U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 4, 2020 ISSN 1223-7027

$\phi-$BIFLATNESS AND $\phi-$BIPROJECTIVITY FOR θ-LAU PRODUCT WITH APPLICATIONS

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For two Banach algebras A and B and a non-zero multiplicative linear functional θ on B, Monfared introduced the $\theta-$Lau product structure $A \times_\theta B$. In this paper, we investigate and study the notions of $\phi-$biprojectivity, $\phi-$biflatness and $\phi-$Johnson amenability of $A \times_\theta B$ and their relation with A and B. As an application, we characterize ϕ-biflatness, ϕ-biprojectivity and ϕ-Johnson amenability for $\theta-$Lau product of Banach algebras related to locally compact groups and discrete semigroups.

Keywords: Banach algebras, ϕ-biflatness, ϕ-biprojectivity, ϕ-Johnson amenability, left ϕ-amenability, θ-Lau product.

MSC2020: Primary 43A07, 46M10; Secondary 43A20, 46H05.

1. Introduction and preliminaries

Johnson studied amenable Banach algebras using virtual diagonals [7]. That is an element $M \in (A\hat{\otimes}A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi^{**}(M)a = a$ for each $a \in A$, where π is the product morphism given by $\pi(a \otimes b) = ab$ for each $a, b \in A$, see [14].

Helemskii studied the structure of Banach algebras through the notions of biflatness and biprojectivity. In fact a Banach algebra is biflat (biprojective) if there exists a bounded A-bimodule morphism $\rho : A \to (A\hat{\otimes}A)^{**}(\rho : A \to A\hat{\otimes}A)$ such that $\pi^{**} \circ \rho(a) = a(\pi \circ \rho(a) = a)$, for all $a \in A$, respectively. It is well-known that a Banach algebra A is amenable if and only if A is biflat and A has a bounded approximate identity, see [14].

Recently some notions of amenability related to a multiplicative linear functional have introduced and studied for Banach algebras. The notions like left ϕ-amenability, left ϕ-contractibility, ϕ-biflatness and ϕ-biprojectivity studied for the group algebras, the measure algebras and the Fourier algebras, for more information about these notions see [1], [8], [12] and [17].

For an arbitrary Banach algebra A, the character space is denoted by $\sigma(A)$ consists of all non-zero multiplicative linear functionals on A and any element of $\sigma(A)$ is called a character. The $\theta-$Lau product was first introduced by Lau [9] for F-algebras. Monfared [11] introduced and investigated θ-Lau product space $A \times_\theta B$, for Banach algebras in general. Indeed for two Banach algebras A and B such that $\sigma(B) \neq \emptyset$ and θ be a non-zero character on B, the Cartesian product $A \times B$ by following multiplication and norm

$$(a, b)(a', b') = (aa' + \theta(b')a + \theta(b)a', bb'),$$
$$\|(a, b)\| = \|a\|_A + \|b\|_B,$$

is a Banach algebra, for all $a, a' \in A$ and $b, b' \in B$. The Cartesian product $A \times B$ with the above properties called the $\theta-$Lau product of A and B which is denoted by $A \times_\theta B$. From

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[11] we identify $A \times \{0\}$ with A, and $\{0\} \times B$ with B. Thus, it is clear that A is a closed two-sided ideal while B is a closed subalgebra of $A \times_\theta B$, and $(A \times_\theta B)/A$ is isometrically isomorphic to B. If $\theta = 0$, then we obtain the usual direct product of A and B. Since direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $B = \mathbb{C}$, the complex numbers, and θ is the identity map on $\mathbb{C}$, then $A \times_\theta B$ is the unitization $A^\sharp$ of A. Note that, by [11, Proposition 2.4], the character space $\sigma(A \times_\theta B)$ of $A \times_\theta B$ is equal to

$$\{(\phi, \theta) : \ \phi \in \sigma(A)\} \bigcup \{(0, \psi) : \ \psi \in \sigma(B)\}.$$

Also, the dual space $(A \times_\theta B)^*$ of $A \times_\theta B$ is identified with $A^* \times B^*$ such that for each $(a, b) \in A \times_\theta B$, $\phi \in \sigma(A)$ and $\psi \in \sigma(B)$ we have

$$\langle (\phi, \psi), (a, b) \rangle = \phi(a) + \psi(b).$$

Now, suppose that A^{**}, B^{**} and $(A \times_\theta B)^{**}$ are equipped with their first Arens products. Then $(A \times_\theta B)^{**}$ is isometrically isomorphic with $A^{**} \times_\theta B^{**}$. Also, for all $(m, n), (p, q) \in (A \times_\theta B)^{**}$ the first Arens product is defined by

$$(m, n)\Box(p, q) = (m\Box p + n(\theta)p + q(\theta)m, n\Box q);$$

see [11, Proposition 2.12]. Note that every $\phi \in \sigma(A)$ has a unique extension to a character on A^{**} is given by $\tilde{\phi}$ where $\tilde{\phi}(m) = m(\phi)$, for all $m \in A^{**}$.

Note that A and B are closed two-sided ideal and closed subalgebra of $L := A \times_\theta B$, respectively. So, we can write $a = (a, 0)$ and $b = (0, b)$ for all $a \in A$ and $b \in B$. Therefore, $L = A \times_\theta B$ is a Banach $A-$bimodule and also is a Banach $B-$bimodule. It has worth to mention that some generalizations of twisted product related to a homomorphism are given recently but by [3] it seems those products are trivial.

