A NOTE ON BERNSTEIN-TYPE RESULTS OF SPACELIKE HYPERSURFACES IN SEMI-RIEMANNIAN WARPED PRODUCTS

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In this paper, making use of the well known generalized maximum principal according to Omori-Yau [20, 24] *and considering the Laplacian of the integral of the warping function* [4, 22]*, we prove some new Bernstein-type theorems concerning complete spacelike hypersurfaces immersed in semi-Riemannian warped products. Our main theorems extend the corresponding results proved by de Lima* [13] *and Aquino-de Lima* [10].

Keywords: Bernstein-type theorem, complete spacelike hypersurface, semi-Riemannian warped product, mean curvature.

MSC2010: Primary 53C42; Secondary 53B30, 53C50, 53Z05.

1. **Introduction**

Historically, spacelike hypersurfaces immersed in semi-Riemannian manifolds are worthy of investigating both from mathematical and physical viewpoints. One of the basic problems in this framework is the existence and uniqueness of complete spacelike hypersurfaces, such as Calabi-Bernstein type problems. In the last years, spacelike hypersurfaces with geometric conditions which are characterized by the mean curvature or the scalar curvature in semi-Riemannian warped products were investigated by many authors. For example, we may cite the works of Albujer-Alías [1], Albujer-Camargo-de Lima [2, 3], L. J. Alías et al. [5, 6, 7], Aquino-de Lima [8], Caminha-de Lima [11] and de Lima-Parente [16] concerning spacelike hypersurfaces with constant mean curvatures in semi-Riemannian warped products. Recently, spacelike hypersurfaces for which the mean curvature is not necessarily constant in semi-Riemannian warped products were also investigated by many authors. For instance, after some rigidity theorems in steady state spacetimes and hyperbolic spaces were obtained by Colares-de Lima [12] and H. F. de Lima [14] respectively, Aquino-de Lima [9] extended these results with bounded higher order

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mean curvature in steady state-type spacetimes and hyperbolic-type spaces. Moreover, by using the generalized maximum principal according to Omori-Yau [20, 24], H. F. de Lima [13] and Aquino-de Lima [10] proved some new Bernstein-type theorems with bounded mean curvature in generalized Robertson-Walker spacetimes and Riemannian warped products respectively.

The object of the present paper is to investigate complete spacelike hypersurfaces immersed in a semi-Riemannian warped product. In fact, by using the well known generalized maximum principal according to Omori-Yau [20, 24] and considering the Laplacian of the integral of the warping function presented by the authors in [4, 21, 22], we obtain some new Bernstein-type results (see section 4) for spacelike hypersurfaces immersed in Riemannian warped products. With regard to complete spacelike hypersurfaces immersed in Lorentzian warped products, we also obtain some new results which extend the related theorem proved in [13]. We remark that in our assumptions we do not need the condition that the derivative of the warping function is positive, for more details see Remark 4.1.

The present paper is organized as follows. We shall first recall some notations and collect some basic facts on semi-Riemannian warped products in a preliminaries section, then some key lemmas used to prove our main Bernstein-type theorems are given in section 3. Later, section 4 is devoted to giving our main theorems and their proofs.

2. **Preliminaries**

In this section, we recall some basic notations and facts that will appear along this paper. Let M^n be a connected, n-dimensional ($n \geq 2$) oriented Riemannian manifold, $I \subseteq \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a positive smooth function. We consider the product differential manifold $I \times M^n$ and denote by π_I and π_M the projections onto the base I and fiber M^n, respectively. A particular class of semi-Riemannian manifolds is the one obtained by furnishing product manifold $I \times M^n$ with the metric

$$\langle v, w\rangle_p = \epsilon\langle(\pi_I)_*v, (\pi_I)_*w\rangle + (f \circ \pi_I(p))^2\langle(\pi_M)_*v, (\pi_M)_*w\rangle, \tag{1}$$

for any $p \in \overline{M}^{n+1}$ and any $v, w \in T_p\overline{M}^{n+1}$, where $\epsilon = \pm 1$. We call such a space semi-Riemannian warped product manifold, f is known as the warping function and we denote the space by $\overline{M}^{n+1} = \epsilon I \times_f M^n$. Notice that $-I \times_f M^n$ is called a generalized Robertson-Walker spacetime (see [6, 19]), in particular, $-I \times_f M^n$ is called a Robertson-Walker spacetime if M^n has constant sectional curvature. From [7, 19] we know that a generalized Robertson-Walker spacetime has constant sectional curvature $\overline{k}$ if and only if the Riemannian fiber M^n has constant sectional curvature k and the warping function f satisfies the following differential equation

