NORM CONVERGENCE ITERATIONS FOR BEST PROXIMITY POINTS OF NON-SELF NON-EXPANSIVE MAPPINGS

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In this paper, we introduce hybrid algorithms for non-self, non-expansive maps on real Hilbert space and prove that the iterative sequence of the algorithm converges strongly to the proximity point of any non-expansive mapping with a nonempty proximity point set. This study is a natural continuation of those of Nakajo and Takahashi on convergence theorems for nonexpansive mappings and nonexpansive semigroups, [J. Math. Anal. Appl., ***279*** *(2003), 372-379].*

Keywords: Non-expansive mapping, P-property, hybrid algorithm, proximity point, norm convergence.

1. Introduction

In the numerical analysis of fixed point, diverse iterative schemes were introduced for various purposes such as:

Fast convergence of one over the other for computational purposes.
Some iterative procedures guarantee the convergence while the others may fail.
To obtain strong convergence instead of weak convergence for application aspects.

That is why a great deal of literature on iterative algorithms for numerical reckoning fixed points of non-expansive mappings has been published: Halpern [8], Nilsrakoo and Saejung [13], Thakur *et al.* [18, 19], Wittmann [20]. These studies have a variety of applications in inverse problems, image recovery, variational inequalities and signal processing: please, see Combettes [4], Mann [9], Xu [21], Yao *et al.* [22, 23, 24, 25] and Youla [26].

In fixed point theory, Mann iteration process is often used to approximate fixed points of several classes of operators [9]. But it does provide only weak convergence sometimes: see Genel and Lindenstrass [6] for example. However, norm convergence is often much more desirable than weak convergence. So, attempts have been made to modify Mann iteration process so that norm convergence is guaranteed. In this regard, Nakajo and Takahashi [11] firstly introduced their hybrid algorithm for non-expansive mappings and proved the strong convergence of iterate sequence to the fixed point of such kind of mappings. Since then several authors have studied the norm convergence of iterates of non-expansive mappings using hybrid algorithms: please, see Opial [14], Sadiq Basha and Veeramani [15], Takahashi [17], Berinde [2], Martinez-Yanes and Xu [10].

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In this paper, we introduce new algorithms and prove some results which assure the norm convergence of iterative sequence to the proximity point set of non-self, non-expansive mappings on Hilbert spaces.

2. Notation and preliminaries

Let H be a real Hilbert space with the inner product $< \cdot, \cdot >$ and the norm $|| \cdot ||$. Let C be a closed subset of H. Recall that a mapping $T\colon C \to H$ is said to be non-expansive if $||Tx - Ty|| \leq ||x - y||$ holds for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T, i.e. $F(T) := \{x \in C : Tx = x\}$.

Let A and B be two nonempty closed convex subsets of H and $T\colon A \to B$ be a non-expansive mapping. Denote by $P_A(x)$ the metric projection of some element x onto A and by $P(A^T)$ denote the set of all proximity points of T on A, that is

$$P(A^T) = \{a \in A : d(a, T(a)) = d(A, B)\},$$

and $w_\omega(x_n)\colon = \{x\colon \exists (x_{n_j}) \subset (x_n), \quad x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$. It is quite natural to see that $P(A^T)$ of T is contained in the closed subset of A given by

$$A_0 := \{a \in A : d(a, b) = d(A, B), \text{ for some } b \in B\}.$$

For details, please see Nashine [12], Shatanawi and Pitea [16].

Definition 2.1 ([16])**.** Let A and B be closed subsets of a metric space (X, d). Then A and B are said to satisfy the P-property if for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ the following implication holds:

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Longrightarrow d(x_1, x_2) = d(y_1, y_2).$$

Definition 2.2 ([5])**.** Let A and B be nonempty closed subsets of a metric space (X, d). Then (A, B) is said to satisfy the UC property if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = d(A, B)$, then $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Now, consider A and B closed subsets of a metric space (X, d) and $T\colon A \to B$ a non-expansive mapping. It is known that the assumption of the weakly compactness of A_0 could give only weak convergence of the sequence to the proximity point set of T. In this sense, please see Haddadi [7], Chen *et al.* [3] and Abkar and Gabeleh [1].

The following result is stated in [1].

