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RECURRENCE AND CHAOS

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Assume that (Σ_2, ρ) is a one sided symbolic space, σ is a sub-shift on Σ_2. In this paper, we proved that there exists $\mathscr{T} \subset PQW(\sigma) \subset \Sigma_2$ such that $\sigma \mid_{\mathscr{T}}$ is $R-T$ chaos, Martelli chaos, distributional chaos, weakly mixing, Xiong-chaos and distributional chaos in a sequence. As an application, we prove that chaos occurs on recurrent sets in exchange economic systems.

Keywords: Chaos, Sub-shift, Quasi-weakly almost periodic point, Exchange economic systems.

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1. Introduction

Since Li and Yorke gave the definition of chaos in a strict mathematical language [8], Scholars in different areas have given different concepts and criteria for identifying chaos in the study of different systems, such as Devaney chaos[6], topological chaos, Ruelle-Takens chaos[12], distributional chaos[14] and so on. The main idea of the above concepts of chaos is the asymptotic and topological structure of the orbits of points, but these chaos are independent of each other. In order to explore the essential properties of chaos, scholars have conducted their research by revealing the inner connection between various chaotic concepts and discussing various complexities such as topological entropy, ergodicity, mixing, etc. After a further study, it is found that for some chaotic systems with zero topological entropy, their uncountable chaotic sets are all contained in an absolute zero measure set, and the absolute zero measure set can not be ignored from the ergodic point of view. This situation we call an artifact. In order to obtain a subsystem that both excludes artifacts and retains the original system's important dynamics, Zhou[23] introduced the concept of measure center, and divided $Li-Yorke$ chaos into three levels with different degrees of complexity: (1) f is chaotic, but f is not chaotic on

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non-wandering sets; (2) f is chaotic on the non-wandering set, but not chaotic on its measure center; (3) f is chaotic on the measure center.

The opinion of measure levels contributes to a deeper understanding of chaotic systems, and shows that the important dynamics of the system are concentrated on the measure center. In order to discuss the measure center and its structure, the concepts of weakly almost periodic points and quasi weakly almost periodic points are introduced by Zhou [23], and the importance on both point sets is illustrated. This paper discuss the chaotic behaviors for subsystem which determined by a quasi-weakly almost periodic point from the perspective of chaos level, and obtain that this subsystem is $R-T$ chaos, Martelli chaos, weakly mixing, Xiong-chaos and distributional chaos. As an application, the chaos property on the recurrence point of sets in the exchange economic systems are discussed.

The paper is organized as follows: In Section 2, we give the basic concepts and lemmas used in this paper. In Section 3, the main theorem and the proof of the theorem are given. In Section 4, as an application of the theorem, it is proved that distributional chaos occurs on some recurrent points sets for the exchange economic system.

2. Basic Definitions and Lemmas

In this paper, (X, d) denote a compact metric space with metric d, and $f : X \to X$ is a continuous map. For all $n \geq 0$, f^n means the n iteration of f.

Definition 2.1. *For any $x \in X$, x is called a periodic point of f, if there exists a positive integer n such that $f^n(x) = x$. The smallest positive integer n that satisfies $f^n(x) = x$ is called its period. The periodic points with period n are called n periodic points. Let $P(f)$ be the collection of periodic points of f.*

Definition 2.2. *For non-empty open set V of X, we denote $N(x,\ V) = \{n \in \mathbb{N} | f^n(x) \in V\}$, where $\mathbb{N}$ denotes the set of positive integers. A subset P of $\mathbb{N}$ is said to be of positive lower density, if*

$$\liminf_{n\to\infty} \frac{\sharp(p \cap \{1, 2, \cdots n\})}{n} > 0;$$

And P is said to be of postive upper density, if

$$\limsup_{n\to\infty} \frac{\sharp(p \cap \{1, 2, \cdots n\})}{n} > 0;$$

where $\sharp(\cdot)$ denotes the cardinality.

Definition 2.3. *For any $x \in X$ is called almost periodic point of f, if for any $\varepsilon > 0$, there exists $N > 0$, such that for any integer $q \geq 1$, there is a integer r, $q \leq r < N + q$ satisfying $d(f^r(x), x) < \varepsilon$. Denote the set of all almost periodic points of f by $A(f)$.*

Definition 2.4. *A point $x \in X$ is called a weakly almost periodic point of f, if for any neighborhood V of x, $N(x,\ V)$ is of positive lower density. Denote the set of all weakly almost periodic points of f by $W(f)$.*

Definition 2.5. *A point $x \in X$ is called a quasi-weakly almost periodic point of f, if for any neighborhood V of x, $N(x,\ V)$ is of positive upper density. Denote the set of all quasi-weakly almost periodic points of f by $QW(f)$.*

Denote that $PQW(f) = QW(f) - W(f)$, $PW(f) = W(f) - A(f)$, $PA(f) = A(f) - P(f)$ is proper quasi-weakly almost periodic points set, proper weakly almost periodic points set and proper almost periodic points set, respectively. For all $x \in X$, $\{x, f(x), f^2(x)\cdots\}$ is call the orbit of x, denote that $orb(x,f)$. For any $y \in X$ is called $\omega - limit$ point of x if there is a sub-sequence of $orb(x,f)$ convergences to y.

Assume that $S = \{0,1\}$, $\Sigma_2 = \{x = x_1x_2\cdots x_i\cdots,\ x_i \in S,\ i = 1,2\cdots\}$.