$$\frac{f''}{f} = \overline{k} = \frac{f'^2 + k}{f^2}. \tag{2}$$

It follows from [17, 18] that the vector field $(f \circ \pi_I)\partial_t$ is conformal and closed (in this sense that its dual 1-form is closed) with conformal factor $\phi = f' \circ \pi_I$, where the prime denotes differentiation with respect to $t \in I$. For $t_0 \in I$, we orient the slice $\Sigma^n_{t_0} := \{t_0\} \times M^n$ by using the unit normal vector field ∂_t, then from [17, 18] we know $\Sigma^n_{t_0}$ has constant r-th mean curvature $H_r = -\epsilon \left(\frac{f'(t_0)}{f(t_0)}\right)^r$ with respect to ∂_t for $0 \leq r \leq n$.

A smooth immersion $\psi : \Sigma^n \to \epsilon I \times_f M^n$ of an n-dimensional connected manifold Σ^n is said to be a spacelike hypersurface if the induced metric via ψ is a Riemannian metric on Σ^n. If Σ^n is oriented by the unit vector field N, one obviously has $\epsilon = \epsilon_{\partial_t} = \epsilon_N$.

In this paper, we consider two particular functions naturally attached to complete spacelike hypersurfaces $\psi : \Sigma^n \to \epsilon I \times_f M^n$, namely the vertical (height) function $h = (\pi_I)|_{\Sigma^n}$ and the support function $\langle N, \partial_t \rangle$. Denoting by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics of $\epsilon I \times_f M^n$ and Σ^n respectively. Thus, by a simple computation we have the gradient of π_I on $\epsilon I \times_f M^n$ given as following

$$\overline{\nabla}\pi_I = \epsilon \langle \overline{\nabla}\pi_I, \partial_t \rangle \partial_t = \epsilon \partial_t. \tag{3}$$

Then the gradient of h on Σ^n is given by

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \epsilon(\partial_t)^\top = \epsilon\partial_t - \langle N, \partial_t \rangle N, \tag{4}$$

where $\partial_t^\top$ denotes the tangent component of ∂_t on Σ^n. We denote by $|\cdot|$ the norm of a vector field on Σ^n, then we get from (2.4) that

$$|\nabla h|^2 = \epsilon \left(1 - \langle N, \partial_t \rangle^2\right). \tag{5}$$

3. **Key Lemmas**

According to [1, 22], a spacelike hypersurface $\psi : \Sigma^n \to \epsilon I \times_f M^n$ is said to be bounded away from the future infinity of $\epsilon I \times_f M^n$ if there exists $\bar{t} \in I$ such that

$$\psi(\Sigma^n) \subset \left\{(t, p) \in \epsilon I \times_f M^n : t \leq \bar{t}\right\}.$$

Analogously, a spacelike hypersurface $\psi : \Sigma^n \to \epsilon I \times_f M^n$ is said to be bounded away from the past infinity of $\epsilon I \times_f M^n$ if there exists $\underline{t} \in I$ such that

$$\psi(\Sigma^n) \subset \{(t, p) \in \epsilon I \times_f M^n : t \geq \underline{t}\}.$$

Finally, Σ^n is said to be bounded away from the infinity of $\epsilon I \times_f M^n$ if it is both bounded away from the past and future infinity of $\epsilon I \times_f M^n$.

Setting $k = 0$ in Lemma 4.1 of [4], we may obtain the Laplacian of the integral of the warping function in a generalized Robertson-Walker spacetime. By using the technique due to L. J. Alías and A. G. Colares [4], the present authors in [22] generalize this result in a semi-Riemannian warped product as follows.

Lemma 3.1 (see [4, 22]). *Let $\psi : \Sigma^n \to \epsilon I \times_f M^n$ be a spacelike hypersurface immersed a semi-Riemannian warped product. If*

$$\sigma(t) = \int_{t_0}^{t} f(s)ds, \tag{6}$$

then

$$\Delta\sigma(h) = \epsilon n \left(f'(h) + f(h)\langle N, \partial_t\rangle H\right), \tag{7}$$

where Δ denotes the Laplacian operator and h is the height function of Σ^n.

The above lemma has been used to obtain some new Bernstein-type theorems in semi-Riemannian warped products by the authors [21, 22]. The following theorem provides a sufficient condition for the Ricci curvature of a spacelike hypersurface in Riemannian warped product to be bounded from below.