The contents of the paper is as follows, in section 2 we study $\phi-$biflatness and $\phi-$biprojectivity of θ-Lau product of Banach algebras. Then we turn our attention to the ϕ-Johnson amenability of θ-Lau product of Banach algebras in section 3. As a conclusion, we characterize $\phi-$biflatness, $\phi-$biprojectivity and ϕ-Johnson amenability of θ-Lau product of Banach algebras related to discrete semigroups or locally compact groups.

2. $\phi-$biflatness and $\phi-$biprojectivity

The usual projections $p_A : L \longrightarrow A$ and $p_B : L \longrightarrow B$ defined by $p_A(a, b) = a$ and $p_B(a, b) = b$. Also, let $q_A : A \longrightarrow L$ and $q_B : B \longrightarrow L$ be the usual injections via $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$. Hence, the mappings q_A and p_B induce the mappings

$$q_A \otimes q_A : A \widehat{\otimes} A \longrightarrow L \widehat{\otimes} L$$

and

$$p_B \otimes p_B : L \widehat{\otimes} L \longrightarrow B \widehat{\otimes} B$$

with

$$(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0)$$

and

$$(p_B \otimes p_B)((a, b) \otimes (c, d)) = b \otimes d,$$

respectively. Clearly, q_A and $q_A \otimes q_A$ are A-bimodule maps and p_B, q_B and $p_B \otimes p_B$ are B-bimodule maps.

Now, suppose that A is unital with unit e. Then define mappings $r_A : L \longrightarrow A$ and $S_B : B \longrightarrow L$ via $r_A(a, b) = a + \theta(b)e$ and $S_B(b) = (-\theta(b)e, b)$, respectively. Also, these maps induce the unique mappings

$$r_A \otimes r_A : L \widehat{\otimes} L \longrightarrow A \widehat{\otimes} A$$

and

$$S_B \otimes S_B : B\widehat{\otimes}B \longrightarrow L\widehat{\otimes}L$$

satisfying

$$(r_A \otimes r_A)((a,b)\otimes(c,d)) = (a+\theta(b)e)\otimes(c+\theta(d)e)$$

and

$$(S_B \otimes S_B)(b\otimes d) = (-\theta(b)e, b)\otimes(-\theta(d)e, d),$$

respectively. It is clear that r_A and $r_A \otimes r_A$ are $A-$bimodule maps and S_B is a $B-$bimodule map. (For more details on the above mappings refer to [4]).

The notion of $\phi-$biprojectivity for Banach algebras first introduced by Sahami and Pourabbas [17]. For a nonzero multiplicative linear functional ϕ on A, the Banach algebras A is called $\phi-$biprojective if there exists a bounded $A-$bimodule morphism $\lambda_A : A \longrightarrow A\widehat{\otimes}A$ such that $\phi\circ\pi_A\circ\lambda_A = \phi$.

Proposition 2.1. *Suppose that A and B are two Banach algebras which A has unit e, $\phi\in\sigma(A)$ and $\theta\in\sigma(B)$. If L is $(\phi,\theta)-$biprojective. Then A is $\phi-$biprojective.*

Proof. Let L be $(\phi,\theta)-$biprojective. Then there exists the $L-$bimodule morphism λ_L : $L\longrightarrow L\widehat{\otimes}L$ such that $(\phi,\theta)\circ\pi_L\circ\lambda_L = (\phi,\theta)$. It is clear that

$$r_A\circ\pi_L = \pi_A\circ(r_A\otimes r_A),\ \phi\circ r_A = (\phi,\theta).$$

Define $\lambda_A : A\longrightarrow A\widehat{\otimes}A$ by $\lambda_A = (r_A\otimes r_A)\circ\lambda_L\circ q_A$. Clearly, λ_A is a bounded $A-$bimodule morphism. Also, we have

$$\begin{aligned}(\phi\circ\pi_A\circ\lambda_A)(a) &= (\phi\circ\pi_A\circ(r_A\otimes r_A)\circ\lambda_L\circ q_A)(a)\\ &= (\phi\circ r_A\circ\pi_L\circ\lambda_L)(a,0)\\ &= ((\phi,\theta)\circ\pi_L\circ\lambda_L)(a,0)\\ &= (\phi,\theta)(a,0)\\ &= \phi(a),\end{aligned}$$

for all $a\in A$. So $\phi\circ\pi_A\circ\lambda_A = \phi$. Thus A is ϕ-biprojective. □

Proposition 2.2. *Suppose that A and B are two Banach algebras which A has unit e and $\psi\in\sigma(B)$. Then L is $(0,\psi)-$biprojective if and only if B is $\psi-$biprojective.*

Proof. Suppose that there exists the $L-$bimodule morphism $\lambda_L : L\longrightarrow L\widehat{\otimes}L$ such that $(0,\psi)\circ\pi_L\circ\lambda_L = (0,\psi)$. Now define the map $\lambda_B : B\longrightarrow B\widehat{\otimes}B$ by $\lambda_B = (p_B\otimes p_B)\circ\lambda_L\circ q_B$. It is easy to see that

$$\pi_B\circ(p_B\otimes p_B) = p_B\circ\pi_L,\ \psi\circ p_B = (0,\psi).$$

Thus we have λ_B is $B-$bimodule map and $\psi\circ\pi_B\circ\lambda_B = \psi$. To see this consider

$$\begin{aligned}\Big(\psi\circ\pi_B\circ\lambda_B\Big)(b) &= \Big(\psi\circ\pi_B\circ(p_B\otimes p_B)\lambda_L\circ q_B\Big)(b)\\ &= \Big(\psi\circ p_B\circ\pi_L\circ\lambda_L\Big)(0,b)\\ &= \Big((0,\psi)\circ\pi_L\circ\lambda_L\Big)(0,b)\\ &= \psi(b),\end{aligned}$$