Definition 2.6. *$\rho : \Sigma_2 \times \Sigma_2 \to R$ is defined as follows: for all $x, y \in \Sigma_2$, $x = x_1x_2\cdots$, $y = y_1y_2\cdots$,*

$$\rho(x,y) = \begin{cases} 0, & x = y, \\ \frac{1}{k}, & x \neq y, \quad and\ k = min\{n \mid x_n \neq y_n\}. \end{cases}$$

Obviously, ρ is a metric on Σ_2. (Σ_2, ρ) is a compact space which is called a one-sided symbol space with 2 symbols.

Definition 2.7. *For any $x = x_1x_2\cdots \in \Sigma_2$, $\sigma : \Sigma_2 \to \Sigma_2$, $\sigma(x_1x_2\cdots) = x_2x_3\cdots$ is called a shift on Σ_2. It is can be proved that σ is continuous.*

Lemma 2.1. [15] *There exists an set E in Σ_2 with the property that for any $x = x_1\cdots$, $y = y_1\cdots \in E$, there are infinitely many n satisfying $x_n = y_n$ and infinitely many m satisfying $x_m \neq y_m$.*

If $y \subset \Sigma_2$ is a closed set and $\sigma(y) \subset y$, then $\sigma|_x : y \to x$ is called a subshift of σ.

For any $x_0 \in X$ is said to be unstable with respect to X, if there exists $r(x_0) > 0$ such that for any $\varepsilon > 0$ there exists $y_0 \in X$ with $d(y_0, x_0) < \varepsilon$ and positive integers n, $d(f^n(y_0), f^n(x_0)) > r(x_0)$.

f is called sensitive dependence on initial value if there exists $\delta > 0$ such that for any $x \in X$ and any neighborhood U_x of x, there is $y \in U_x$ and $n > 0$, satisfying $d(f^n(x), f^n(y)) > \delta$. δ is called the sensitive constants of f.

Definition 2.8. *A continuous map $f : X \to X$ is called $R - T$ chaos if*

(1) f is topological transitive on X.
(2) f is sensitive dependence on initial value.

Definition 2.9. *A continuous map $f : X \to X$ is called $Maurtelli$ chaos, if*

(1) $\omega(x_0, f) = X$.
(2) x_0 is unstable with respect to $orb(x_0, f)$

Definition 2.10. *Let $f : X \to X$ be a continuous map, and $\{p_i\}$ be a increasing sequence of positive integers. For a subset C of X, if $A \subset C$ and any continuous mapping $F : A \to X$, there exists $q_j \subset \{p_i\}$ such that for for any $x \in A$, $\lim_{j\to\infty} f^{q_j}(x) = F(x)$, then f is said to be chaotic in the sense of Xiong on C with respect to the sequence $\{p_i\}$.*

Lemma 2.2. [20] *Let X be a separable locally compact metric space containing at least two points, $f : X \to X$ is continuous, then f is topologically weakly mixed if and only if there exists a C-dense F_σ-subset of X which is chaotic in the sense of Xiong with respect to the natural number sequence.*

Definition 2.11. *Assume that (X, d), (Y, ρ) are compact metric space, $f : X \to X$, $g : Y \to Y$ are continuous map. If there exists a continuous surjective map $\Phi : X \to Y$ such that $\Phi \circ f = g \circ \Phi$, then Φ is said to be a topological semi-conjugation from f to g.*

Lemma 2.3. [24] *Assume that $f : X \to X$, $g : Y \to Y$ are continuous maps, Φ is a topological semi-conjugation from f to g, then*
(1) $\Phi(A(f)) = A(g)$.
(2) $\Phi(W(f)) = W(g)$.
(3) $\Phi(QW(f)) = QW(g)$.

Definition 2.12. *Let $(X,\ f)$ be a dynamical system, $x, y \in X$, for all $t > 0$ and any positive integer n, take*

$$\xi(f,\ x,\ y, t) = |\{i|d(f^i(x), f^i(y)) < t, 0 \le i < n\}|,$$

where $|\cdot|$ denotes the cardinality of the set. Assume that

$$F_{x,y}(t, f) = \liminf_{n\to\infty} \frac{1}{n}\xi_n(f,\ x,\ y, t),$$

$$F^*_{x,y}(t, f) = \limsup_{n\to\infty} \frac{1}{n}\xi_n(f,\ x,\ y, t).$$

If there exist $x,\ y \in X$ such that the pair $(x,\ y)$ satisfies:

$$F^*_{x,y} \equiv 1 \text{ and there exists } t > 0, F_{x,y}(t, f) = 0 \tag{1}$$

or

$$F^*_{x,y} \equiv 1 \text{ and for all } t > 0, F_{x,y}(t, f) < F^*_{x,y}(t, f) \tag{2}$$

or

$$\text{For all } t \text{ in some interval }, F_{x,y}(t, f) < F^*_{x,y}(t, f) \tag{3}$$

Then (x, y) exhibits distributional chaos of type 1,2,3, respectively. A dynamical system $(X,\ f)$ is said to be distributional chaos of type k (abbreviated as DC_k, where $k = 1, 2, 3$) if there exists an uncountable subset $D \subset X$ such that for any two different points in D, the point pair is distributional chaos of type k.