Lemma 3.2 (Proposition 3.1 of [8]). *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product which satisfies convergence condition $k \geq \sup_I(f'^2 - ff'')$, where k denotes the sectional curvature of the fiber M^n. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface with both mean curvature H and second fundamental form A bounded. If $\frac{f''}{f}$ is bounded on Σ^n, then the Ricci curvature of Σ^n is bounded from below.*

With regard to the boundness of the Ricci curvature of a spacelike hypersurface in Lorentzian warped products, we shall make use of the following result proved by H. F. de Lima and U. L. Parente in [15].

Lemma 3.3 (Proposition 3.2 of [15]). *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a generalized Robertson-Walker spacetime whose sectional curvature of the fiber satisfies $k \geq \sup_I(ff'' - f'^2)$, where k denotes the sectional curvature of the fiber M^n. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface with bounded mean curvature H. If $\frac{f''}{f}$ is bounded from below on Σ^n, then the Ricci curvature of Σ^n is bounded from below.*

Furthermore, we also need the following well known generalized maximum principal due to Omori-Yau [20, 24].

Lemma 3.4 (see [20, 24]). *Let Σ^n denote an n-dimensional complete Riemannian manifold whose Ricci curvature tensor is bounded from below. Then, for any $\mathcal{C}^2$-function $u : \Sigma^n \to \mathbb{R}$ with $u^* = \sup_{\Sigma^n} u < \infty$, there exists a sequence of points $\{p_k\}_{k\in N}$ in Σ^n satisfying the following properties:*

$$\text{(i)} \lim_{k\to\infty} u(p_k) = \sup_{\Sigma^n} u, \quad \text{(ii)} \lim_{k\to\infty} |\nabla u|(p_k) = 0, \quad \text{(iii)} \lim_{k\to\infty} \Delta u(p_k) \leq 0. \tag{8}$$

Equivalently, for any $\mathcal{C}^2$-function $u : \Sigma^n \to \mathbb{R}$ with $u_* = \inf_{\Sigma^n} u > -\infty$, there exists a sequence of points $\{q_k\}_{k\in N}$ in Σ^n satisfying the following properties:

$$\text{(i)} \lim_{k\to\infty} u(q_k) = \inf_{\Sigma^n} u, \quad \text{(ii)} \lim_{k\to\infty} |\nabla u|(q_k) = 0, \quad \text{(iii)} \lim_{k\to\infty} \Delta u(q_k) \geq 0. \tag{9}$$

4. **Main results**

Making use of some lemmas shown in section 3, we may present our main results and their proofs as follows.

Theorem 4.1. *Let* $\overline{M}^{n+1} = I \times_f M^n$ *be a Riemannian warped product whose fiber has constant sectional curvature* k *satisfying*

$$k \geq \sup_I (f'^2 - ff''). \tag{10}$$

Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete spacelike hypersurface bounded away from the infinity of* $\overline{M}^{n+1}$*. Suppose that the mean curvature* H *of* Σ^n *satisfies*

$$H \geq \frac{f'}{f}(h). \tag{11}$$

If the second fundamental form of Σ^n *is bounded and*

$$|\nabla h| \leq \inf_{\Sigma^n} \left(H - \frac{f'}{f}(h) \right), \tag{12}$$

then Σ^n *is slice of* $I \times_f M^n$*.*

Proof. Throughout this section, for a spacelike hypersurface Σ^n immersed in $I \times_f M^n$, we may consider N being the orientation of the hypersurface Σ^n such that its angle function satisfies $\langle N, \partial_t \rangle < 0$. Then, by applying the Cauchy-Schwarz inequality, we obtain

$$-1 \leq \langle N, \partial_t \rangle < 0. \tag{13}$$

Setting $\epsilon = 1$ then it follows from Lemma 3.1 that

$$\Delta \sigma(h) = nf(h) \left(\frac{f'}{f}(h) + \langle N, \partial_t \rangle H \right). \tag{14}$$

It is easy to verify the following identity (see also equation (23) of [22]):

$$\Delta \sigma(h) = f'(h)|\nabla h|^2 + f(h)\Delta h. \tag{15}$$

Following [3, 8], let S_2 denote the second elementary symmetric function on the eigenvalues of shape operator A of spacelike hypersurface Σ^n, and $H_2 = \frac{2}{n(n-1)} S_2$ denote the mean value of S_2. Elementary algebra implies that