for all $b \in B$. Moreover,

$$\begin{aligned}\lambda_B(by) &= (p_{_B} \otimes p_{_B}) \circ \lambda_L \circ q_{_B}(by)\\ &= (p_{_B} \otimes p_{_B}) \circ \lambda_L \circ (q_{_B}(b) \cdot y)\\ &= (p_{_B} \otimes p_{_B})(\lambda_L \circ q_{_B}(b) \cdot y)\\ &= ((p_{_B} \otimes p_{_B}) \circ \lambda_L \circ q_{_B}(b)) \cdot y\\ &= \lambda_B(b) \cdot y,\end{aligned}$$

for all $b, y \in B$. Similarly, $\lambda_B(yb) = y \cdot \lambda_B(b)$ for all $b, y \in B$.

For converse, suppose that B is $\psi-$biprojective. Then there exists a bounded B-bimodule morphism $\lambda_B : B \longrightarrow B\widehat{\otimes}B$ such that $\psi \circ \pi_B \circ \lambda_B = \psi$. Define the map $\lambda_L : L \longrightarrow L\widehat{\otimes}L$ by

$$\lambda_L(a, b) := (S_{_B} \otimes S_{_B}) \circ \lambda_B(b),$$

for all $a \in A$ and $b \in B$. It is easy to see that

$$\pi_L \circ (S_{_B} \otimes S_{_B}) = S_{_B} \circ \pi_B, \quad (0, \psi) \circ S_{_B} = \psi, \quad ((S_{_B} \otimes S_{_B}) \circ \lambda_B(b)) \cdot x = 0,$$

for all $b \in B$ and $x \in A$. So, these relations conclude that λ_L is a $L-$bimodule morphism and L is $(0, \psi)-$biprojective. Therefore,

$$(0, \psi) \circ \pi_L \circ \lambda_L = (0, \psi).$$

□

Remark 2.1. *Note that $(\phi, \theta)-$biprojectivity of L implies that B is $\theta-$biprojective. To see this, we know that there exists a $L-$bimodule map $\lambda_L : L \longrightarrow L\widehat{\otimes}L$ such that*

$$(\phi, \theta) \circ \pi_L \circ \lambda_L = (\phi, \theta).$$

Hence, it is clear that

$$p_{_B} \circ \pi_L = \pi_B \circ (p_{_B} \otimes p_{_B}), \quad r_A \circ \pi_L = \pi_A \circ (r_A \otimes r_A), \quad \theta \circ p_{_B} = (0, \theta).$$

Define $\lambda_B : B \longrightarrow B\widehat{\otimes}B$ by $\lambda_B := (p_{_B} \otimes p_{_B}) \circ \lambda_L \circ q_{_B}$. Since $\Big((\phi, 0) \circ \pi_L \circ \lambda_L\Big)(0, b) = 0$, we have that

$$\begin{aligned}\Big(\theta \circ \pi_B \circ \lambda_B\Big)(b) &= \langle(\phi, \theta), (0, b)\rangle - \Big((\phi, 0) \circ \pi_L \circ \lambda_L\Big)(0, b)\\ &= \theta(b),\end{aligned}$$

for all $b \in B$. Thus, B is $\theta-$biprojective. Moreover, if e is unit for A, then

$$\Big(\theta \circ p_{_B} \circ \pi_L \circ \lambda_L\Big)(e, 0) = 1.$$

So, we can define λ_B as following

$$\lambda_B(b) := b \cdot ((p_{_B} \otimes p_{_B}) \circ \lambda_L(e, 0)),$$

for all $b \in B$. It implies that B is $\theta-$biprojective.

Recently, in [17], Sahami and Pourabbas introduced and studied the new concept of $\phi-$biflatness for Banach algebras. In fact a Banach algebra A is called $\phi-$biflat if there exists a bounded $A-$bimodule morphism $\lambda_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \lambda_A = \phi$, where $\tilde{\phi}(F) = F(\phi)$ for all $F \in A^{**}$.

Proposition 2.3. *Suppose that A and B are two Banach algebras. Let $\theta \in \sigma(B)$ and $\phi \in \sigma(A)$. If L is $(\phi, \theta)-$biflat, then A is $\phi-$biflat.*

Proof. By hypothesis there exists the bounded $L-$bimodule morphism $\lambda_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ such that $\widetilde{(\phi,\theta)} \circ \pi_L^{**} \circ \lambda_L = (\phi,\theta)$. Define a bounded $A-$bimodule morphism $\lambda_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ by $\lambda_A := (r_A \otimes r_A)^{**} \circ \lambda_L \circ q_A$. It is clear that

$$(r_A \otimes r_A)^*(\phi \circ \pi_A) = (\phi,\theta) \circ \pi_L.$$

Therefore it concludes that

$$\begin{aligned} \langle \tilde{\phi} \circ {\pi_A}^{**} \circ \lambda_A, a \rangle &= \langle \lambda_A(a), {\pi_A}^*(\phi) \rangle \\ &= \langle \lambda_L(a,0), (r_A \otimes r_A)^*(\phi \circ \pi_A) \rangle \\ &= \phi(a), \end{aligned}$$

for all $a \in A$. Thus, A is $\phi-$biflat. □

Proposition 2.4. *Suppose that A and B are two Banach algebras which A is unital and $\psi,\theta \in \sigma(B)$. Then L is $(0,\psi)-$biflat if and only if B is $\psi-$biflat.*