Theorem 2.1 ([1])**.** *Let (A, B) be a pair of nonempty, closed and convex subsets of a Banach space X such that A_0 is nonempty. Let $T\colon A \to B$ be a non-expansive mapping such that $T(A_0) \subset B_0$. Suppose the pair (A, B) has the P-property and A is weakly compact. Then T has at least one best proximity point in A provided that one of the following conditions is satisfied:*

1. *T is weakly continuous.*
2. *T satisfies the proximal property.*

Since the norm convergence is more desirable, we introduce new algorithms and prove some results which assure the norm convergence of the iterative sequence to the proximity point set on Hilbert space.

Lemma 2.1 ([13])**.** *In a Hilbert space the following hold, for $u, v \in H$:*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2 < u - v, v >, \tag{2.1}$$

$$\|\alpha u+(1-\alpha)v\|^2=\alpha\|u\|^2+(1-\alpha)\|v\|^2-\alpha(1-\alpha)\|u-v\|^2, \quad \alpha\in[0,1]. \tag{2.2}$$

Lemma 2.2 ([10]). *Let K be a closed and convex subset of real Hilbert space H. Then $z=P_K(x)$ if and only if the relation holds:*

$$< x-z, y-z >\leq 0, \quad for\ all\ y\in K.$$

Lemma 2.3 ([10]). *Let C be a closed convex subset of a real Hilbert space H and let $T\colon C\to C$ be a nonexpansive mapping such that $F(T)\neq\emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n\rightharpoonup z$ and $\|x_n-Tx_n\|\to 0$, then $z=Tz$.*

Lemma 2.4 ([10]). *Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $x\in H$. Let $q=P_K(x)$. If $\{x_n\}$ is such that $w_\omega(x_n)\subset K$ and satisfies the condition*

$$\|x_n-x\|\leq\|x-q\|\ for\ all\ n\in\mathbb{N},$$

then $x_n\to q$.

In this paper, we prove the norm convergence of the iterative sequence to the proximity point set of non-expansive map using new hybrid algorithm. Moreover, the sequence converges to the nearest proximity point of the non-expansive map.

3. **Algorithm and norm convergence to proximity pair**

In this section, the norm convergence of the iterate sequence to the proximity point of non-self, non-expansive mappings will be proved using hybrid algorithms.

Algorithm 3.1.
$x_0\in A_0$ *arbitrarily.*
$y_n=\alpha_n x_n+(1-\alpha_n)P_A(T(x_n)),\ n\in\mathbb{N}\cup\{0\}$
$C_n=\{z\in A_0:\|y_n-z\|\leq\|x_n-z\|\}$
$Q_n=\{z\in A_0:< x_n-z, x_n-x_0 >\leq 0\}$
$x_{n+1}=P_{(C_n\cap Q_n)}(x_0)$.
where $\alpha_n\in[0,a]$ for some $a\in[0,1)$.

Theorem 3.1. *Let A and B be nonempty closed and convex subsets of H which satisfy P-property. Let $T\colon A_0\to B_0$ be a non-expansive mapping such that $P(A^T)$ is a nonempty convex subset of A_0. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm* 3.1 *converge to a proximity point in A. In particular, $\{x_n\}$ and $\{y_n\}$ converge to q, where $q=P_{P(A^T)}(x_0)$.*

Proof. Choose $x_0\in A_0$ arbitrarily. It is clear that C_n and Q_n are closed and convex subsets of A.

Now we claim that $P(A^T)\subset C_n$.

Let $u\in P(A^T)$. Then, we have

$$\begin{aligned}\|y_n-u\|&=\|\alpha_n x_n+(1-\alpha_n)P_A(T(x_n))-u\|\\&\leq\|\alpha_n(x_n-u)\|+\|(1-\alpha_n)(P_A(T(x_n))-u)\|\\&\leq\alpha_n\|x_n-u\|+(1-\alpha_n)\|P_A(T(x_n))-u\|\end{aligned}$$

Since $\|P_AT(x_n)-T(x_n)\|=d(A,B)$ and $\|u-T(u)\|=d(A,B)$, using the P-property we obtain that $\|P_A(T(x_n))-u\|=\|T(x_n)-T(u)\|$. Therefore, the above inequality becomes

$$\begin{aligned}\|y_n-u\|&\leq\alpha_n\|x_n-u\|+(1-\alpha_n)\|T(x_n)-T(u)\|\\&\leq\alpha_n\|x_n-u\|+(1-\alpha_n)\|x_n-u\|\\&=\|x_n-u\|.\end{aligned}$$

Now, we prove that $P(A^T) \subset Q_n$ by induction.

