t-PRIME SUBMODULES

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Let R be a commutative ring with identity. For $t \in \mathbb{N}$, a proper submodule N of an R-module M is called a t-prime submodule if $rm \in N$ $(r \in R, m \in M)$, then $m \in N$ or $r^t \in (N :_R M)$. We show that any maximal t-prime submodule, with respect to inclusion, is prime and a proper submodule is a t-prime submodule if and only if its quotient module is t-torsion free. We obtain some characterizations of t-prime submodules. Also various properties of t-prime submodules are investigated. We provide several examples which illustrate our results.

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1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. (When the ring R has an identity, say 1, an R-module M is called unitary, if for any $x \in M$, $1x = x$.) Also $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ will denote, respectively, the set of positive integers, the ring of integers and the field of rational numbers. If N is an R-submodule of M, the annihilator of the R-module M/N is defined by $Ann_R(M/N) = (N :_R M) = \{r \in R | rM \subseteq N\}$. Thus the annihilator of M, is denoted by $Ann_R(M)$, which is the same as $(0 :_R M)$. For a subset X of an R-module M, $(X :_R M) = \{r \in R | rM \subseteq X\}$, which is a subset of R. Suppose that I is an ideal of R. We denote the radical of I by $\sqrt{I} = \{a \in R | a^n \in I \ \ for \ \ some \ \ n \in \mathbb{N}\}$.

Recall that a proper submodule N of M is called prime ($primary$) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N :_R M)$ ($r^n \in (N :_R M)$, for some $n \in \mathbb{N}$)(see [4], [7], [8]). Several generalizations of prime and primary submodules, like strongly prime, weakly prime, semi prime, quasi prime submodules were studied in literature. Also the notions 2-prime ideals and graded 2-prime submodules were defined and studied (see [5], [11], [2]). Recall that, a proper ideal P of a ring R is called a 2-prime ideal, if $a, b \in R$ with $ab \in P$, implies $a^2 \in P$ or $b^2 \in P$ (see [5]).

We fix $t \in \mathbb{N}$. In this paper, we introduce and study the notions of t-prime submodules and t-torsion free modules. Furthermore we introduce the notion of a t-prime ideal which generalize the 2-prime ideals. In Section 2, we investigate some properties of t-prime submodules and also obtain some basic structural results. We show that a t-prime submodule is a s-prime submodule, for any $s \geq t$. Moreover we show that any maximal t-prime submodule, with respect to inclusion, is prime. On the other hand, it is shown that a proper submodule N of M is a t-prime submodule if and only if M/N is a t-torsion free module. We establish several connections between t-prime submodules and other notions in modules theory. To this end, it is shown that, for ring extension $f : R \to S$, such that S is a free R-module

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and N is a submodule of an R-module M, N is a t-prime submodule of M if and only if $N \bigotimes_R S$ is a t-prime R-submodule of $M \bigotimes_R S$. In Section 3, we provide several examples, for the notions and the necessary conditions of some propositions in the former sections.

2. t-Prime Submodules

In this section, we introduce the notion t-prime submodule, for $t \in \mathbb{N}$ and give some characterizations of it. Moreover, various properties of t-prime submodules are proved.

Definition 2.1. *A proper submodule N of a module M over a commutative ring R is said to be a t-prime submodule, if for $a \in R$ and $x \in M$ with $ax \in N$, then $x \in N$ or $a^t \in (N :_R M)$. Also a proper ideal I of R is called a t-prime ideal, if for $r, s \in R$ with $rs \in I$, then $r^t \in I$ or $s^t \in I$.*

Lemma 2.1. *(i) Any proper subspace of a vector space is a t-prime submodule.*
(ii) Any prime submodule is a t-prime submodule.

Proof. (i) Let V be a F-vector space and W be a proper subspace of V. Assume that $a \in F$ and $x \in V$ such that $ax \in W$. Since F is a field, $a = 0$ or a has an inverse in F. If $a = 0$, then $a^t = 0 \in (W :_F V)$. If a has an inverse in F, then $x = a^{-1}(ax) \in W$. Therefore W is a t-prime submodule.
(ii) Let N be a prime submodule of an R-module M. Assume that $r \in R$ and $x \in M$ such that $rx \in N$. Since N is a prime submodule, $x \in N$ or $r \in (N :_R M)$. If $r \in (N :_R M)$, then $r^t \in (N :_R M)$. Thus N is a t-prime submodule. □

Proposition 2.1. *Let M be a finitely generated or faithfully flat R-module. Then M has a t-prime submodule.*

Proof. By hypothesis, M has a maximal submodule N. Then N is a prime submodule and therefore by Lemma 2.1(ii), N is a t-prime submodule. □

Let I be a non-empty subset of R. We denote $\sqrt[t]{I} = \{r \in R | r^t \in I\}$.
Note that a proper submodule N of an R-module M is a t-prime submodule, if for $a \in R$ and $x \in M$ with $ax \in N$, then $x \in N$ or $a \in \sqrt[t]{(N :_R M)}$.

