and Ω_2 are measurable. Assume that S_j be the frame operator of c-Bessel mapping $f_j : \Omega_j \longrightarrow H$, $j = 1, 2$. Then for all real numbers a and b, the mapping $g : \Omega \longrightarrow H$ defined by*

$$g(\omega) = \begin{cases} f_1(\omega) + S_1^a f_1(\omega), & \omega \in \Omega_1 \\ f_2(\omega) + S_2^b f_2(\omega), & \omega \in \Omega_2 \end{cases},$$

is a c-frame for H.

Proof. Let $a, b \in \mathbb{R}$, then for each $h \in H$,

$$\begin{aligned}
&(\int_{\Omega_1} |\langle h, (f_1 + S_1^a f_1)(\omega)\rangle|^2 d\mu(\omega))^{\frac{1}{2}} \\
\leq &(\int_{\Omega_1} |\langle h, f_1(\omega)\rangle|^2 d\mu(\omega))^{\frac{1}{2}} + (\int_{\Omega_1} |\langle h, S_1^a f_1(\omega)\rangle|^2 d\mu(\omega))^{\frac{1}{2}} \\
\leq &\sqrt{B}\|h\| + \sqrt{B}\|S_1^a h\| \leq \sqrt{B}(1 + \|S_1^a\|)\|h\|,
\end{aligned}$$

similarly,

$$(\int_{\Omega_2} |\langle h, (f_2 h + S_2^b f_2)\rangle|^2 d\mu(\omega))^{\frac{1}{2}} \leq \sqrt{B}(1 + \|S_2^b\|)\|h\|.$$

Thus g is a c-Bessel mapping with respect to (Ω, μ).

If $\tilde{S}_1$ is the frame operator of $f_1 + S_1^a f_1$, then for each $h, k \in H$,

$$\begin{aligned}
\langle \tilde{S}_1 h_1, k\rangle &= \int_{\Omega_1} \langle h, (f_1 + S_1^a f_1)(\omega)\rangle \langle (f_1 + S_1^a f_1)(\omega), k\rangle d\mu(\omega) \\
&= \langle (S_1 + 2S_1^{1+a} + S_1^{1+2a})h, k\rangle.
\end{aligned}$$

Hence $\tilde{S}_1 = S_1 + 2S_1^{1+a} + S_1^{1+2a} \geq S_1$. Similarly, for $\tilde{S}_2$, the frame operator of $f_2 + S_2^b f_2$, we have $\tilde{S}_2 \geq S_2$. Hence, $\tilde{S}$, the frame operator of g satisfies

$$\tilde{S} \geq S_1 + S_2 = S.$$

Therefore, g is a c-frame for H. □

4. **Orthonormal mappings**

We want to define some kind of mappings that in discrete case are orthonormal bases. Actually, our purpose here is to define a mapping $f : \Omega \longrightarrow H$ that has the similar properties to an orthonormal basis of H.

Definition 4.1. *Suppose that (Ω, μ) is a measure space. A mapping $f : \Omega \longrightarrow H$ is called an* orthonormal mapping *with respect to (Ω, μ) for H, if:*
(i) For each $h \in H$, $\omega \longmapsto \langle h, f(\omega)\rangle$ is measurable,
(ii) for almost all $\nu \in \Omega$,

$$\int_\Omega \varphi(\omega)\langle f(\omega), f(\nu)\rangle d\mu(\omega) = \varphi(\nu), \;\; \varphi \in L^2(\Omega, \mu),$$

(ii) for each $h \in H$, $\int_\Omega |\langle h, f(\omega)\rangle|^2 d\mu(\omega) = \|h\|^2$.