It is obvious to note that $P(A^T) \subset A_0$. Let $Q_0 = A_0$ and we assume that $P(A^T) \subset Q_n$ is true. Since $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ (such an element exits since C_n and Q_n are closed and convex), by Lemma 2.2 we have that $< x_{n+1} - z, x_{n+1} - x_0 > \leq 0$ for all $z \in C_n \cap Q_n$, in particular for all $P(A^T)$. Therefore, $P(A^T) \subset Q_{n+1}$. Hence the induction holds true.

Since $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $P(A^T) \subset C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|q - x_0\|, \text{ where } q = P_{P(A^T)}(x_0). \tag{3.1}$$

Hence, $\{x_n\}$ is bounded.

Since $x_{n+1} \in Q_n$ and by using Lemma 2.1, we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

which in turn gives that

$$\sum_{n=0}^{\infty} \|x_{n+1} - x_n\|^2 \leq \|q - x_0\|^2 - \|x_1 - x_0\|^2.$$

By the definition of C_n, we get that

$$\begin{aligned} \|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \end{aligned}$$

Therefore, $\|y_n - x_n\| \to 0$. Also,

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n) P_A(T(x_n)) - x_n\| \\ &= \|(1 - \alpha_n)(P_A(T(x_n)) - x_n)\| \\ &= (1 - \alpha_n)\|P_A(T(x_n)) - x_n\| \end{aligned}$$

Since $\{\alpha_n\}$ does not converge to 1, we get $\|P_A(T(x_n)) - x_n\| \to 0$ as $n \to \infty$.

Now, define $S \colon A_0 \to A_0$ as $S(x) = P_A(Tx)$ for all $x \in A_0$. By using P-property, we get that S is nonexpansive and $P(A^T) = F(S)$. Hence, by Lemma 2.3 we obtain that $w_\omega(x_n) \subset F(S)$. This, together with (3.1) and Lemma 2.4, guarantees that $\{x_n\}$ converges strongly to a fixed point of S (say p). Therefore, $\{x_n\}$ converges to the point $p \in A_0$, which satisfies $d(p, T(p)) = d(A, B)$. Therefore, $p \in P(A^T)$ and hence

$$\|x_0 - p\| \geq \|x_0 - q\| = d(x_0, P(A^T)).$$

Also, using equation (3.1) it follows

$$\begin{aligned} \|x_0 - p\| &= \lim_{n \to \infty} \|x_0 - x_n\| \\ &\leq \|q - x_0\| \\ &\leq d(x_0, P(A^T)). \end{aligned}$$

Therefore, $\|x_0 - p\| = d(x_0, P(A^T))$. Hence, $p = q$ and the proof is now complete. □

Algorithm 3.2.
$x_0 \in A_0$ *arbitrarily.*
$y_n = \alpha_n P_B(x_n) + (1 - \alpha_n) T(x_n),\ n \in \mathbb{N} \cup \{0\}$
$C_n = \{z \in A_0 : \|y_n - z\| \leq \|x_n - z\| + d(A, B)\}$
$Q_n = \{z \in A_0 :< x_n - z, x_n - x_0 > \leq 0\}$
$x_{n+1} = P_{(C_n \cap Q_n)}(x_0)$
where, $\alpha_n \in [0, 1]$, $\alpha_n \to 0$.

Lemma 3.1. *C_n generated in Algorithm 3.2 is convex.*

Proof. From the definition of C_n, we get

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + \big(d(A,B)\big)^2 + 2d(A,B)\|x_n - z\|. \tag{3.2}$$

By using Lemma 2.1, relation (2.1), and equation (3.2), we get

$$\begin{aligned}\|y_n - x_n\|^2 &= \|(y_n - z) - (x_n - z)\|^2 \\ &= \|y_n - z\|^2 - \|x_n - z\|^2 - 2 < y_n - x_n, x_n - z > \\ &\leq \|x_n - z\|^2 + \big(d(A,B)\big)^2 + 2d(A,B)\|x_n - z\| \\ &\qquad\qquad - \|x_n - z\|^2 - 2 < y_n - x_n, x_n - z > \\ &= \big(d(A,B)\big)^2 + 2d(A,B)\|x_n - z\| - 2 < y_n - x_n, x_n - z > .\end{aligned}$$

Therefore, the relation in C_n is equivalent to

$$\|y_n - x_n\|^2 - \big(d(A,B)\big)^2 - 2d(A,B)\|x_n - z\| + 2 < y_n - x_n, x_n - z > \leq 0.$$

From the above inequality, it is clear that C_n is convex. □

Theorem 3.2. *Let A and B be nonempty closed and convex subsets of H which satisfy P-property. Let $T\colon A_0 \to B_0$ be a non-expansive mapping such that $P(A^T)$ is a nonempty convex subset of A_0. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.2 converges to a proximity pair in $A \times B$. In particular $\{(x_n, y_n)\}$ converges to $(q, T(q))$, where $q = P_{P(A^T)}(x_0)$.*

Proof. Choose $x_0 \in A_0$ arbitrarily. It is clear that C_n and Q_n are closed and convex subsets of A.