Definition 2.2. *(See [9]) A proper ideal I of R is called semiprime, if whenever $a^n \in I$ for $a \in R$ and $n \in \mathbb{N}$, then $a \in I$.*

Lemma 2.2. *Let I and J be ideals of R and $t \in \mathbb{N}$. Then the following statements hold:*
(i) $I \subseteq \sqrt[t]{I}$.
(ii) $I = R$ if and only if $\sqrt[t]{I} = R$.
(iii) If $I \subseteq J$, then $\sqrt[t]{I} \subseteq \sqrt[t]{J}$.
(iv) $\sqrt[t]{I \cap J} = \sqrt[t]{I} \cap \sqrt[t]{J}$.
(v) For all $s \geq t$, $\sqrt[t]{I} \subseteq \sqrt[s]{I}$.
(vi) $\sqrt[t]{\sqrt[s]{I}} = \sqrt[s]{\sqrt[t]{I}} = \sqrt[ts]{I}$.
(vii) $\sqrt[t]{I} \subseteq \sqrt{I}$.
(viii) $\sqrt[t]{\sqrt{I}} = \sqrt{I}$.
(ix) If I is a semiprime or radical ideal, then $\sqrt{I} = \sqrt[s]{I} = I$, for any $s \in \mathbb{N}$.

Proof. The proofs of parts (i), (iii), (iv), (v) and (vii) are clear.
(ii) Assume that $I = R$. Then by part (i), $\sqrt[t]{I} = R$. Now assume that $\sqrt[t]{I} = R$. Then $1 = 1^t \in I$ and so $I = R$.
(vi) $a \in \sqrt[s]{\sqrt[t]{I}}$ if and only if $a^s \in \sqrt[t]{I}$ if and only if $a^{st} \in I$ if and only if $a \in \sqrt[st]{I}$. So

$\sqrt[s]{\sqrt[t]{I}} = \sqrt[ts]{I}$. Similarly $\sqrt[t]{\sqrt[s]{I}} = \sqrt[ts]{I}$.
(viii) By part (i), $\sqrt{I} \subseteq \sqrt[t]{\sqrt{I}}$. Let $r \in \sqrt[t]{\sqrt{I}}$. Then $r^t \in \sqrt{I}$ and so there exists $n \in \mathbb{N}$ such that $r^{tn} = (r^t)^n \in I$. Thus $r \in \sqrt{I}$ and hence $\sqrt[t]{\sqrt{I}} \subseteq \sqrt{I}$. Therefore $\sqrt[t]{\sqrt{I}} = \sqrt{I}$.
(ix) Let I be a semiprime or radical ideal. Then $I = \sqrt{I}$ and so by part (viii), $\sqrt[s]{I} = \sqrt[s]{\sqrt{I}} = \sqrt{I} = I$, for any $s \in \mathbb{N}$. □

Definition 2.3. *(See* [13]*) The ring R (with identity, not necessarily commutative) is called a von Neumann regular ring, if for any element $a \in R$, there exists $b \in R$ such that $a = aba$.*

Proposition 2.2. *(i) A t-prime submodule is a primary submodule.*
(ii) A primary submodule N of an R-module M such that $(N :_R M)$ is a semiprime or radical ideal is a t-prime submodule.
(iii) A t-prime submodule is a s-prime submodule, for any $s \geq t$.
(iv) A t-prime submodule N of an R-module M such that $(N :_R M)$ is a semiprime or radical ideal; is a s-prime submodule, for any $s \in \mathbb{N}$.
(v) If the ring is a von Neumann regular ring, then the notions prime, t-prime and primary submodules coincide.