Proposition 2.1. *If f is DC_1, then f also exhibites DC_2, DC_3, DC'_2.*

For more information about DC_2', please refer to the [22].

Lemma 2.4. [11] *Let f be a continuous map from the compact metric space (X,d) to itself, and $\Phi : X \to \Sigma_2$ be a topological semiconjugate from f to σ. If there exists $y \in \Sigma_2$ such that $orb(y,\sigma) \neq \Sigma_2$ and $\Phi^{-1}(\{y\}) = 1$, then f is distributional chaos.*

Lemma 2.5. [16] *For a positive integer n, the continuous map f^n is distributional chaos if and only if f is distributional chaos.*

In order to have a better understanding of the inner relationship between Li-Yorke chaos and distributional chaos, Wang introduced the concept of Distributional chaos in a sequence in 2007.

$$F_{xy}(t,\{p_i\}) = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{[0,t)}(d(f^{\{p_i\}}(x), f^{\{p_i\}}(y))),$$

$$F^*_{xy}(t,\{p_i\}) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{[0,t)}(d(f^{\{p_i\}}(x), f^{\{p_i\}}(y))).$$

Definition 2.13. *$D \subset X$ is called a distributional chaos set in sequence $\{p_i\}$ if for any $x, y \in D, x \neq y$ such that*

(1) $\exists \delta > 0,\ F_{xy}(\delta,\{p_i\}) = 0.$

(2) $\forall t > 0,\ F^*_{xy}(t,\{p_i\}) = 1.$

where(x,y) is called a pair of distributional chaos in a sequence. If f has an uncountable scramble set in which any pair is distributional chaos in a sequence, then f is said to be distributional chaos in a sequence.

Lemma 2.6. [19] *Weak mixing implies distributional chaos in a sequence.*

The original definition of topological entropy was proposed by Adle, Konheim et al. in the [1], which is a measure to describe the complexity of a system.

Definition 2.14. *Let (X,f) be a compact system, α, β be an open cover of X, and denote that $N(\alpha)$ is the smallest of the cardinality of the sub-covers of α, and let*

$$\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\},$$

$$f^{-1}(\alpha) = \{f^{-1}(A) : A \in \alpha\},$$

$$ent(f) = sup_\alpha \lim_{n\to\infty} \frac{1}{N} logN(\alpha \vee f^{-1}(\alpha) \vee \cdots \vee f^{-(n-1)}(\alpha)).$$

where α takes all open covers of X, and we call $ent(f)$ the topological entropy of f.

Definition 2.15. *A discrete system (X,f) is called mean Li-Yorke chaotic if there exists an uncountable subset S of X such that for any $x,\ y \in S$ with $x \neq y$, one has*

$$\liminf_{N\to+\infty}\frac{1}{N}\sum_{k=1}^{N}d(f^kx,f^ky)=0$$

and

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{k=1}^{N}d(f^kx,f^ky)>0.$$

Lemma 2.7. [7] *If a topological dynamical system (X, f) has positive topological entropy, then it is multi-variant mean Li-Yorke chaotic.*

Lemma 2.8. [4] *Let $f : I \to I$ be a continuous map and $I = [a, b](0 \leq a < b)$, if $ent(f) > 0$, then the compact subset $\Lambda \subset I$ satisfies:*

(1) $f(\Lambda) = \Lambda$.
(2) there exists N and continuous surjective map $\Phi : \Lambda \to \Sigma$ such that $\Phi \circ f^N |_\Lambda = \sigma \circ \Phi |_\Lambda$.

Lemma 2.9. [10] *Suppose that $f : [a, b] \to [a, b](0 \leq a < b < \infty)$ is continuous and satisfies:*

(1) there exists $m \in (a, b)$ such that f is strictly increasing in $[a, m]$ and strictly decreasing in $[m, b]$.
(2) $f(a) \geq a,\ f(b) < b$ and for all $x \in (a, m)$, $f(x) > x$.
(3) there exists unique $\mathbf{z} \in (m, b)$ such that $f(\mathbf{z}) = \mathbf{z},\ f^2(m) < m,\ f^3(m) < \mathbf{z}$, therefore $ent(f) > 0$.

Definition 2.16. *Suppose that a finite sequence of length n in S denoted by $A = a_1 \cdots a_n$ (i.e. $|A| = n$).*

Denote that

$$[A] = \{x \in \Sigma_k | x_i = a_i, 0 \leq i \leq n\}$$

Obviously, [A] is the set of (Σ_2, σ) that is both open and closed.

Definition 2.17. *f is said to be topologically transitive if for any nonempty open sets U, V of X, there exits $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. The point where the orbit is dense in X is called the transitive point of f, and is denoted as Tran(f).*

Definition 2.18. *f is said to be topological weakly mixing if f_2 is transitive, i.e. for any nonempty open sets U_1, U_2, V_1 and V_2, there exists a positive integer n such that$f^n(U_i) \cap V_i \neq \emptyset$, $i = 1, 2$.*

Definition 2.19. *f is said to be topological mixing if for any nonempty open sets U and V, there exists a positive integer N such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.*

Lemma 2.10. [21] *Let $(X,\ f)$ be a compact transitive system, if $Tran(f) \cap (QW(f) - A(f)) \neq \emptyset$, then f is $R - T$ chaos.*

Lemma 2.11. [20] *Suppose that $f : I \to I$ is continuous and $ent(f) > 0$, then there exist $\Lambda \subset I$ and $n > 0$ such that $f^n : \Lambda \to \Lambda$ is topological mixing.*