Proof. First suppose that L is $(0,\psi)-$biflat. Then there exists a bounded $L-$bimodule morphism $\lambda_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ such that $\widetilde{(0,\psi)} \circ \pi_L^{**} \circ \lambda_L = (0,\psi)$. But, we know that $\pi_B^*(\psi) = \psi \circ \pi_B$. Now, define $\lambda_B : B \longrightarrow (B\widehat{\otimes}B)^{**}$ by

$$\lambda_B := (p_B \otimes p_B)^{**} \circ \lambda_L \circ q_B.$$

It is easy to see that ${\pi_L}^*((0,\psi)) = (p_B \otimes p_B)^*(\psi \circ \pi_B)$. So, we obtain

$$\begin{aligned} \langle \tilde{\psi} \circ {\pi_B}^{**} \circ \lambda_B, b \rangle &= \langle {\pi_B}^{**} \circ \lambda_B(b), \psi \rangle \\ &= \langle \lambda_B(b), \psi \circ \pi_B \rangle \\ &= \langle \lambda_L((0,b)), (p_B \otimes p_B)^*(\psi \circ \pi_B) \rangle \\ &= \psi(b), \end{aligned}$$

for all $b \in B$. To prove the only if part, suppose that B is $\psi-$biflat. Then there exists the bounded $B-$bimodule morphism $\lambda_B : B \longrightarrow (B\widehat{\otimes}B)^{**}$ such that $\tilde{\psi} \circ \pi_B^{**} \circ \lambda_B = \psi$. By an easy calculation, we have

$$(S_B \otimes S_B)^*((0,\psi) \circ \pi_L) = {\pi_B}^*(\psi).$$

Define the map $\lambda_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ via

$$\lambda_L := (S_B \otimes S_B)^{**} \circ \lambda_B \circ p_B.$$

Hence, it is easy to see that λ_L is a bounded $L-$bimodule morphism and $\widetilde{(0,\psi)} \circ {\pi_L}^{**} \circ \lambda_L = \psi$. It follows that L is $(0,\psi)-$biflat. □

Note that in the proof of Proposition 2.4 (if part), if we define λ_B as $\lambda_B = (p_B \otimes p_B)^{**} \circ \lambda_L \circ S_B$, then we can see that B is $\psi-$biflat.

3. $\phi-$Johnson amenability

The notion of $\phi-$Johnson amenability for Banach algebras is defined by Sahami and Pourabbas, see [17]. A Banach algebra A is called ϕ-Johnson amenable (ϕ-Johnson contractible), if there exists an element $m \in (A\widehat{\otimes}A)^{**}$ ($m \in A\widehat{\otimes}A$) such that $a \cdot m = m \cdot a$ and $\tilde{\phi} \circ \pi^{**}(m) = 1$ ($\phi \circ \pi(m) = 1$) for all $a \in A$, respectively. By [17, Lemma 2.1], the Banach algebra A is $\phi-$Johnson amenable if and only if there exists a bounded net $(m_\alpha) \in A\widehat{\otimes}A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \longrightarrow 0$ and $\phi \circ \pi(m_\alpha) \longrightarrow 1$, for all $a \in A$. In this section, we consider hereditary properties of $\phi-$Johnson amenability for Banach algebras. Next, we turn our attention to the structure of $\sigma(A \times_\theta B)$. We study $\phi-$Johnson amenability for $A \times_\theta B$ and obtain it's relationship with $\phi-$Johnson amenability of A and B.

Proposition 3.1. *Let A and B be two Banach algebras, $\psi \in \sigma(B)$ and $T : A \longrightarrow B$ be a continuous homomorphism. Then we have the following statements.*
a) If T has dense range, $0 \in Im(T)$ and A is $(\psi \circ T)-$Johnson amenable, then B is $\psi-$Johnson amenable.
b) If T is a bijective mapping and B is $\psi-$Johnson amenable, then A is $(\psi \circ T)-$Johnson amenable.

Proof. The proof is clear by [17, Lemma 2.1] and [8, Proposition 3.5]. □

Recall that if I is a closed two-sided ideal of Banach algebra A and $\phi \in \sigma(A)$, then $\phi_{|_I} \in \sigma(I)$. Also, if $\varphi \in \sigma(I)$, then it has an extension $\overline{\varphi} \in \sigma(A)$. Moreover, if $\phi \in \sigma(A)$, then $\widehat{\phi} : \frac{A}{I} \longrightarrow \mathbb{C}$ defined by $\widehat{\phi}(a+I) = \phi(a)$ is a character on $\frac{A}{I}$. Hence, by the Proposition 3.1, $\widehat{\phi}-$Johnson amenability of A/I implies the $\phi-$Johnson amenability of A.

Corollary 3.1. *Suppose that I is a closed two-sided ideal in Banach algebra A and $\phi \in \sigma(A)$. Then $\phi-$Johnson amenability of A implies the $\widehat{\phi}-$Johnson amenability of A/I.*

We know that the quotient map $q : A\times_\theta B \longrightarrow (A \times_\theta B)/A$ is continuous epimorphism and $(A \times_\theta B)/A$ is isometrically isomorphic to B. So, similar to Proposition 3.1 we have the following result.

Proposition 3.2. *Suppose that A and B are two Banach algebras, $\phi \in \sigma(A)$ and $\theta \in \sigma(B)$. If $A \times_\theta B$ is $(\phi, \theta)-$Johnson amenable, then*
(a) A is $\phi-$Johnson amenable, provided that A is unital.
(b) B is $\theta-$Johnson amenable.