Proof. (i) By Lemma 2.2 (vii), it is clear.
(ii) Let N be a primary submodule of an R-module M such that $(N :_R M)$ is a semiprime or radical ideal. Assume that $r \in R$ and $x \in M \backslash N$ with $rx \in N$. Since N is primary, we have $r \in \sqrt{(N :_R M)}$. So $r^s \in (N :_R M)$, for some $s \in \mathbb{N}$. Thus $r \in (N :_R M)$ and hence by Lemma 2.2 (i), $r \in \sqrt[t]{(N :_R M)}$. Therefore N is a t-prime submodule.
(iii) Let N be a t-prime submodule of an R-module M and $rx \in N$, for $r \in R$ and $x \in M \backslash N$. Since N is t-prime, $r \in \sqrt[t]{(N :_R M)}$. So by Lemma 2.2 (v), $r \in \sqrt[s]{(N :_R M)}$, for any $s \geq t$. Therefore N is a s-prime submodule, for any $s \geq t$.
(iv) Let N be a t-prime submodule of an R-module M such that $(N :_R M)$ is a semiprime or radical ideal. Assume that $r \in R$ and $x \in M \backslash N$ with $rx \in N$. Since N is t-prime, $r \in \sqrt[t]{(N :_R M)}$. So by Lemma 2.2 (ix), $r \in \sqrt[s]{(N :_R M)}$, for any $s \in \mathbb{N}$. Therefore N is a s-prime submodule.
(v) As the notions prime and primary submodules coincide, when the ring is a von Neumann regular ring, by Lemma 2.1(ii), then the proof follows by part (i). □

Proposition 2.3. *Every direct summand of a torsion free module, is a t-prime submodule.*

Proof. Let N be a direct summand of a torsion free R-module M. Then there exists an R-submodule L of M such that $M = N \oplus L$. Let $rx \in N$, for $r \in R$ and $x \in M$. So there exist $n \in N$ and $\ell \in L$ such that $x = n + \ell$. Hence $r\ell = 0$ and as M is torsion free $\ell = 0$ or $r = 0$. If $\ell = 0$, then $x = n \in N$ and if $r = 0$ then $r \in \sqrt[t]{(N :_R M)}$. Therefore N is a t-prime submodule. □

Proposition 2.4. *Let N be a t-prime submodule of an R-module M. Then*
(i) $(N :_R M)$ is a t-prime ideal of R.
(ii) $\sqrt{(N :_R M)}$ is a prime ideal of R.

Proof. (i) Let $rs \in (N :_R M)$, for $r, s \in R$. Then $rsM \subseteq N$ and so, for $x \in M \backslash N$, $rsx \in N$. Therefore, N is t-prime, $sx \in N$ or $r \in \sqrt[t]{(N :_R M)}$. Hence $s \in \sqrt[t]{(N :_R M)}$ or $r \in \sqrt[t]{(N :_R M)}$.
(ii) Let $rs \in \sqrt{(N :_R M)}$, for $r, s \in R$. Then there exists $n \in \mathbb{N}$ such that $r^n s^n \in (N :_R M)$. So by part (i), $r^{nt} \in (N :_R M)$ or $s^{nt} \in (N :_R M)$. Thus $r \in \sqrt{(N :_R M)}$ or $s \in \sqrt{(N :_R M)}$. It follows that $\sqrt{(N :_R M)}$ is a prime ideal of R. □

Theorem 2.1. *Let M be an R-module and N be a proper submodule of M. Then the following statements are equivalent:*
(i) N is a t-prime submodule of M.
(ii) If for an ideal I of R and a submodule L of M, $IL \subseteq N$, then $L \subseteq N$ or $I \subseteq \sqrt[t]{(N :_R M)}$.
(iii) $N = (N :_M r)$ or $r \in \sqrt[t]{(N :_R M)}$, for any $r \in R$.
(iv) $Rx \subseteq N$ or $(N :_R x) \subseteq \sqrt[t]{(N :_R M)}$, for any $x \in M$.
(v) $N = \{m \in M | rm \in N\}$, for all $r \in R \backslash \sqrt[t]{(N :_R M)}$.
(vi) $N = \{m \in M | Jm \subseteq N\}$, for any ideal J of R such that $J \not\subseteq \sqrt[t]{(N :_R M)}$.
(vii) $(N :_R m) \subseteq \sqrt[t]{(N :_R M)}$, for any $m \in M \backslash N$.
(viii) $(N :_R L) \subseteq \sqrt[t]{(N :_R M)}$, for any submodule L of M such that $N \subset L$.
(ix) $ann_R(m + N) \subseteq \sqrt[t]{(N :_R M)}$, for all $m \in M \backslash N$.
(x) $Z_R(M/N) \subseteq \sqrt[t]{(N :_R M)}$.
(xi) $N = \{m \in M | rm \in N,\ for\ some\ r \in R \backslash \sqrt[t]{(N :_R M)}\}$.