Example 4.1. *Let H be a separable Hilbert space and $\{e_i\}_{i\in I}$ be an orthonormal basis for H. Let $\Omega = I$ and μ be a counting measure on Ω. Then $f : I \longrightarrow H$ defined by $f(i) = e_i$, $i \in I$, is an orthonormal mapping with respect to (Ω, μ) for H.*

Example 4.2. *Let $\Omega = \{a, b, c\}$, $\Sigma = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$ and $\mu : \Sigma \longrightarrow [0, \infty]$ be a measure such that $\mu(\emptyset) = 0$, $\mu(\{a, b\}) = 1$, $\mu(\{c\}) = 1$ and $\mu(\Omega) = 2$. Assume that H is a 2 dimensional Hilbert space with an orthonormal basis $\{e_1, e_2\}$. Now, define*

$$f : \Omega \longrightarrow H$$

$$f(a) = e_1, \;\; f(b) = e_1, \;\; f(c) = e_2,$$

or, equivalently, $f = e_1\chi_{\{a,b\}} + e_2\chi_{\{c\}}$. For each $h \in H$,

$$\langle f, h\rangle = \langle e_1, h\rangle\chi_{\{a,b\}} + \langle e_2, h\rangle\chi_{\{c\}},$$

so $\omega \longmapsto \langle h, f(\omega)\rangle$ is measurable. For each $\omega \in \Omega$, we have $\langle f(\omega), f(a)\rangle = \chi_{\{a,b\}}(\omega)$. Now, suppose

$$S = \sum_{i=1}^{3} \alpha_i \chi_{A_i}$$

is a simple function where $A_1 = \{a, b\}$, $A_2 = \{c\}$ and $A_3 = \Omega$. So

$$\int_\Omega S(\omega)\langle f(\omega), f(a)\rangle d\mu(\omega) = S(a).$$

Similarly, for $\nu = b, c$, we have

$$\int_\Omega S(\omega)\langle f(\omega), f(\nu)\rangle d\mu(\omega) = S(\nu).$$

Therefore, for each $\nu \in \Omega$ and $\varphi \in L^2(\Omega, \mu)$,

$$\int_\Omega \varphi(\omega)\langle f(\omega), f(\nu)\rangle d\mu(\omega) = \varphi(\nu).$$

It is easy to show that $\int_\Omega |\langle h, f(\omega)\rangle|^2 d\mu(\omega) = \|h\|^2$. Hence f is an orthonormal mapping with respect to (Ω, μ) for H.

It is clear that every orthonormal mapping is a c-Riesz basis.

The following proposition shows that a c-frame g can be characterized as $g = Vf$, where f is an orthonormal mapping and V is a bounded and onto operator on H.

Proposition 4.1. *Let f be an orthonormal mapping with respect to (Ω, μ) for H and g be a c-frame with respect to (Ω, μ) for H. Then there exists a bounded and onto operator $V : H \longrightarrow H$ such that $g = Vf$. Moreover, V is invertible if g is a c-Riesz basis for H and V is unitary if g is an orthonormal mapping for H.*

Proof. Let $V : H \longrightarrow H$ be weakly defined by

$$\langle Vh, k\rangle = \int_{\Omega} \langle h, f(\omega)\rangle\langle g(\omega), k\rangle d\mu(\omega), \quad h, k \in H.$$

We have

$$\begin{aligned}\|Vh\| &= \sup_{\|k\|=1} |\langle Vh, k\rangle| \\ &\leq \sup_{\|k\|=1} (\int_{\Omega} |\langle h, f(\omega)\rangle|^2 d\mu(\omega))^{\frac{1}{2}} (\int_{\Omega} |\langle g(\omega), k\rangle|^2 d\mu(\omega))^{\frac{1}{2}} \\ &\leq \sqrt{B}\|h\|,\end{aligned}$$

where B is an upper frame bound of c-frame g. So V is a well defined and bounded operator. Now, we show that V is onto. Let T_f and T_g be the synthesis operators of c-frames f and g, respectively. For each $\nu \in \Omega$,

$$\begin{aligned}\langle Vf(\nu), k\rangle &= \int_{\Omega} \langle f(\nu), f(\omega)\rangle\langle g(\omega), k\rangle d\mu(\omega) \\ &= \overline{\int_{\Omega} \langle k, g(\omega)\rangle\langle f(\omega), f(\nu)\rangle d\mu(\omega)} \\ &= \overline{\langle k, g(\nu)\rangle} = \langle g(\nu), k\rangle, \quad k \in H,\end{aligned}$$

so $Vf = g$. Let $h \in H$. Then there exists a $\varphi \in L^2(\Omega, \mu)$ such that $T_g\varphi = h$. Letting $k' = T_f\varphi$, for each $k \in H$, we have

$$\begin{aligned}\langle Vk', k\rangle &= \langle T_f\varphi, V^*k\rangle = \int_{\Omega} \varphi(\omega)\langle f(\omega), V^*k\rangle d\mu(\omega) \\ &= \int_{\Omega} \varphi(\omega)\langle Vf(\omega), k\rangle d\mu(\omega) = \int_{\Omega} \varphi(\omega)\langle g(\omega), k\rangle d\mu(\omega) \\ &= \langle h, k\rangle.\end{aligned}$$