Lemma 2.12. *Suppose that $f : I \to I$is continuous and $ent(f) > 0$, then there exist $\Lambda \subset I$ such that $f : \Lambda \to \Lambda$ is topological weakly mixing.*

Proof. According to Lemma 2.11, it is easy to draw this conclusion. □

3. Proofs of the main theorems

Theorem 3.1. *Let (Σ_2, ρ) is a one-sided symbol space, σ is a shift on (Σ_2, ρ), there exists subset $\mathscr{T} \subset PQW(\sigma)$ such that*

(1) $\sigma|_{\mathscr{T}}$ is $R-T$ chaos.
(2) $\sigma|_{\mathscr{T}}$ is $Martelli$ chaos.
(3) $\sigma|_{\mathscr{T}}$ is distributional chaos.
(4) $\sigma|_{\mathscr{T}}$ is DC_2, DC_3, DC_2'.
(5) $\sigma|_{\mathscr{T}}$ is topological weakly mixing.
(6) $\sigma|_{\mathscr{T}}$ is chaotic in sense of Xiong.
(7) $\sigma|_{\mathscr{T}}$ is distributional chaos in a sequence.

Proof. Constructing $\mathscr{T} \subset PQW(\sigma) \subset \Sigma_2$. Let $\mathscr{T}$ be a collection of sequences in an one-sided symbolic dynamical system consisting of two symbols 0, 1. $\mathscr{T}$ consists of all sequences as follows:

$a = A_1B_1C_1A_2B_2C_2 \cdots A_nB_nC_n \cdots$, where $A_1 = 01, B_1 = 00$ or $11, C_1 = 0101$ and for $n \geq 2$, $A_nB_nC_n$ are constructed inductively as follows.

For the sake of convenience, we denote by $\overbrace{0 \cdots 0}^{p}$ the arrangement of p many symbols 0 and by $\overbrace{1 \cdots 1}^{p}$ the arrangement of p many symbols 1.

Denote $D_m = A_1B_1C_1A_2B_2C_2 \cdots A_mB_mC_m$ for $1 \leq m \leq n-1$.

(i) $A_n = E_n^1E_n^2 \cdots E_n^{n-1}$ where $|E_n^1|$ is a multiple of $|D_1|$ with $|D_{n-1}| \leq |E_n^1| < |D_{n-1}| + |D_1|$, and E_n^1 is a repeated arrangement of D_1 (that is $E_n^1 = D_1D_1 \cdots D_1$). Inductively, for $2 \leq m \leq n-1$, $|E_n^m|$ is a multiple of $|D_m|$ with $|D_{n-1}E_2^1E_n^2 \cdots E_n^{m-1}| \leq |E_n^m| < |D_{n-1}E_n^1E_n^2 \cdots E_n^{m-1}| + |D_m|$, and E_n^m is a repeated arrangement of $|D_m|$.

(ii) $|B_n| = n \cdot |D_{n-1}A_n|$, and $|B_n|$ is a repeated arrangement of symbol 0 or a repeated arrangement of symbol 1, that is,

$$B_n = \overbrace{0 \cdots 0}^{n \cdot |D_{n-1}A_n|} \text{ or } \overbrace{1 \cdots 1}^{n \cdot |D_{n-1}A_n|} .$$

(iii) $C_n = F_n^1F_n^2 \cdots F_n^n$. where $|F_n^1|$ is a multiple of 2 with $n \cdot |D_{n-1}A_nB_n| \leq |F_n^1| < n \cdot |D_{n-1}A_nB_n| + 2$, and F_n^1 is a repeated arrangement of 01. Inductively, for $2 \leq m \leq n$, $|F_n^m|$ is a multiple of $2m$ with $n \cdot |D_{n-1}A_nB_nF_n^1F_n^2 \cdots F_n^{m-1}| \leq |F_n^m| < n.\ |D_{n-1}A_nB_nF_n^1F_n^2 \cdots F_n^{m-1}| + 2m$, and $|F_n^m|$ is a repeated arrangement of $\overbrace{0 \cdots 0}^{m}\overbrace{1 \cdots 1}^{m}$.

According to Lemma 2.1, one can choose a uncountable subset $E \subset \Sigma_2$ such that for all $x = x_1 \cdots x_n \cdots,\ y = y_1 \cdots y_m \cdots \in E,\ x \neq y$, there are infinite $n,\ m$ such that $x_n = y_m,\ x_n \neq y_m$. Define $\varphi : E \to \Sigma_2$ such that for all $x = x_1 \cdots x_n \cdots \in E,\ \varphi(x) = A_1B_1C_1A_2B_2C_2 \cdots A_iB_iC_i \cdots$, for all $i = 1, 2, 3 \cdots$.

$$B_i = \begin{cases} \underbrace{000\cdots 0}_{i|D_{i-1}A_i|}, & \text{if} x_i = 0, \\ \underbrace{111\cdots 1}_{i|D_{i-1}A_i|}, & \text{if} x_i = 1. \end{cases}$$