Proof. $(i) \Rightarrow (ii)$ Let N be a t-prime submodule and $IL \subseteq N$, for an ideal I of R and submodule L of M. If $L \not\subseteq N$, then there exists $x \in L \backslash N$. For any $r \in I$, $rx \in IL \subseteq N$. So as N is t-prime, $r \in \sqrt[t]{(N :_R M)}$. Therefore $I \subseteq \sqrt[t]{(N :_R M)}$.
$(ii) \Rightarrow (iii)$ Assume that for $r \in R$, $N \neq (N :_M r)$. So there exists $x \in (N :_M r) \backslash N$. Put $L = (N :_M r)$ and $I = Rr$. Then $IL \subseteq N$ and by part (ii), $r \in \sqrt[t]{(N :_R M)}$.
$(iii) \Rightarrow (iv)$ Assume that for $x \in M$, $Rx \not\subseteq N$. For any $r \in (N :_R x)$, $rx \in N$ and so $x \in (N :_M r) \backslash N$. Thus by part (iii), $r \in \sqrt[t]{(N :_R M)}$. Therefore $(N :_R x) \subseteq \sqrt[t]{(N :_R M)}$.
$(iv) \Rightarrow (v)$ Let $r \in R \backslash \sqrt[t]{(N :_R M)}$ and $m \in M$ with $rm \in N$. If $m \notin N$, then as $r \in (N :_R m)$, by part (iv), $r \in \sqrt[t]{(N :_R M)}$, which is a contradiction. Hence $m \in N$ and therefore $N = \{m \in M | rm \in N\}$.
$(v) \Rightarrow (vi)$ Let J be an ideal of R such that $J \not\subseteq \sqrt[t]{(N :_R M)}$. Then there exists $r \in J \backslash \sqrt[t]{(N :_R M)}$. So by part (v), $N = \{m \in M | rm \in N\}$. Hence $N \subseteq \{m \in M | Jm \subseteq N\} \subseteq \{m \in M | rm \in N\} = N$. Therfore $N = \{m \in M | Jm \subseteq N\}$.
$(vi) \Rightarrow (vii)$ Let for $m \in M \backslash N$, there exists $r \in (N :_R m) \backslash \sqrt[t]{(N :_R M)}$. Then $Rrm \subseteq N$ and by part (vi), $m \in N$, which is a contradiction. Therefore $(N :_R m) \subseteq \sqrt[t]{(N :_R M)}$.
$(vii) \Rightarrow (viii)$ Let L be a submodule of M such that $N \subset L$. Then there exists $m \in L \backslash N$. So by part (vii), $(N :_R m) \subseteq \sqrt[t]{(N :_R M)}$ and hence $(N :_R L) \subseteq (N :_R m) \subseteq \sqrt[t]{(N :_R M)}$. Therefore $(N :_R L) \subseteq \sqrt[t]{(N :_R M)}$.
$(viii) \Rightarrow (ix)$ Let $m \in M \backslash N$ such that $rm \in N$. Then $r \in (N :_R Rm + N)$. So by part (viii), $r \in \sqrt[t]{(N :_R M)}$. Therefore $ann_R(m + N) \subseteq \sqrt[t]{(N :_R M)}$.
$(ix) \Rightarrow (x)$ Let $r \in Z_R(M/N)$. Then there exists $m \in M \backslash N$ such that $rm \in N$. So $r \in Ann_R(m + N)$ and hence by part (ix), $r \in \sqrt[t]{(N :_R M)}$. Therefore $Z_R(M/N) \subseteq \sqrt[t]{(N :_R M)}$.
$(x) \Rightarrow (xi)$ Let $r \in R \backslash \sqrt[t]{(N :_R M)}$ such that $rm \in N$. If $m \notin N$, then by part (x), $r \in Z_R(M/N)$, which is a contradiction. So $m \in N$ and therefore $N = \{m \in M | rm \in N,\ for\ some\ r \in R \backslash \sqrt[t]{(N :_R M)}\}$.
$(xi) \Rightarrow (i)$ Let $r \in R$ and $m \in M$ such that $rm \in N$. If $r \notin \sqrt[t]{(N :_R M)}$, then by part (xi), $m \in N$. Thus N is a t-prime submodule of M. □

Proposition 2.5. *Let N be a submodule of an R-module M such that $(N :_R M)$ is a semiprime or radical ideal of R. Then N is a prime submodule if and only if N is a t-prime submodule.*

Proof. Let N be a prime submodule. Then by Lemma 2.1(ii), N is a t-prime submodule. For the converse, assume that N is a t-prime submodule. Then by Lemma 2.2(ix), N is a prime submodule. □

Proposition 2.6. *Let N be a t-prime submodule of an R-module M and I be a non-empty subset of R such that $I \not\subseteq (N :_R M)$. Then $(N :_M I)$ is a t-prime submodule of M.*