Proof. (a) Let $A \times_\theta B$ be $(\phi, \theta)-$Johnson amenable. Then there exists a net (U_α) in $(A \times_\theta B)\widehat{\otimes}(A \times_\theta B)$ such that $(a, b) \cdot U_\alpha - U_\alpha \cdot (a, b) \longrightarrow 0$ and $((\phi, \theta) \circ \pi)(U_\alpha) \longrightarrow 1$. Moreover, we can write $U_\alpha = u_\alpha + m_\alpha + n_\alpha + \nu_\alpha$ such that $u_\alpha \in A\widehat{\otimes}A$, $m_\alpha \in A\widehat{\otimes}B$, $n_\alpha \in B\widehat{\otimes}A$ and $\nu_\alpha \in B\widehat{\otimes}B$ for all α. Now, if $b = 0$, then

$$a \cdot u_\alpha + a \cdot n_\alpha - u_\alpha \cdot a - m_\alpha \cdot a \longrightarrow 0,$$
$$a \cdot m_\alpha + a \cdot \nu_\alpha \longrightarrow 0,\ n_\alpha \cdot a + \nu_\alpha \cdot a \longrightarrow 0,\ (1)$$
$$(\phi \circ \pi)(u_\alpha) + (\phi \circ \pi)(m_\alpha) + (\phi \circ \pi)(n_\alpha) + (\theta \circ \pi)(\nu_\alpha) \longrightarrow 1$$

for all $a \in A$. Suppose that A has the unit e. From (1) we conclude that

$$m_\alpha + e \cdot \nu_\alpha \longrightarrow 0,\ n_\alpha + \nu_\alpha \cdot e \longrightarrow 0$$

and

$$(\phi \circ \pi)(m_\alpha) + (\phi \circ \pi)(e \cdot \nu_\alpha) \longrightarrow 0,\ (\phi \circ \pi)(n_\alpha) + (\phi \circ \pi)(\nu_\alpha \cdot e) \longrightarrow 0.$$

Hence, the above facts give that

$$\begin{aligned} a \cdot (u_\alpha - [\theta(\pi(v_\alpha))e \otimes e]) - (u_\alpha - [\theta(\pi(v_\alpha))e \otimes e]) \cdot a & \\ = a \cdot u_\alpha - [\theta(\pi(v_\alpha))a \otimes e] - u_\alpha \cdot a + [\theta(\pi(v_\alpha))e \otimes a] & \\ = a \cdot u_\alpha + a \cdot n_\alpha - u_\alpha \cdot a - m_\alpha \cdot a & \\ \longrightarrow \quad 0. & \end{aligned}$$

Also, we have

$$\begin{aligned} (\phi \circ \pi)(u_\alpha - \theta(\pi(\nu_\alpha))e \otimes e) &= (\phi \circ \pi)(u_\alpha) - (\theta \circ \pi)(\nu_\alpha) + (\phi \circ \pi)(m_\alpha) \\ &\quad +(\phi \circ \pi)(e \cdot \nu_\alpha) + (\phi \circ \pi)(n_\alpha) + (\phi \circ \pi)(\nu_\alpha \cdot e) \\ &= (\phi \circ \pi)(u_\alpha) + (\phi \circ \pi)(m_\alpha) \\ &\quad +(\phi \circ \pi)(n_\alpha) + (\theta \circ \pi)(\nu_\alpha) \\ &\longrightarrow 1. \end{aligned}$$

So, A is $\phi-$Johnson amenable.

(b) From these facts that $A \times_\theta B$ is $(\phi, \theta)-$Johnson amenable and $(A \times_\theta B)/A$ is isometrically isomorphic to B, we have B is $\theta-$Johnson amenable. $\square$

Proposition 3.2 concludes that if $A \times_\theta B$ be $(0, \psi)-$Johnson amenable, then B is $\psi-$Johnson amenable. In general, we do not know whether the reverse of Proposition 3.2 in the certain case is correct or not. So, we formulate it as a question.

Question. If B is $\psi-$Johnson amenable, then is $A \times_\theta B$, $(0, \psi)-$Johnson amenable?

Proposition 3.3. *Suppose that A and B are two Banach algebras, $\phi \in \sigma(A)$ and $\theta \in \sigma(B)$. If A is $\phi-$Johnson amenable, then $A \times_\theta B$ is $(\phi, \theta)-$Johnson amenable.*

Proof. There exists a net (u_α) in $A\widehat{\otimes}A$ such that $a \cdot u_\alpha - u_\alpha \cdot a \longrightarrow 0$ and $(\phi \circ \pi_A)(u_\alpha) \longrightarrow 1$ for all $a \in A$. Also, $u_\alpha = \sum_{i=1}^{\infty} a_i^\alpha \otimes a_i'^\alpha$ for some $a_i^\alpha, a_i'^\alpha \in A$. By definition the net as $U_\alpha := \sum_{i=1}^{\infty} (a_i^\alpha, 0) \otimes (a_i'^\alpha, 0) \subseteq (A \times_\theta B)\widehat{\otimes}(A \times_\theta B)$. Therefore, by using this net the Banach algebra $A \times_\theta B$ is $(\phi, \theta)-$Johnson amenable. $\square$

4. Some Applications

Suppose that A is a Banach algebra and $\phi \in \sigma(A)$. Then the Banach algebra A is called left ϕ-amenable (left ϕ-contractible) if there exists a bounded net (m_α) in A (an element m in A) such that $am_\alpha - \phi(a)m_\alpha \to 0$ ($am = \phi(a)m$) and $\phi(m_\alpha) \to 1$ ($\phi(m) = 1$) for all $a \in A$, respectively, see [8] and [12].