According to [18], it follows that $a = A_1B_1C_1A_2B_2C_2 \cdots A_nB_nC_n \cdots \in PQW(\sigma)$, suppose $\mathscr{T} = \omega(a, \sigma)$, then $\mathscr{T} \subset PQW(\sigma)$ and $\mathscr{T} \subset \varphi(E)$. In fact, if $y \in \mathscr{T}$ and $y = \sigma^n(a)$, then $y \in PQW(\sigma)$. Assume that $\lim\limits_{k\to\infty} \sigma^{n_k}(a)$, denote $\sigma^{n_k}(a) = a_{n_k}$, as $a \in PQW(\sigma)$, we have $a_{n_k} \in PQW(\sigma)$, as $a_{n_k} \to y(k \to \infty)$, by the uniform continuity of σ, for all $\varepsilon > 0$, there exists k_0, when $k > k_0$, there are $\rho(a_{n_k}, y) < \frac{\varepsilon}{4}$, $\rho(\sigma^n(a_{n_k}), \sigma^n(y)) < \frac{\varepsilon}{4}$, and $v(a_{n_k}, \frac{\varepsilon}{2}) \subset \nu(y, \varepsilon)$.

Next, we will prove that if $k > k_0$, $\sigma^n(a_{n_k}) \in v(a_{n_k},\ \frac{\varepsilon}{4})$, then $\sigma^n(y) \in \nu(a_{n_k},\ \frac{\varepsilon}{2})$. Suppose $\sigma^n(a_{n_k}) \in v(a_{n_k},\ \frac{\varepsilon}{4}) \Rightarrow \rho(a_{n_k}, \sigma^n(a_{n_k})) < \frac{\varepsilon}{4}$, $\rho(\sigma^n(y), a_{n_k}) < \rho(\sigma^n(y), \sigma^n(a_{n_k})) + \rho(\sigma^n(a_{n_k}), a_{n_k}) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$, hence $\sigma^n(y) \in \nu(a_{n_k}, \frac{\varepsilon}{2}) \subset \nu(y, \varepsilon)$. As $a_{n_k} = \sigma^{n_k}(a) \in PQW(f)$, so

$$\limsup_{n\to\infty} \frac{1}{n}[\{n|\sigma^n(a_{n_k}) \in v(a_{n_k}, \frac{\varepsilon}{2})\} \cap \{1, 2 \cdots\}] > 0,$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n}[\{n|\sigma^n(y) \in v(y, \varepsilon)\} \cap \{1, 2 \cdots\}] > 0.$$

i.e. $y \in PQW(f) \Rightarrow \omega(a, \sigma) \subset PQW(\sigma)$.

(1) For the conclusion (1) of theorem 3.1, it is obvious that $\sigma \mid_{\mathscr{T}}$ is topological transitive by the construction of $\mathscr{T}$, and it is enough to prove that $\sigma \mid_{\mathscr{T}}$ has initial value sensitive dependence.
Let $\sigma_0 = \frac{1}{4}$, for all $x \in \mathscr{T},\ x = x_1 \cdots x_n \cdots$, v is any neighbourhood of x, it is enough to prove that if there exists $y = y_1 \cdots y_n \cdots \in v$ such that $x \neq y$. Let $k = min\{n > 0 | x_n \neq y_n\} + 1$, then $\rho(\sigma^k(x), \sigma^k(y)) > \frac{1}{4}$, So we just need to prove the existence of y. We discuss the following in two cases.

 (i) Assume that there exists n such that $x = \sigma^n(a)$, as $x \in PQW(\sigma)$ and the definition of $PQW(\sigma)$, for all $\varepsilon > 0$ and any neighborhood of x, $V(x, \varepsilon) = \{n|\sigma^n(x) \in V(x, \varepsilon)\}$ is the positive upper density set, then for all $q \geq 1$, there exists $r \geq q$ such that $\rho(\sigma^r(x), x) < \varepsilon$, i.e. $\rho(\sigma^r(\sigma^n(a)), x) < \varepsilon$.
 Let $y = \sigma^r(\sigma^n(a)) = \sigma^{r+n}(a)$, then $y \in PQW(\sigma)($ as $\sigma(QW(\sigma) = QW(\sigma)))$ and $y \neq x$, else if $x = y$, then $\sigma^{n+r}(a) = \sigma^r(x) = y = x$, that is $x \in p(\sigma)$, which is a contradiction with $x \in PQW(\sigma)$.

Hence, for any r, there exists k such that $\sigma^{r+n}(a)$ is different from the $(k+1)$th symbol of x, therefore $\rho(\sigma^k(\sigma^{r+1}(a)), \sigma^k(x)) = \rho(\sigma^k(y), \sigma^k(x)) = 1 > \frac{1}{4}$.

(ii) If for all n, $\sigma^n(a) \neq x, x \in \mathscr{T}$, there exists an increasing sequence $\{m_i\}$, $m_i \to \infty$, such that $\lim\limits_{i\to\infty} \sigma^{m_i}(a) = x$. Thus for any $\varepsilon > 0$, there exists i_0, $\rho(\sigma^{m_i}(a), x) < \varepsilon$ if $i \geq i_0$. Let $y = \sigma^{m_{i_0}}(a)$, as for any $n \geq 0$, $\sigma^n(a) \neq x$, thus for $i > i_0$, $\sigma^{m_i}(a) \neq x$, there exist $k(i)$, $i > i_0$ such that $\sigma^{m_{i_0}}(a)$ is different from the $(k+1)$th symbol of x. Therefore $\rho(\sigma^k(\sigma^{m_{i_o}})(a), \sigma^k(a)) = \rho(\sigma^k(y), \sigma^k(x)) = 1 > \frac{1}{4}$.