$$|A|^2 = nH^2 + n(n-1)(H^2 - H_2). \tag{16}$$

By using the Cauchy-Schwarz inequality, we obtain from (4.7) that

$$|A|^2 \geq nH^2. \tag{17}$$

Taking into account (4.8) and the assumption that the second fundamental form A is bounded on Σ^n, we see that the mean curvature H is also bounded on Σ^n. Notice that Σ^n is bounded away from the infinity of $I \times_f M^n$ and (4.1) holds, then it follows from Lemma 3.2 that the Ricci curvature of Σ^n is bounded from below. Making use again of the hypothesis that Σ^n is bounded away from the infinity of $I \times_f M^n$, we know that the height function h is bounded on Σ^n. Thus, applying Lemma 3.4 on smooth function h on spacelike hypersurface Σ^n, there exists a sequence of points $\{q_k\}_{k\in N}$ in Σ^n satisfying the following properties:

$$\text{(i)} \lim_{k\to\infty} h(q_k) = \inf_{\Sigma^n} h, \quad \text{(ii)} \lim_{k\to\infty} |\nabla h|(q_k) = 0, \quad \text{(iii)} \lim_{k\to\infty} \Delta h(q_k) \geq 0. \tag{18}$$

Notice that the warping function f is positive on I, then by using the second and third terms of (4.9) in (4.6) we obtain that

$$\lim_{k\to\infty} \Delta\sigma(h)(q_k) \geq 0. \tag{19}$$

On the other hand, using the second term of (4.9) in (2.5) (in this context $\epsilon = 1$) and applying (4.4) we obtain

$$\lim_{k\to\infty} \langle N, \partial_t\rangle(q_k) = -1. \tag{20}$$

Taking into account the first term of (4.9), (4.10), (4.11) and (4.2) in (4.5), then we obtain the following inequality

$$0 \leq \lim_{k\to\infty} \Delta\sigma(h)(q_k) = nf(\inf_{\Sigma^n} h) \lim_{k\to\infty} \left(\frac{f'}{f}(h) - H\right)(q_k) \leq 0. \tag{21}$$

Since the warping function f is positive on I, it is easy to obtain from the above inequality that

$$\lim_{k\to\infty} \left(\frac{f'}{f}(h) - H\right)(q_k) = 0. \tag{22}$$

Finally, it follows from inequality (4.2) and equation (4.13) that

$$\inf_{\Sigma^n} \left(H - \frac{f'}{f}(h)\right) = 0. \tag{23}$$

Putting (4.14) into (4.3) yields that $|\nabla h| \equiv 0$, i.e., h is a constant on Σ^n. Thus, we prove that Σ^n is a slice of $I \times_f M^n$. □

Theorem 4.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Robertson-Walker spaetime whose fiber has constant sectional curvature k satisfying*

$$k \geq \sup_I (ff'' - f'^2). \tag{24}$$

Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface bounded away from the infinity of $\overline{M}^{n+1}$. Suppose that the mean curvature H of Σ^n satisfies

$$H \leq \frac{f'}{f}(h). \tag{25}$$

If the second fundamental form of Σ^n is bounded and

$$|\nabla h| \leq \inf_{\Sigma^n} \left(\frac{f'}{f}(h) - H\right), \tag{26}$$

then Σ^n is slice of $-I \times_f M^n$.

Proof. In this case, for a spacelike hypersurface Σ^n immersed in $-I \times_f M^n$, we may consider N being the orientation of the hypersurface Σ^n such that its angle function satisfies $\langle N, \partial_t \rangle < 0$. By using the reverse Cauchy-Schwarz inequality we have

$$\langle N, \partial_t \rangle|_{\Sigma^n} \leq -1. \tag{27}$$

Now letting $\epsilon = -1$ then it follows from Lemma 3.1 that

$$\Delta\sigma(h) = -nf(h)\left(\frac{f'}{f}(h) + \langle N, \partial_t \rangle H\right). \tag{28}$$

Notice that relations (4.6), (4.7) and (4.8) still hold in this context. Since the second fundamental form A is bounded on Σ^n, we see from (4.8) that the mean curvature H is also bounded on Σ^n. As Σ^n is bounded away from the infinity of $-I \times_f M^n$ and (4.15) holds, then it follows from Lemma 3.3 that the Ricci curvature of Σ^n is bounded from below. Moreover, by the assumption that Σ^n is bounded away from the infinity of $-I \times_f M^n$, we see that the height function h is bounded on Σ^n. Thus, applying Lemma 3.4 on smooth function h on hypersurface Σ^n yields that there exists a sequence of points $\{q_k\}_{k \in N}$ in Σ^n satisfying (4.9). As the warping function f is positive, then putting the second and third terms of (4.9) into (4.6) we obtain inequality (4.10). Moreover, using the second term of (4.9) in (2.5) (in this context $\epsilon = -1$) and (4.18) we get equation (4.11).