Example 4.1. *The set $C^1[0,1]$ consists of all differentiable functions which its first derivation is continuous. With the point-wise product $C^1[0,1]$ becomes a Banach algebra. Also, $\sigma(C^1[0,1]) = \{\phi_t : t \in [0,1]\}$, where $\phi_t(f) = f(t)$ for all $t \in [0,1]$. In this example, we claim that $C^1[0,1] \times_\theta C^1[0,1]$ is neither $(\phi_t, \theta)-$biflat nor $(0, \phi_t)-$biflat. We assume in contradiction that $C^1[0,1] \times_\theta C^1[0,1]$ is $(\phi_t, \theta)-$biflat or $(0, \phi_t)-$biflat, where $\phi_t(f) = f(t)$ for each $t \in [0,1]$. It is clear that 1 is an identity for $C^1[0,1]$. So, $C^1[0,1]$ is ϕ_t-biflat. Therefore, there exists a $C^1[0,1]-$bimodule morphism $\rho : C^1[0,1] \longrightarrow (C^1[0,1]\hat{\otimes}C^1[0,1])^{**}$ such that*

$$\tilde{\phi_t} \circ \pi^{**} \circ \rho(f) = \phi_t(f)$$

for all $f \in C^1[0,1]$. Put $m = \rho(1)$, we have

$$f \cdot m = f\rho(1) = \rho(f1) = \rho(1f) = \rho(1)f = m \cdot f,$$

and

$$\tilde{\phi_t} \circ \pi^{**}(m) = \tilde{\phi_t} \circ \pi^{**} \circ \rho(1) = \phi_t(1) = 1,$$

for all $f \in C^1[0,1]$. It follows that $C^1[0,1]$ is ϕ_t-Johnson amenable. Thus, [17, Proposition 2.2]*, implies that $C^1[0,1]$ is left ϕ_t-amenable which is impossible by* [8, Example 2.5].

Recall that the Banach algebra A is called character biflat (character biprojective) if A is ϕ-biflat (ϕ-biprojective) for each $\phi \in \sigma(A)$, respectively, see [15].

Proposition 4.1. *Suppose that G is a locally compact group and $M(G)$ is the measure algebra over G. Let $\theta \in \Delta(M(G))$. Then $M(G) \times_\theta M(G)$ is character biflat if and only if G is a discrete amenable group.*

Proof. Let $M(G) \times_\theta M(G)$ be character biflat. Since $M(G)$ is unital. As in the previous example, $M(G)$ is left and right ϕ_t-amenable for all $\phi \in \sigma(M(G))$ (By placing $m = \rho(e)$ where e is the unit of $M(G)$). We know that $M(G)$ has a bounded approximate identity, thus $M(G)$ is character amenable. So, [10, Corollary 2.5] implies that G is a discrete amenable group.

For converse, let G be a discrete amenable group. Then $M(G) = \ell^1(G)$. Hence by using Johnson Theorem $\ell^1(G)$ is amenable. Therefore, [2, Corollary 2.1] finishes the proof. $\square$

Proposition 4.2. *Let G be a locally compact group. Then $M(G) \times_\theta M(G)$ is character biprojective if and only if G is finite.*

Proof. Let $M(G) \times_\theta M(G)$ be character biprojective. Then by Proposition 2.1, $M(G)$ is character biprojective ($M(G)$ is unital). So, by [17, Lemma 3.2], $M(G)$ is $\phi-$Johnson contractible for all $\phi \in \sigma(M(G))$. Also, using the same arguments as in [17, Proposition 2.2] implies that $M(G)$ is left $\phi-$contractible for all $\phi \in \sigma(M(G))$. We know that $M(G)$ is unital, therefore $M(G)$ is character contractible. From [12, Corollary 6.2], we have G is a finite group. Converse is clear. □

It is well-known that the Fourier algebra $A(G)$ over a locally compact group G is a commutative Banach algebra. Also, $\sigma(A(G)) = \{\phi_g : g \in G\}$, where $\phi_g(f) = f(g)$.

Theorem 4.1. *Let G be a locally compact group. Then $M(G) \times_\theta A(G)$ is character biprojective if and only if G is a finite group.*

Proof. Let $M(G) \times_\theta A(G)$ be character biprojective. Since $M(G)$ has an identity and $A(G)$ is commutative, for each $\gamma \in \sigma(M(G) \times_\theta A(G))$ there exists an element $a_\gamma \in M(G) \times_\theta A(G)$ such that $aa_\gamma = a_\gamma a$ and $\gamma(a_\gamma)$=1, for all $a \in M(G) \times_\theta A(G)$. Similar to the proof of [16, Theorem 3.2], we can see that $M(G) \times_\theta A(G)$ is γ-Johnson contractible, for all $\gamma \in \sigma(M(G) \times_\theta A(G))$. Following the arguments in [17, Proposition 2.2], one can see that $M(G) \times_\theta A(G)$ is left γ-contractible, for all $\gamma \in \sigma(M(G) \times_\theta A(G))$. Applying Proposition 2.1, leads to that $M(G)$ is character biprojective. Since $M(G)$ is unital, character biprojectivity of $M(G)$ implies the character contractibility of $M(G)$. Using [12, Corollary 6.2] gives that G is a finite group. □

Remark 4.1. *In the sequel, suppose that G is a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions*