In summary, $\sigma|_{\mathscr{T}}$ is $R-T$ chaos.

(2) For the conclusion (2) of Theorem 3.1, firstly, according to the proof of ii , $\sigma|_{\mathscr{T}}$ is sensitive. Secondly, by the construction of $\mathscr{T}$, for any a, $b \in \mathscr{T}$, $a = A_1B_1C_1A_2B_2C_2 \cdots A_nB_nC_n \cdots$, $b = A_1'B_1'C_1'A_2'B_2'C_2' \cdots A_n'B_n'C_n' \cdots$, there exist infinitely many n and m such that $B_n \neq B_n'$ and $B_m \neq B_m'$, thus for any $x \in \mathscr{T}$, there exists $r(x_0) = \frac{1}{4} > 0$ and for any $\varepsilon > 0$, there exist $y_0 \in \mathscr{T}$ and n such that $\rho(x_0, y_0) < \varepsilon$, $\rho(\sigma^n(x_0), \sigma^n(y_0)) > \frac{1}{4}$, i.e. for any $x_0 \in \mathscr{T}$, $orb^+(x_0)$ is unstable with respect to $\mathscr{T}$.

In summary, $\sigma|_{\mathscr{T}}$ is *Martelli* chaos.

(3) For the conclusion (3) of Theorem 3.1, according to [18], $\sigma|_{\mathscr{T}}$ is distributional chaos.

(4) For the conclusion (4) of Theorem 3.1, according to Proposition 2.1, $\sigma|_{\mathscr{T}}$ is DC_2, DC_3, DC_2'.

(5) For the conclusion (5) of Theorem 3.1, as $\omega(a,\sigma) = \overline{\omega(a,\sigma)}$ and $f(\omega(a,\sigma)) = \omega(a,\sigma)$, then $(\omega(a,\sigma), f)$ is the sub-system of (Σ_2, f). Let V_1^ω,V_2^ω,U_1^ω,U_2^ω be nonempty open set of $\omega(a,\sigma)$, U_1, U_2 are nonempty open set of Σ_2 and $\omega(a,\sigma) \cap U_i \neq \emptyset$, $i = 1,2$. Next we will prove there exists $m \in N$ such that $\sigma^m(V_i^\omega) \cap U_i^\omega \neq \emptyset$, $i = 1,2$.

Because $\omega(a,\sigma)$ is transitive, there exist $m_i \in N$, $i = 1,2$ such that $\sigma^{m_i}(V_i^\omega) \cap U_i^\omega \neq \emptyset$, $i = 1,2$. Also because V_i^ω, $i = 1,2$ are open sets in Σ_2. Hence there is $([x_1^i x_2^i \cdots x_{N_i}^i]) \subset V_i^\omega$, $i = 1,2$ and $\sigma^{N_i}([x_1^i \cdots x_{N_i}^i]) = \Sigma_2$, $i = 1,2$, $\sigma^{N_i}(V_i^\omega) = \Sigma_2$, thus $\sigma^{N_i}(V_i^\omega) \cap U_i^\omega = U_i^\omega \neq \emptyset$, $i = 1,2$. Take $m = max\{N_1, N_2\}$, then $\sigma^m(V_i^\omega) \cap U_i^\omega \neq \emptyset$, $i = 1,2$, that is $\omega(a,\sigma)$ is topological weakly mixing.

(6) For the conclusion (6) of Theorem 3.1, according to Lemma 2.2, $\sigma|_{\mathscr{T}}$ is Xiong-chaos .

(7) For the conclusion (7) of Theorem 3.1, according to [19], $\sigma|_{\mathscr{T}}$ is distributional chaos in a sequence.

□

Theorem 3.2. *Let (X,d) be compact metric space and $f : X \to X$ be continuous map, $\Phi : X \to \Sigma_2$ is topological conjugate between f and σ. If there exists $x \in \Sigma_2$ such that $orb^+(x) \neq \Sigma_2$ and $\sharp(\Phi^{-1}[x]) = 1$, then*

(1) $f|_{PQW(f)}$ *is* DC_1.
(2) $f|_{PW(f)}$ *is* DC_1.
(3) $f|_{A(f)}$ *is* DC_1.

Proof. For the conclusion (1) of Theorem 3.2, $\Phi : X \to \Sigma_2$ is the topological semi-conjugate of f and σ. According to Lemma 2.3, for any $y \in \mathscr{T} \subset PQW(\sigma)$, there exists $x \in PQW(f)$ such that $\Phi(x) = y$. Let $D = \{x|\Phi(x) = y, \ \forall y \in \mathscr{T}\}$, then $D \subset PQW(f)$ and there exists $y \in PQW(\sigma) \subset \Sigma_2$ such that $orb^+(y) \neq \Sigma_2$ and $\Phi^{-1}\{(y)\} = \sharp\{x\} = 1$. By Lemma 2.4, $f|_{PQW(f)}$ is distributional chaos.