Hence, $Vk' = h$ and consequetly V is onto.
Let g be a c-Riesz basis for H and $Vk = 0$. We have

$$\langle Vk, h\rangle = \int_{\Omega} \langle k, f(\omega)\rangle\langle g(\omega), h\rangle d\mu(\omega) = \langle T_f^*k, T_g^*h\rangle, \quad h \in H,$$

so $Vk = T_gT_f^*k = 0$, then by Theorem 2.1, $k = 0$ and it implies that V is one-to-one. Hence V is invertible.
If g is an orthonormal mapping for H, for each $h \in H$, we have

$$\|h\|^2 = \int_{\Omega} |\langle h, g(\omega)\rangle|^2 d\mu(\omega) = \|V^*h\|^2,$$

therefore $VV^* = I$, and since V is invertible so it is unitary. □

Now, we show that every c-frame can be written as a sum of three orthonormal mappings.

Proposition 4.2. *If g is a c-frame with respect to (Ω, μ) for H and if f is an orthonormal mapping with respect to (Ω, μ) for H, then there exist orthonormal mappings ψ, γ, ϕ and a constant α such that $g = \alpha(\psi + \gamma + \phi)$.*

Proof. By Proposition 4.1, there is a bounded and onto operator $V : H \longrightarrow H$ such that $g = Vf$, and by Proposition 1.3, we have $V = \alpha(U_1 + U_2 + U_3)$, where each U_j, $j = 1, 2, 3$, is a unitary operator and α is a constant. So $g = Vf = \alpha(U_1 f + U_2 f + U_3 f)$. For each $\varphi \in L^2(\Omega, \mu)$, $\nu \in \Omega$ and $j = 1, 2, 3$, we have

$$\int_\Omega \varphi(\omega)\langle U_j f(\omega), U_j f(\nu)\rangle d\mu(\omega) = \int_\Omega \varphi(\omega)\langle f(\omega), f(\nu)\rangle d\mu(\omega).$$

Also

$$\int_\Omega |\langle h, U_j f(\omega)\rangle|^2 d\mu(\omega) = \|U_j^* h\|^2 = \|h\|^2.$$

Thus each $U_j f$, $j = 1, 2, 3$, is an orthonormal mapping. Putting $\psi = U_1 f$, $\gamma = U_2 f$, $\phi = U_3 f$, the proof is complete. □

We can give some conditions that a c-frame can be written as a sum of two orthonormal mappings instead of three. The following proposition states the necessary and sufficient conditions on a c-frame to write it as a linear combination of two orthonormal mappings.

Proposition 4.3. *Let f be an orthonormal mapping and g be a c-frame for H. Then g can be written as a linear combination of two orthonormal mappings for H if and only if g is a c-Riesz basis for H.*

Proof. If g is a c-Riesz basis for H, by Proposition 4.1, there is an invertible operator $V : H \longrightarrow H$ such that $g = Vf$ and by Proposition 1.3, $V = aU_1 + bU_2$, for some constants a and b and unitary operators U_1 and U_2. So $g = Vf = aU_1 f + bU_2 f$. By a similar way in the proof of Proposition 4.2, we can prove that $U_1 f$ and $U_2 f$ are orthonormal mappings.
Conversely, suppose that there are orthonormal mappings ψ and γ and constants a and b such that $g = a\psi + b\gamma$. By Proposition 4.1, there is an onto operator V and there are unitary operators K and R such that $g = Vf$, $\psi = Kf$ and $\gamma = Rf$. Since $g = a\psi + b\gamma$ and f is an orthonormal mapping, we have $V = aK + bR$. So by Proposition 1.3, V is an invertible operator. If $T_g\varphi = 0$, then for each $h \in H$

$$\begin{aligned} 0 =& \langle T_g\varphi, h\rangle = \int_\Omega \varphi(\omega)\langle g(\omega), h\rangle d\mu(\omega) \\ =& \int_\Omega \varphi(\omega)\langle Vf(\omega), h\rangle d\mu(\omega) \\ =& \int_\Omega \varphi(\omega)\langle f(\omega), V^* h\rangle d\mu(\omega) \\ =& \langle VT_f\varphi, h\rangle. \end{aligned}$$