(i) *$S^1(G)$ is dense in $L^1(G)$,*
(ii) *$S^1(G)$ with a norm $||\cdot||_{S^1(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S^1(G)}$ for every $f \in S(G)$,*
(iii) *for $f \in S^1(G)$ and $y \in G$, we have $L_y(f) \in S(G)$ the map $y \mapsto L_y(f)$ from G into $S^1(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,*
(iv) *$||L_y(f)||_{S^1(G)} = ||f||_{S^1(G)}$ for every $f \in S^1(G)$ and $y \in G$.*

It is well-known that $S^1(G)$ always has a left approximate identity. We remind that a Segal algebra is a left ideal of $L^1(G)$ and for a Segal algebra $S^1(G)$ it has been shown that

$$\Delta(S^1(G)) = \{\phi_{|_{S^1(G)}} | \phi \in \Delta(L^1(G))\},$$

for more information see [13] *and* [1, Lemma 2.2].

Proposition 4.3. *Suppose that G is a locally compact group and $\varphi, \theta, \psi \in \sigma(S^1(G))$. If $S^1(G) \times_\theta S^1(G)$ is either $(\varphi, \theta)-$biflat or $(0, \psi)-$biflat, then G is amenable.*

Proof. It is well-known that $S^1(G)$ has a left approximate identity. Therefore, $S^1(G) \times_\theta S^1(G)$ has a left approximate identity. Hence, [5, Theorem 3.1] follows that $S^1(G) \times_\theta S^1(G)$ is either left (φ, θ) or $(0, \psi)-$amenable. Therefore, $S^1(G)$ is left $\varphi-$amenable (or left $\psi-$amenable). Thus [1, Corollary 3.4] follows that G is an amenable group. □

Example 4.2. *Let S be the left zero semigroup. That is a semigroup S such that $st = t$ for all $s, t \in S$. It is easy to see that for the semigroup algebra $\ell^1(S)$, $fg = \phi_S(g)f$ for all $f, g \in \ell^1(S)$, where ϕ_S is the augmentation character on $\ell^1(S)$. Note that $\ell^1(S)$ only has a right unit and $\sigma(\ell^1(S)) = \{\phi_S\}$. Suppose that $|S| \geq 2$. Then we claim that $\ell^1(S) \times_{\phi_S} \ell^1(S)$ is not $(\phi_S, \phi_S)-$Johnson amenable. We assume in a contradiction that $\ell^1(S) \times_{\phi_S} \ell^1(S)$ is $(\phi_S, \phi_S)-$Johnson amenable, then $\ell^1(S)$ is ϕ_S-Johnson amenable. So,* [17, Proposition 2.2] *implies that $\ell^1(S)$ is left ϕ_S-amenable. Applying* [8, Theorem 1.4], *follows that there exists a bounded net (m_α) in $\ell^1(S)$ such that*

$$am_\alpha - \phi_S(a)m_\alpha \longrightarrow 0, \qquad \phi_S(m_\alpha) = 1,$$

for all $a \in \ell^1(S)$. It implies that

$$am_\alpha - \phi_S(a)m_\alpha = a - \phi_S(a)m_\alpha \longrightarrow 0.$$

Suppose that $s_1 \neq s_2 \in S$ and $\delta_{s_1} \neq \delta_{s_2} \in \ell^1(S)$. So, replace a with δ_{s_1} and δ_{s_2}. It follows that

$$\delta_{s_1} - \phi_{s_0}(\delta_{s_1})m_\alpha = \delta_{s_1} - m_\alpha \longrightarrow 0, \qquad \delta_{s_2} - \phi_{s_0}(\delta_{s_2})m_\alpha = \delta_{s_2} - m_\alpha \longrightarrow 0.$$

Therefore, $m_\alpha \longrightarrow \delta_{s_1}$ and $m_\alpha \longrightarrow \delta_{s_2}$ which is a impossible.

Theorem 4.2. *Suppose that the semigroup S has a unit. Then $\ell^1(S) \times_\theta \ell^1(S)$ is either (ϕ_S, θ)- biprojective or $(0, \phi_S)-$biprojective if and only if S is finite, where ϕ_S is the augmentation character.*

Proof. We know that $\ell^1(S) \times_\theta \ell^1(S)$ has the unit. Thus, by [17, Lemma 3.2], $\ell^1(S) \times_\theta \ell^1(S)$ is $(\phi_S, \theta), (0, \phi_S)-$Johnson contractible. Similar to Proposition 3.2, we can see that $\ell^1(S)$ is ϕ_S-Johnson contractible. So, there exists $m' \in \ell^1(S)\hat{\otimes}\ell^1(S)$ such that $a \cdot m' = m' \cdot a$ and $\phi_S(\pi(m')) = 1$, for all $a \in \ell^1(S)$. Define $T : \ell^1(S)\hat{\otimes}\ell^1(S) \longrightarrow \ell^1(S)$ by $T(a \otimes b) := \phi_S(b)a$, for all $a, b \in \ell^1(S)$. It is easy to see that T is linear and continuous which satisfies

$$aT(m') = T(a \cdot m') = T(m' \cdot a) = \phi_S(a)m', \qquad \phi_S \circ T(m') = \phi_S \circ \pi(m') = 1,$$

for all $a \in \ell^1(S)$. Similarly, if we define $T(a \otimes b) := \phi_S(a)b$, then we can find an element $m'' \in \ell^1(S)\hat{\otimes}\ell^1(S)$ such that