Similarly, the conclusion (2)(3) holds. □

Theorem 3.3. *Let* $f \in C^o(I)$*, if* $ent(f) > 0$*, then*
(1) $f|_{PQW(f)}$ *is* DC_1.
(2) $f|_{PW(f)}$ *is* DC_1.
(3) $f|_{PA(f)}$ *is* DC_1.
(4) $f|_{\mathscr{T}}$ *is topological weakly mixing.*
(5) $f|_{\mathscr{T}}$ *is chaos in sense of Xiong.*
(6) $f|_{\mathscr{T}}$ *is mean* $Li - Yorke$ *chaos.*

Proof. (1) If $ent(f) > 0$, according to [4], there exist $N > 0$ and a compact subset $\Lambda \subset I$ such that $f^N(\Lambda) = \Lambda$ and continuous surjective $\Phi : \Lambda \to \Sigma_2$ satisfy $\Phi \circ f^N|_\Lambda = \sigma \circ \Phi|_\Lambda$. By theorem 3.1, there is sub-system consisting of proper quasi weakly almost periodic sets $\mathscr{T} \subset PQW(\sigma) \subset \Sigma_2$ and $\sigma|_{\mathscr{T}}$ is DC_1.

According to Lemma 2.3, for every $y \in \mathscr{T} \subset PQW(\sigma)$ there is $x \in PQW(f^N)$ such that $\Phi(x) = y$. Let $D = \{x|x \in \mathscr{T} \subset PQW(f^N), \ \Phi(x) = y, \ y \in \mathscr{T} \subset PQW(\sigma)\}$, then $D \subset \Lambda$. It is obvious that if $y \in \mathscr{T}$, $orb^+(y) \neq \Sigma_2, \Phi^{-1}(y) = \sharp\{x\} = 1$. Therefore $f^N|_D$ is DC_1. By Lemma 2.5, $f|_D$ is DC_1.

(2) Similarly, conclusion (2)(3) of 3.3 are hold.
(3) By Lemma 2.12, $f|_{\mathscr{T}}$ is topological weakly mixing.
(4) By [20] and (4) of Theorem3.3, $f|_{\mathscr{T}}$ is chaos in sense of Xiong.
(5) According to Lemma 2.7, $f|_{\mathscr{T}}$ mean $Li - Yorke$ chaos.

□

4. Application of an Economic Model

An economic system is an evolutionary system, which is the object of study of dynamical systems. After an economic mathematical model is built, it can be judged by dynamical systems method whether it is simple or complex in some sense. They have different perspectives and focus, such as chaos, entropy, hybridity, etc. In this paper, we focus on the complexity of economic models from the perspective of entropy and chaos.

First, we give the economic model under discussion, which is described in more detail in [5, 2, 3, 9, 13].

Suppose that there is an economic model with two individuals $A,\ B$ and two goods $x,\ y$, where the preference functions of $A,\ B$ are $x^\alpha y^{1-\alpha}$, $0<\alpha<1$ and $x^\beta y^{1-\beta}$, $0<\beta<1$, respectively. Also, the endowments of $A,\ B$ are $(0,x^0)$ and $(0,y^0)$, where $x^0>0$ and y^0, respectively. Let p be the price of x with respect to y, then the excess demand function for good x is

$$Z(p)=\frac{\beta y^0}{p}-(1-\alpha)x^0. \tag{4}$$

It is easy to find the fix point of $Z(p)$, $p^*=\frac{\beta y^0}{(1-\alpha)x^0}$, let

$$p(t+1)=p(t)+\gamma\ Z(p(t)). \tag{5}$$

where $\gamma>0$ is the rate of price adjustment (which we assume to be constant here), and from $p(t+1)=f(p(t))$, the equation (5) can be written as

$$f(p)=p+\gamma\ Z(p)=p+\gamma(\frac{\beta y^0}{p}-(1-\alpha)x^0).$$

It is easy to obtain that $p=\bar{p}=\sqrt{\gamma\beta y^0}$ is a minimal point of f and the minimal value $f(\bar{p})=2\sqrt{\gamma\beta y^0}-\gamma(1-\alpha)x^0$.

In order to make function f has defined for any $p>0$, it demands for all $p>0$, $f(p)>0$, so only need the minimum $f(\bar{p})=2\sqrt{\gamma\beta y^0}-\gamma(1-\alpha)x^0>0$, that is to say

$$4>\frac{\gamma((1-\alpha)x^0)^2}{\beta y^0}. \tag{6}$$

Let $K=\frac{\gamma((1-\alpha)x^0)^2}{\beta y^0}$, then $K=\bar{p}^2/p^*$. The following fixes the values of all parameters except γ:

$$\beta y^0=1,\ (1-\alpha)x^0=6.$$

Then $K=36\gamma$, the formula (5) reduces to

$$p(t+1)=p(t)+\frac{(\frac{1}{p(t)}-6)K}{36} \tag{7}$$

From $q=\frac{1}{p}$ and formula (7),

$$q(t+1)=\frac{1}{p(t+1)}=\frac{36q(t)}{36+K(q(t)-6)q(t)}=g(q(t)),$$

Denote $g_k(q)=\frac{36q}{36+kq^2-6kq}$, in the following we discuss the chaoticity of $g_k(\cdot)$.

Theorem 4.1. *There exists* $2.78<k<4$ *and the interval* I_k*, such that* $g_k:I_k\to I_k$*, then:*

(1) g_k *has uncountable distributional chaos sets in* $PA(g_k)$*.*

(2) g_k *has uncountable distributional chaos sets in* $PW(g_k)$*.*

(3) g_k *has uncountable distributional chaos sets in* $PQW(g_k)$.