Taking into account the first term of (4.9), (4.10), (4.11) and (4.16) in (4.19) then we obtain the following inequality

$$0 \leq \lim_{k \to \infty} \Delta\sigma(h)(q_k) = -nf(\inf_{\Sigma^n} h) \lim_{k \to \infty} \left(\frac{f'}{f}(h) - H\right)(q_k) \leq 0. \tag{29}$$

Since the warping function f is positive, it is easy to obtain from the above inequality that (4.13) holds. Finally, it follows from inequality (4.16) and equation (4.13) that

$$\inf_{\Sigma^n}\left(\frac{f'}{f}(h) - H\right) = 0. \tag{30}$$

Putting (4.21) into (4.17) yields that $|\nabla h| \equiv 0$, i.e., h is a constant on Σ^n. Thus, we prove that Σ^n is a slice of $-I \times_f M^n$. □

Following the above arguments we now present another characterization for a complete spacelike hypersurface in $I \times_f M^n$ to be a slice.

Theorem 4.3. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product whose fiber has constant sectional curvature k satisfying* (4.1)*. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface bounded away from the infinity of $\overline{M}^{n+1}$. Suppose that the mean curvature H of Σ^n satisfies* (4.16)*. If the second fundamental form of Σ^n is bounded and inequality* (4.17) *holds, then Σ^n is slice of $I \times_f M^n$.*

Proof. As shown in the proof of Theorem 4.1, in this case, the Ricci curvature of Σ^n is bounded from below. Using the assumption that Σ^n is bounded away from the infinity of $I \times_f M^n$, we know that the height function h is bounded on Σ^n. Thus, applying Lemma 3.4 on smooth function h on hypersurface Σ^n, there exists a sequence of points $\{p_k\}_{k\in N}$ in Σ^n satisfying the following properties:

$$\text{(i)} \lim_{k\to\infty} h(p_k) = \sup_{\Sigma^n} h, \quad \text{(ii)} \lim_{k\to\infty} |\nabla h|(p_k) = 0, \quad \text{(iii)} \lim_{k\to\infty} \Delta h(p_k) \leq 0. \tag{31}$$

Putting the second and third terms of (4.22) into relation (4.6) we obtain

$$\lim_{k\to\infty} \Delta\sigma(h)(p_k) \leq 0. \tag{32}$$

From (2.5), (4.18) and the second term of (4.22), it is easy to see that $\lim_{k\to\infty} \langle N, \partial_t\rangle(p_k) = -1$ holds in this context. Taking into account this relation, (4.23) and the first term of (4.22) in (4.5), then we obtain the following inequality

$$0 \geq \lim_{k\to\infty} \Delta\sigma(h)(p_k) = nf(\sup_{\Sigma^n} h) \lim_{k\to\infty} \left(\frac{f'}{f}(h) - H\right)(p_k) \geq 0. \tag{33}$$

As the warping function f is positive on I, it is easy to see from the above inequality that

$$\lim_{k\to\infty} \left(\frac{f'}{f}(h) - H\right)(p_k) = 0.$$

Finally, it follows from inequality (4.16) and the above equation that (4.21) is true.

Putting (4.21) into (4.17) we may obtain $|\nabla h| \equiv 0$, i.e., h is a constant on Σ^n. Thus, we prove that Σ^n is a slice of $I \times_f M^n$. □

Similarly, we now give another Bernstein-type result concerning complete spacelike hypersurfaces immersed in generalized Robertson-Walker spacetimes, omitting its proof since it is similar to that of Theorem 4.2 and 4.3.

Theorem 4.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Robertson-Walker spacetime whose fiber has constant sectional curvature k satisfying* (4.15)*. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface bounded away from the infinity of $\overline{M}^{n+1}$. Suppose that the mean curvature H of Σ^n satisfies* (4.2)*. If the second fundamental form of Σ^n is bounded and inequality* (4.3) *holds, then Σ^n is slice of $-I \times_f M^n$.*

Remark 4.1. *We observe that the proofs of Theorem* 1.1 *of* [13]*, Theorem* 4.2 *and* 4.3 *of* [10] *depend on the following formula*

$$\Delta f(h) = f''(h)|\nabla h|^2 + f'(h)\Delta h$$

for complete spacelike hypersurfaces in semi-Riemannian warped products. To assure the Laplacian of the warping function being nonnegative, these results mentioned above require the condition that the derivative of the warping function is positive. However, in assumptions of our main results, such a condition is unnecessary. Furthermore, we refer the reader to [17, 18] *for some examples of semi-Riemannian warped products whose derivatives of the warping functions are not necessarily to be positive.*

Acknowledgement

This work was supported by the National Natural Science Foundation of China (No. 11371076) and the Fundamental Research Funds for the Central Universities (No. DUT14ZD208).

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