$$m''a = \phi_S(a)m'', \qquad \phi_S(m'') = 1,$$

for all $a \in \ell^1(S)$. Put $m = m' \otimes m''$. Thus,

$$a \cdot m = am' \otimes m'' = \phi_S(a)m' \otimes m'' = m' \otimes m''\phi_S(a) = m' \otimes m''a = m \cdot a,$$

$$\phi_S(\pi(m)) = \phi_S(m^{'}m^{''}) = \phi_S(m^{'})\phi_S(m^{''}) = 1,$$

for all $a \in \ell^1(S)$. Putting $a = \delta_s$, we have that $\delta_s m = \phi_S(\delta_s)m' \otimes m'' = m, \qquad m\delta_s = m.$ Therefore,

$$\delta_s \pi_{\ell^1(S)}(m) = \pi_{\ell^1(S)}(\delta_s m) = \pi_{\ell^1(S)}(m) = \pi_{\ell^1(S)}(m\delta_s) = \pi_{\ell^1(S)}(m)\delta_s.$$

Thus, we find an element $f = \pi_{\ell^1(S)}(m) \in \ell^1(S)$ such that $\delta_s f(x) = f(xs) = f(x) = f(sx) = f\delta_s(x)$, for all $s, x \in S$. If $x = e$ is a unit for S, then we have that $f(s) = f(e)$. It gives that f is a constant function in $\ell^1(S)$. Using $\phi_S(\pi_{\ell^1(S)}(m)) = \phi_S(f) = 1$ implies that $f \neq 0$. Therefore, S must be a finite semigroup. □

Equip $\mathbb{N}$ with the multiplication $m \vee n = \max\{m, n\}$ for all $m, n \in \mathbb{N}$. So, $\mathbb{N}$ with this multiplication denoted by $\mathbb{N}_\vee$ becomes a semigroup. It is clear that $\ell^1(\mathbb{N}_\vee)$ is a commutative Banach algebra and $\delta_1 * \delta_n = \delta_n = \delta_n * \delta_1$ for all $n \in \mathbb{N}$. Thus, $\ell^1(\mathbb{N}_\vee)$ is unital.

Corollary 4.1. *The Banach algebra $\ell^1(\mathbb{N}_\vee) \times_\theta \ell^1(\mathbb{N}_\vee)$ is neither $(\phi_{\mathbb{N}_\vee}, \theta)$-biprojective nor $(0, \phi_{\mathbb{N}_\vee})$-biprojective, where $\phi_{\mathbb{N}_\vee}$ is the augmentation character on $\ell^1(\mathbb{N}_\vee)$.*

Proof. We assume conversely that $\ell^1(\mathbb{N}_\vee) \times_\theta \ell^1(\mathbb{N}_\vee)$ is either $(\phi_{\mathbb{N}_\vee}, \theta)$-biprojective or $(0, \phi_{\mathbb{N}_\vee})$-biprojective. Since $\mathbb{N}_\vee$ has unit, the previous Theorem gives us that $\mathbb{N}_\vee$ must be finite which is impossible. □

Let A be a Banach algebra and $\phi \in \sigma(A)$. A Banach algebra A is called $\phi-$inner amenable if there exists a bounded net (e_α) in A such that $ae_\alpha - e_\alpha a \to 0$ and $\phi(e_\alpha) \to 1$ for all $a \in A$, see [6].

Example 4.3. *Let* $A=\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}: \ a,b,d\in\mathbb{C}\}$ *be a matrix algebra. Then* $\phi : A\longrightarrow\mathbb{C}$ *defined by* $\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix})=d$ *is a character on A. We claim that* $A\times_\theta A$ *is neither* (ϕ,θ)*-biflat nor* $(0,\phi)-$*biflat, where* $\theta\in\sigma(A)$*. Suppose in contradiction that* $A\times_\theta A$ *is either* (ϕ,θ)*-biflat or* $(0,\phi)$*-biflat. Since* $A\times_\theta A$ *is unital, therefore* $A\times_\theta A$ *is either* (ϕ,θ)*-inner amenable or* $(0,\phi)-$*inner amenable. So, by* [17, Proposition 3.3], $A\times_\theta A$ *is* $(\phi,\theta),(0,\phi)-$*Johnson amenable. Applying Proposition 3.2, we have A is* $\phi-$*Johnson amenable. Using* [17, Proposition 2.2] *implies that A is left and right* ϕ*-amenable. Define* $J:=\{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}: \ b,d\in\mathbb{C}\}$ *and* $\phi_{|_J}\neq 0$*. It is easy to see that J is a closed ideal of A. Since A is left* ϕ*-amenable, from* [8, Lemma 3.1] *we have that J is* $\phi_{|_J}-$*amenable. Applying* [8, Theorem 1.4]*, there exists a bounded net* (u_α) *in J such that* $ju_\alpha-\phi(j)u_\alpha\longrightarrow 0$ *and* $\phi(u_\alpha)=1$ *for all* $j\in J$*. Assume that* $j=\begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$ *and* $u_\alpha=\begin{pmatrix} 0 & w_\alpha \\ 0 & v_\alpha \end{pmatrix}$*, for some* $j_1,j_2,w_\alpha,v_\alpha\in\mathbb{C}$*. Thus,*

$$ju_\alpha-\phi(j)u_\alpha=\begin{pmatrix} 0 & j_1w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix}-\begin{pmatrix} 0 & j_2w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix}\longrightarrow 0.$$

It gives that $j_1v_\alpha-j_2w_\alpha\longrightarrow 0$*. If we put* $j_1=1$ *and* $j_2=0$*, then the contradiction reveals.*

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions. The authors are thankful to university of Kurdistan and Ilam university respectively, for their supports.

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