(4) The subsystem $(I_k^* \cdot g \mid_{I_k^*})$ *consisting of* $PQW(g_k)$ *is a* $R-T$ *chaos.*

(5) $g_k \mid_{\mathscr{T}\subset PQW(g_k)}$ *is topological weakly mixing on a subsystem consisting of* $PQW(g_k)$.

(6) $g_k \mid_{\mathscr{T}\subset PQW(g_k)}$ *is chaos in sense of Xiong on a subsystem consisting of* $PQW(g_k)$.

(7) $g_k \mid_{\mathscr{T}\subset PQW(g_k)}$ *is mean* $Li-Yorke$ *chaos.*

Proof. (1) To prove that the conclusions of the above (1) to (6) hold, according to theorem 3.3, it is enough to prove that $ent(g_k) > 0$ on the interval $I_k(2.78 < k < 4)$. The following proves that $q = \frac{6}{\sqrt{k}}$ is the maximum point of $g_k(q)$. Because

$$g'_k(q) = (\frac{36q}{36 + q^2k - 6kq})' = \frac{36(36 + q^2k - 6kq) - (2qk - 6k)36q}{(36 + q^2k - 6kq)^2}.$$

Let $g'_k(q) = 0$, and solve $\overline{q}_k = \sqrt{\frac{36}{k}} = \frac{6}{\sqrt{k}}$, so $\overline{q}_k = \frac{6}{\sqrt{k}}$ is the stationary point of $g_k(q)$.

When $q < \overline{q}_k$, it is obvious that there is $g'_k(q) > 0$, i.e., $g'_k(q)$ is monotonically increasing when $q < \overline{q}_k$. Similarly, $q > \overline{q}_k$, we have $g'_k(q) < 0$, that is, $g'_k(q)$ is monotonically decreasing when $q > \overline{q}_k$.

Thus, $\overline{q}_k$ is a maximal point of $g_k(q)$, and since $\overline{q}_k$ is the only stationary point of $g_k(q)$, $\overline{q}_k$ is a maximal point of $g_k(q)$ and the maximum value is $b(k) = g_k(\overline{q}_k)$.

Let $b(k) = g_k(\overline{q}_k), a(k) = min\{g_k(b(k))\overline{q}_k\}$, since g_k has a unique fix point 6, so $\overline{q}_k \neq g_k(\overline{q}_k)$. Therefore, we can set $I_k = [a(k), b(k)]$, and obviously $g_k(\cdot) : I_k \to I_k$ is continuous.

It follows that $ent(g_k) > 0$, for which it is sufficient to prove that

(i) there exists $m \in (a(k), b(k))$ such that g_k in increasing on $(a(k), m)$ and decreasing on $(m, b(k))$.

For this purpose we let $g_k(b(k)) < \overline{q}_k$, i.e., $\frac{6(2-\sqrt{k})}{\sqrt{k}(5+2k-6\sqrt{k})} < \frac{6}{\sqrt{k}}$, the solution gives $2.78 < k < 4$ and $I_k = [g_k(b(k)), g_k(\overline{q}_k)]$. Let $m = \overline{q}_k$, then

$$a(k) = g_k(b(k)) < \overline{q}_k = m < 6 = q = g_k(q) < g_k(\overline{q}_k) = b(k), \quad (8)$$

$g_k(q)$ is strictly increasing on $[a(k), m]$ and strictly decreasing on $[m, b(k)]$.

(ii) Prove that $g_k(a(k)) > a(k), g_k(b(k)) < b(k)$ and for all $q \in (a(k), m)$, $g_k(q) > q$.

Let $g(q) = q, f(q) = g_k(q) - q$, $q = 6$ is the only fix point of $g_k(\cdot)$, so $q = 6$ is the only real root of f. And $g(q)$ and $g_k(q)$ have a unique intersection at $q = 6$, and by $m = \overline{q}_k = \frac{6}{\sqrt{k}} <$

$q = 6$ we know that $g_k(q)$ is monotonically increasing on $[a(k), m]$ and has no intersection. So, $g_k(q)$ must be above $g(q)$. Therefore, $f(q) = g_k(q) - g(q) > 0$. That is, for any $q \in [a(k),\ \overline{q}_k]$, there is $g_k(q) > q, g_k(a(k)) > a(k)$. And by equation (8) we know that $6 = q = g_k(q) < b(k)$, so we know that $g_k(b(k)) < b(k)$.

(iii) It proves that $g_k^2(m) < m, g_k^3(m) < 6$, where 6 is the only fix point of $g_k(q)$. By Lemma 2.9 we have $ent(f) > 0$.

(2) For conclusion (7), let $q \in (QW(g_k) - A(g_k))$, then $q \in R(g_k)$, and thus $q \in \omega(q, g_k) = \overline{orb(q, g_k)}$. Let $I_k^* = \omega(q, g_k)$, then the dynamical system (I_k^*, g_k) is topological transitive. By Lemma 2.10, we know that the subsystem (I_k^*, g_k) is $R - T$ chaos.

□

The result shows that it is unlikely that a real economic system will experience chaos in the exact set of periodic points, and if it does, it will be an illusion. Therefore, some approximation is generally considered, such as almost periodic, weakly almost periodic, and quasi weakly almost periodic. And these recurrent points are generally the points that indicate the stability of the system. But the discussion of this economic system shows us that a simple exchange economic system is also very complex, and even in the seemingly stable set of recurrent points, chaos, mixing and other complex phenomena can occur.

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