Proof. Let $r \in R$ and $x \in M$ such that $rx \in (N :_M I)$. Then $Irx \subseteq N$. Since N is t-prime, $Ix \subseteq N$ or $r^t \in (N :_R M)$. If $Ix \subseteq N$, then $x \in (N :_M I)$. If $r^t \in (N :_R M)$, then $r^t \in ((N :_M I) :_R M)$. Therefore $(N :_M I)$ is a t-prime submodule. □

Corollary 2.1. *Let N be a t-prime submodule of an R-module M. Then for any $r \in R$, $(N :_M r) = M$ or $(N :_M r)$ is a t-prime submodule of M.*

Proof. It is enough to put $I = Rr$ in Proposition 2.6. □

Theorem 2.2. *Any maximal t-prime submodule, with respect to inclusion, is prime.*

Proof. Let N be a maximal t-prime submodule of an R-module M. Assume that $r \in R\backslash(N :_R M)$ and $x \in M$ with $rx \in N$. Then $(N :_M r) \neq M$. By Corollary 2.1, $(N :_M r)$ is a t-prime submodule. Now as $N \subseteq (N :_M r)$ and N is a maximal t-prime submodule, $N = (N :_M r)$ and so $x \in N$. Therefore N is a prime submodule. □

Corollary 2.2. *Let M be a finitely generated R-module that has a t-prime submodule. Then M has a prime submodule.*

Recall from [12] that a proper submodule N of an R-module M is said to be a n-submodule, if $r \in R\backslash\sqrt{Ann_R(M)}$ and $x \in M$ with $rx \in N$, then $x \in N$ (For more information see [1]).

Proposition 2.7. *Let N be a submodule of an R-module M. Then the following statements hold.*
(i) If N is t-prime, then N is a n-submodule if and only if $\sqrt[t]{(N :_R M)} \subseteq \sqrt{Ann_R(M)}$.
(ii) If N is a n-submodule and $\sqrt{Ann_R(M)} \subseteq \sqrt[t]{(N :_R M)}$, then N is a t-prime submodule.

Proof. (i) Let N be a n-submodule and $a \in \sqrt[t]{(N :_R M)}$. Then $a^tM \subseteq N$. Since N is a n-submodule, it is proper. So there exists $m \in M\backslash N$ and hence $a^tm \in N$. Now as N is a n-submodule and $m \notin N$, $a^t \in \sqrt{Ann_R(M)}$. Thus $a \in \sqrt{Ann_R(M)}$. Therefore $\sqrt[t]{(N :_R M)} \subseteq \sqrt{Ann_R(M)}$. For the converse assume that $\sqrt[t]{(N :_R M)} \subseteq \sqrt{Ann_R(M)}$ and $r \in R$ and $x \in M$ with $rx \in N$. As N is t-prime, $x \in N$ or $a \in \sqrt[t]{(N :_R M)}$ which implies $a \in \sqrt{Ann_R(M)}$. Therefore N is a n-submodule.
(ii) Let $r \in R$ and $x \in M$ with $rx \in N$. Since N is a n-submodule, $x \in N$ or $r \in \sqrt{Ann_R(M)}$ which implies $r \in \sqrt[t]{(N :_R M)}$. Thus N is a t-prime submodule. □

Corollary 2.3. *Let N be a submodule of an R-module M such that $\sqrt[t]{(N :_R M)} = \sqrt{Ann_R(M)}$. Then N is a t-prime submodule if and only if N is a n-submodule.*

Recall that a submodule N of an R-module M is called a pure submodule, if for any $r \in R$, $(rM) \cap N = rN$ [3].

Proposition 2.8. *Let N be a proper submodule of a torsion free R-module M such that $\sqrt[t]{(N :_R M)} = 0$. Then N is a pure submodule if and only if N is a t-prime submodule.*

Proof. Let N be a t-prime submodule. It is clear that for any $r \in R - \{0\}$, $rN \subseteq (rM) \cap N$. Assume that $r \in R$ and $x \in M$ with $rx \in N$. If $x \notin N$, then as N is t-prime, $r \in \sqrt[t]{(N :_R M)} = 0$, which is a contradiction. So $x \in N$ and therefore $(rM) \cap N = rN$. This means N is a pure submodule. Now let N be a pure submodule, $r \in R - \{0\}$ and $x \in M$ with $rx \in N$. Then $rx \in (rM) \cap N = rN$. Since M is torsion free, $x \in N$ and therefore N is a t-prime submodule. □

Proposition 2.9. *(i) Let $\{N_i\}_{i\in I}$ be a nonempty set of t-prime submodules of an R-module M such that $(N_i :_R M) = (N_j :_R M)$, for any $i, j \in I$. Then $\bigcap_{i\in I} N_i$ is a t-prime submodule.*
(ii) Let $\{N_i\}_{i\in I}$ be a chain of t-prime submodules of a finitely generated R-module M. Then $\bigcup_{i\in I} N_i$ is a t-prime submodule of M.