Now we claim that C_n is nonempty subset of A_0.

Let $u \in P(A^T)$, that is $\|u - T(u)\| = d(A,B)$.

$$\begin{aligned}\|y_n - u\| &= \|\alpha_n P_B(x_n) + (1-\alpha_n)T(x_n) - u\| \\ &\leq \|\alpha_n(P_B(x_n) - u)\| + \|(1-\alpha_n)(T(x_n) - u)\| \\ &\leq \alpha_n\|P_B(x_n) - u\| + (1-\alpha_n)\|T(x_n) - u\| \\ &\leq \alpha_n\|P_B(x_n) - T(u)\| + \alpha_n\|T(u) - u\| + (1-\alpha_n)\|T(x_n) - T(u)\| \\ &\qquad\qquad + (1-\alpha_n)\|T(u) - u\| \\ &\leq \alpha_n\|P_B(x_n) - T(u)\| + \alpha_n d(A,B) + (1-\alpha_n)\|x_n - u\| \\ &\qquad\qquad + (1-\alpha_n)d(A,B).\end{aligned}$$

Since $\|P_B(x_n) - x_n\| = \|u - T(u)\| = d(A,B)$, using the P-property we obtain that $\|P_B(x_n) - T(u)\| = \|x_n - u\|$. Therefore, the above inequality becomes

$$\begin{aligned}\|y_n - u\| &\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\|x_n - u\| + d(A,B) \\ &= \|x_n - u\| + d(A,B).\end{aligned}$$

Using the induction principle, it is easy to prove that $P(A^T) \subset Q_n$ and hence Q_n is a nonempty subset of A_0.

Since $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $P(A^T) \subset C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|q - x_0\|, \text{ where } q = P_{P(A^T)}(x_0). \tag{3.3}$$

Hence, $\{x_n\}$ is bounded.

Since, $x_{n+1} \in Q_n$ and by using Lemma 2.1, we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2,$$

which in turn gives that

$$\sum_{n=0}^{\infty} \|x_{n+1} - x_n\|^2 \leq \|q - x_0\|^2 - \|x_1 - x_0\|^2.$$

By the definition of C_n, we get that

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\| + d(A, B) + \|x_{n+1} - x_n\|.\end{aligned}$$

Therefore as $n \to \infty$, we obtain that $\|y_n - x_n\| \to d(A, B)$.

Also,

$$\begin{aligned}y_n - x_n &= \alpha_n P_B(x_n) + (1 - \alpha_n)T(x_n) - x_n \\ &= \alpha_n(P_B(x_n) - T(x_n)) + (T(x_n) - x_n).\end{aligned}$$

From the above equality, we obtain

$$\|T(x_n) - x_n\| \leq \|y_n - x_n\| + \alpha_n\|P_B(x_n) - T(x_n)\|.$$

Therefore, $\|T(x_n) - x_n\| \to d(A, B)$ as $n \to \infty$. Since every Hilbert space satisfies UC property and $\|P_B x_n - x_n\| \to d(A, B)$, we get $\|P_B x_n - T(x_n)\| \to 0$ as $n \to \infty$. Therefore, $\|P_A(T(x_n)) - x_n\| \to 0$ as $n \to \infty$.

Now, define $S\colon A_0 \to A_0$ as $S(x) = P_A(Tx)$ for all $x \in A_0$. By using P-property, we get S is nonexpansive and $P(A^T) = F(S)$. Hence, by Lemma 2.3 we obtain the inclusion $w_\omega(x_n) \subset F(S)$. This, together with (3.1) and Lemma 2.4, guarantees that $\{x_n\}$ converges strongly to a fixed point of S (say p). Therefore, $\{x_n\}$ converges to the point $p \in A_0$, which satisfies $d(p, T(p)) = d(A, B)$. Hence,

$$\|x_0 - p\| \geq \|x_0 - q\| = d(x_0, P(A^T)).$$

Also, using equation (3.3), we get

$$\begin{aligned}\|x_0 - p\| &= \lim_{n\to\infty} \|x_0 - x_n\| \\ &\leq \|q - x_0\| \\ &\leq d(x_0, P(A^T)).\end{aligned}$$