So $VT_f\varphi = 0$ and by Proposition 2.1, $\varphi = 0$. Hence, by Proposition 2.1, g is a c-Riesz basis for H. □

By an orthonormal mapping we can get a Parseval c-frame as follows:

Proposition 4.4. *If V is a co-isometry on H and f is an orthonormal mapping for H, then Vf is a Parseval c-frame for H.*

Proof. By assumption, V^* is an isometry, so for all $h \in H$,

$$\int_\Omega |\langle h, Vf(\omega)\rangle|^2 d\mu(\omega) = \|V^*h\|^2 = \|h\|^2.$$

□

Every bounded operator L on H has a representation in the form $L = UP$ (called the *polar decomposition* of L), where U is a partial isometry, P is a positive operator and $kerL = kerU$. Also, every positive operator P on H with $\|P\| < 1$ can be written in the form $P = \frac{1}{2}(W + W^*)$, where $W = P + i\sqrt{1-P^2}$ is unitary.

Via an orthonormal mapping it is possible to describe a c-frame as a multiple of a sum of two Parseval c-frames.

Theorem 4.1. *If f is an orthonormal mapping for H, then every c-frame g for H is as a linear combination of two Parseval c-frames for H.*

Proof. By Proposition 4.1, there is a bounded and onto operator $V : H \longrightarrow H$ such that $g = Vf$. We have $V = \frac{\|V\|}{2}Z(W + W^*)$, where W is a unitary and Z is a partial isometry, which is a co-isometry. So ZW and ZW^* are co-isometry. We obtain $g = Vf = \frac{\|V\|}{2}(ZW + ZW^*)f$ and by Proposition 4.4, $(ZW)f$ and $(ZW^*)f$ are Parseval c-frames for H. □

Following the above theory, one shows that a c-frame can be represented as a linear combination of an orthonormal mapping and a c-Riesz basis.

Theorem 4.2. *If f is an orthonormal mapping for H, then every c-frame g for H is a sum of an orthonormal mapping for H and a c-Riesz basis for H.*

Proof. If g is a c-frame for H, then by Proposition 4.1, there is a bounded and onto operator $V : H \longrightarrow H$ such that $g = Vf$. For each $0 < \epsilon < 1$, define operator $L : H \longrightarrow H$ by $L = \frac{5}{6}I + \frac{1}{6}(1-\varepsilon)\frac{V}{\|V\|}$. Then $\|I - L\| < 1$, so L is invertible. We now write the polar decomposition of L as $L = UP$, where U is a partial isometry, P is a positive operator and $kerL = kerU$. Since L is invertible, U is a unitary operator (Since a necessary and sufficient condition that U be an isometry is that L be 1-1, and a necessary and sufficient condition that U be a co-isometry is that L has dense range [17]). Also, $\|L\| \leq 1$ implies that $\|P\| \leq 1$, and hence $P = \frac{1}{2}(W + W^*)$, where W is a unitary operator. Now we have,

$$L = \frac{1}{2}(UW + UW^*),$$

where UW, UW^* are unitary. Now, we have

$$V = \frac{6\|V\|}{(1-\epsilon)}[\frac{1}{2}(UW + UW^*) - \frac{5}{6}I] = \frac{3\|V\|}{(1-\epsilon)}[UW + R],$$

where $R = UW^* - \frac{5}{3}I$. Since UW is unitary, UWf is an orthonormal mapping. Since

$$\|I - \frac{-1}{2}R\| = \|\frac{1}{6}I + \frac{1}{2}UW^*\| < 1,$$

thus $\frac{-1}{2}R$ is an invertible operator and consequently R is invertible. Therefore, Rf is a c-frame for H and by Proposition 2.1, Rf is a c-Riesz basis for H. □

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