Proof. (i) Let $r \in R$ and $x \in M$ such that $rx \in \bigcap_{i\in I} N_i$. As $(\bigcap_{i\in I} N_i :_R M) = \bigcap_{i\in I}(N_i :_R M) = (N_j :_R M)$, for all $j \in I$, if $r \notin \sqrt[t]{(\bigcap_{i\in I} N_i :_R M)}$, then $r \notin \sqrt[t]{(N_j :_R M)}$, for all $j \in I$. Since N_j is a t-prime submodule, $x \in N_j$, for any $j \in I$. So $x \in \bigcap_{i\in I} N_i$ and therefore $\bigcap_{i\in I} N_i$ is a t-prime submodule.
(ii) Let $rx \in \bigcup_{i\in I} N_i$, for $r \in R$ and $x \in M$. Then $rx \in N_k$ for some $k \in I$. Since N_k is t-prime, we conclude that $x \in N_k \subseteq \bigcup_{i\in I} N_i$ or $r^t \in (N_k :_R M) \subseteq (\bigcup_{i\in I} N_i :_R M)$. Therefore $\bigcup_{i\in I} N_i$ is a t-prime submodule. □

Definition 2.4. *An R-module M is said to be a t-torsion free R-module, if $r \in R$ and $x \in M$ with $rx = 0$, then $x = 0$ or $r \in \sqrt[t]{Ann_R(M)}$.*

It is clear that any torsion free R-module, is a t-torsion free module.

Theorem 2.3. *Let N be a proper submodule of an R-module M. Then N is a t-prime submodule if and only if M/N is a t-torsion free R-module.*

Proof. Let N be a t-prime submodule, $r \in R$ and $\overline{x} \in M/N$ with $r\overline{x} = \overline{0}$. Then $rx \in N$ and since N is t-prime, $x \in N$ or $r^t \in (N :_R M) = Ann_R(M/N)$. If $x \in N$, then $\overline{x} = \overline{0}$. If $r^t \in (N :_R M) = Ann_R(M/N)$, then $r \in \sqrt[t]{Ann_R(M/N)}$. So M/N is a t-torsion free R-module. Now assume that M/N is a t-torsion free R-module. Let $r \in R$ and $x \in M$ with $rx \in N$. Then $r\overline{x} = \overline{0}$ and so $\overline{x} = \overline{0}$ or $r \in \sqrt[t]{Ann_R(M/N)}$. If $\overline{x} = \overline{0}$, then $x \in N$. If $r \in \sqrt[t]{Ann_R(M/N)}$, then $r \in \sqrt[t]{(N :_R M)}$. Therefore N is a t-prime submodule. □

Proposition 2.10. *Let $f : R \to S$ be a ring extension such that S is a free R-module and N be a submodule of an R-module M. Then N is a t-prime submodule of M if and only if $N \bigotimes_R S$ is a t-prime R-submodule of $M \bigotimes_R S$.*

Proof. Let $\{y_i\}_{i\in I}$ be a basis for S as an R-module. It is clear that $(N :_R M) = (N \bigotimes_R S :_R M \bigotimes_R S)$. Assume that N is a t-prime submodule of M. Let $r \in R$ and $\alpha = \sum_{i\in J}(m_i \bigotimes y_i) \in M \bigotimes_R S$, (for finite subset J of I) with $r\alpha \in N \bigotimes_R S$. Since $\{y_i\}_{i\in I}$ is a basis for S, we have $rm_i \in N$, for any $i \in J$. If $\alpha \notin N \bigotimes_R S$, then there exists $i_1 \in J$ such that $m_{i_1} \notin N$. Since N is t-prime, $r \in \sqrt[t]{(N :_R M)}$ and hence $r \in \sqrt[t]{(N \bigotimes_R S :_R M \bigotimes_R S)}$. Therefore $N \bigotimes_R S$ is a t-prime submodule of $M \bigotimes_R S$. Now for the converse, let $N \bigotimes_R S$ be a t-prime submodule of $M \bigotimes_R S$, $r \in R$ and $x \in M$ with and $rx \in N$. So $r(x \bigotimes y_1) \in N \bigotimes_R S$. Since $N \bigotimes_R S$ is t-prime, $r \in \sqrt[t]{(N \bigotimes_R S :_R M \bigotimes_R S)}$ or $x \bigotimes y_1 \in N \bigotimes_R S$. Hence $r \in \sqrt[t]{(N :_R M)}$ or $x \in N$. Therefore N is a t-prime submodule of M. □

Let N be a submodule of an R-module M. Consider $N[x] = \{a_0 + a_1x + ... + a_nx^n | n \in \mathbb{N},\ a_i \in N\}$. It is clear that $M[x]$ is an R-module and $N[x]$ is a submodule of $M[x]$. Hence we have the following direct consequence.