Therefore, $\|x_0 - p\| = d(x_0, P(A^T))$, hence $p = q$, and this completes the proof. □

4. **Numerical Computation**

Consider $A = \left\{(0, a) | a \in \left[\frac{3}{2}, \frac{5}{2}\right]\right\}$ and $B = \left\{(1, a) | a \in \left[\frac{3}{2}, \frac{5}{2}\right]\right\}$. Define $T\colon A \to B$ by the rule $T(0, x) = (1, 4 - x)$. It is clear that T is a nonexpansive mapping and $(0, 2)$ is the best proximity point of T. Let us choose an arbitrary element $x_0 = (0, \frac{5}{2})$. For different selection of $\{\alpha_n\}$ the tabular column at rows $1, 2, 3, 4$ gives the values of x_5, x_{25}, x_{50}, x_{100} terms of the generated sequence $\{x_n\}$ respectively.

x_n	$\{\alpha_n\} = \{\frac{1}{n+1}\}$	$\{\alpha_n\} = \{\frac{3}{4}\}$	$\{\alpha_n\} = \{0.95\}$
x_5	$(0, 2.000694e+00)$	$(0, 2.000000e+00)$	$(0, 2.000000e+00)$
x_{25}	$(0, 2.015625e+00)$	$(0, 2.000000e+00)$	$(0, 2.000000e+00)$
x_{50}	$(0, 2.118652e+00)$	$(0, 2.000000e+00)$	$(0, 2.000000e+00)$
x_{100}	$(0, 2.386890e+00)$	$(0, 2.038472e+00)$	$(0, 2.002960e+00)$

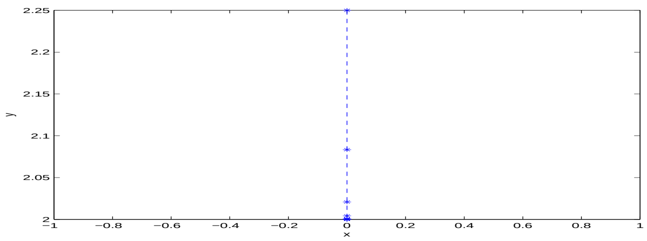

FIGURE 1. Convergence of sequence for $x_0 = (0, \frac{5}{2})$ and $\alpha_n = \frac{1}{n+1}$

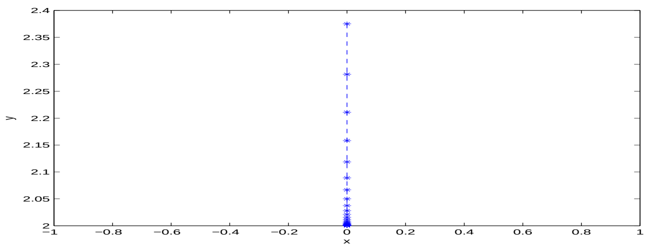

FIGURE 2. Convergence of sequence for $x_0 = (0, \frac{5}{2})$ and $\alpha_n = 0.75$

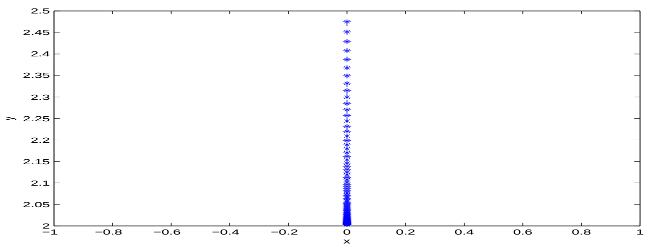

FIGURE 3. Convergence of sequence for $x_0 = (0, \frac{5}{2})$ and $\alpha_n = 0.95$

From the figures it is clear that the convergence rate is fast as $\{\alpha_n\}$ goes near 0.

Our future work will give results comparing the fast convergence of sequence $\{x_n\}$ obtained using our hybrid algorithm.

5. **Conclusion**

In this paper, we introduced hybrid algorithms for non-self, non-expansive maps on real Hilbert space and prove that the iterative sequence of algorithm converges strongly to the proximity point of any mapping in this class with nonempty proximity point set. This study is a natural continuation of those of Nakajo and Takahashi [11], and also of Opial [14], Sadiq Basha and Veeramani [15], Takahashi [17], Berinde [2].

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