Corollary 2.4. *Let N be a submodule of an R-module M. Then N is a t-prime submodule of M if and only if $N[x]$ is a t-prime R-submodule of $M[x]$.*

Proposition 2.11. *Let N and L be submodules of an R-module M and I be an ideal of R. Then the followings hold:*
(i) If N and L are t-prime submodules such that $I \nsubseteq \sqrt[t]{(N \cup L :_R M)}$ and $IN = IL$, then $N = L$.
(ii) If IN is a t-prime submodule of M such that $I \nsubseteq \sqrt[t]{(IN :_R M)}$, then $IN = N$.

Proof. (i) Since $I \nsubseteq \sqrt[t]{(N \cup L :_R M)}$, then by Lemma 2.2 (iii), $I \nsubseteq \sqrt[t]{(N :_R M)}$. Also since N is a t-prime submodule and $IL \subseteq N$, by Theorem 2.1 (i)⇒(ii), we get that $L \subseteq N$. Likewise, $N \subseteq L$. Therefore $N = L$.
(ii) Since IN is a t-prime submodule and $IN \subseteq IN$, we conclude that $N \subseteq IN$, so this completes the proof. □

Definition 2.5. *An R-module M is called a t-prime module, if the zero submodule is t-prime.*

The next proposition provides a useful characterization of t-prime modules.
Recall that a proper submodule N of M is said to be an r-submodule, if for $a \in R$, $m \in M$ and whenever $am \in N$ with $ann_M(a) = 0$, then $m \in N$ [6].

Proposition 2.12. *Let M be an R-module. Then the following conditions are equivalent.*
(i) M is a t-prime module.
(ii) $Z_R(M) = \sqrt[t]{Ann_R(M)}$.
(iii) Any r-module is a t-prime submodule.

Proof. $(i) \Rightarrow (ii)$ Let $r \in Z_R(M)$. Then there exists a nonzero $x \in M$ such that $rx = 0$. Since zero is a t-prime submodule, $r \in \sqrt[t]{Ann_R(M)}$. For the converse, let $r \in \sqrt[t]{Ann_R(M)}$. Then $r^tM = 0$. For any $x \in M \backslash \{0\}$, we have $r^tx = 0$. Let $s \in \mathbb{N}$ be the smallest integer with $r^sx = 0$. It follows that $r(r^{s-1}x) = 0$ and $r^{s-1}x \neq 0$. Hence $r \in Z_R(M)$.
$(ii) \Rightarrow (iii)$ Let N be an r-submodule of M and $rx \in N$, for $r \in R$ and $x \in M$. If $r \notin \sqrt[t]{Ann_R(M)}$, then by part (ii), $ann_M(r) = 0$. Now, since N is an r-submodule, $x \in N$.
$(iii) \Rightarrow (i)$ Since the zero submodule is an r-submodule, by part (iii) the zero submodule is a t-prime submodule. □

Proposition 2.13. *Let N be a proper submodule of a torsion-free R-module M. Then the following conditions are equivalent.*
(i) N is a t-prime submodule.
(ii) $rN = N \cap rM$, for any $r \in R \backslash \sqrt[t]{(N :_R M)}$.
(ii) $N = (N :_M r)$, for any $r \in R \backslash \sqrt[t]{(N :_R M)}$.

Proof. $(i) \Rightarrow (ii)$ Let N be a t-prime submodule and $r \in R \backslash \sqrt[t]{(N :_R M)}$. It is clear that $rN \subseteq N \cap rM$. Let $x \in M$. Then $rx \in N$ and so by part (i) and since $r \notin \sqrt[t]{(N :_R M)}$, we have $x \in N$. Therefore $rN = N \cap rM$.
$(ii) \Rightarrow (iii)$ Let $r \in R \backslash \sqrt[t]{(N :_R M)}$. It is clear that $N \subseteq (N :_M r)$. Consider $x \in M$ such that $rx \in N$. Then by part (ii), $rx \in rN$ and since M is torsion free, $x \in N$. Therefore $N = (N :_M r)$.
$(iii) \Rightarrow (i)$ Assume that $r \in R \backslash \sqrt[t]{(N :_R M)}$ and $x \in M$ with $rx \in N$. Then $x \in (N :_M r)$ and so by part (iii), $x \in N$. Therefore N is a t-prime submodule. □

Proposition 2.14. *Let N be a proper submodule of a torsion-free R-module M. If $rN = N$, for any $r \in R \backslash \sqrt[t]{(N :_R M)}$, then N is a t-prime submodule.*

Proof. Assume that $r \in R \backslash \sqrt[t]{(N :_R M)}$ and $x \in M$ with $rx \in N$. Then $rx \in rN$ and since M is torsion free, $x \in N$. Therefore N is a t-prime submodule. □

Proposition 2.15. *Let N be a t-prime submodule of an R-module M. Then $N = (0 :_M Ann_R(N))$ or $Ann_R(N) \subseteq \sqrt[t]{(N :_R M)}$.*

Proof. Assume that $Ann_R(N) \nsubseteq \sqrt[t]{(N :_R M)}$ and $x \in (0 :_M Ann_R(N))$. Then for $a \in Ann_R(N) \backslash \sqrt[t]{(N :_R M)}$, $ax = 0$. Since N is t-prime, $x \in N$. Therefore $N = (0 :_M Ann_R(N)$. □

Theorem 2.4. *Let M be an R-module. The following conditions are eqivalent.*
(i) Any proper submodule is a t-prime submodule.
(ii) Any proper cyclic submodule is a t-prime submodule.

Proof. (i) $\Rightarrow$ (ii) It is clear.
(ii) $\Rightarrow$ (i) Let N be a proper submodule of M, $r \in R$ and $x \in M$ with $rx \in N$. Then there exists $n \in N$, such that $rx = n$. Hence $rx \in Rn$ and Rn is a proper submodule of M. So by part (ii), $x \in Rn$ or $r^t \in (Rn :_R M)$. If $x \in Rn$, then $x \in N$. If $r^t \in (Rn :_R M)$, then $r^t \in (N :_R M)$. Therefore N is a t-prime submodule. □

Proposition 2.16. *(i) Let $\{P_i\}_{i\in I}$ be a nonempty set of prime submodules of an R-module M. If $\bigcap_{i\in I} P_i$ is a t-prime submodule, then $\bigcap_{i\in I} P_i$ is a prime submodule.*
(ii) Let $\{P_i\}_{i\in I}$ be a nonempty set of primary submodules of an R-module M. If $\bigcap_{i\in I} P_i$ is a t-prime submodule, then $\bigcap_{i\in I} P_i$ is a primary submodule.

Proof. (i) Let $r \in R$ and $x \in M$ with $rx \in \bigcap_{i\in I} P_i$. If $r \notin (\bigcap_{i\in I} P_i :_R M)$, then as $(P_i :_R M)$ are prime ideals; $r^t \notin (\bigcap_{i\in I} P_i :_R M)$. Since $\bigcap_{i\in I} P_i$ is a t-prime submodule, we have $x \in \bigcap_{i\in I} P_i$. Therefore $\bigcap_{i\in I} P_i$ is a prime submodule.
(ii) The proof is similar to part (i). □

Recall from [10], the intersection of all prime submodules contains N, denoted $rad(N)$, is called the radical of N. If there is no prime submodule containing N, $rad(N) = M$.
By Lemma 2.1(ii) and Proposition 2.16(i), we arrive at the following corollary.

Corollary 2.5. *Let N be a submodule of an R-module M. Then $rad(N)$ is a t-prime submodule if and only if $rad(N)$ is a prime submodule.*

3. **Examples**

This Section is devoted to examples. We provide several examples which illustrate our results mentioned in previous sections.

Example 3.1. *(i) $4\mathbb{Z}$ as a $\mathbb{Z}$-submodule of $\mathbb{Z}$ is a 2-prime submodule, but it is not a prime submodule.*
(ii) $8\mathbb{Z} \oplus 4\mathbb{Z}$ as a $\mathbb{Z}$-submodule of $\mathbb{Z} \oplus \mathbb{Z}$ is not a 2-prime submodule.

Example 3.1 (i) shows that the converse of Proposition 2.3 is not true in general.

Example 3.2. *Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then every proper submodule of M is a t-prime submodule. (It is clear that every proper submodule of M is prime and so according to Example 2.1(ii), every proper submodule of M is a t-prime submodule.)*

Now we have an example which shows that there exists an R-module that does not have a t-prime submodule.

Example 3.3. *Let p be any prime number and M be the Prüfer p-group i.e., $M = \mathbb{Z}_{p^\infty}$ and $R = \mathbb{Z}$. Then we show that any proper submodule of M is not a t-prime submodule. It is clear that the zero is not a t-prime submodule. Let N be a proper submodule of M which is a t-prime submodule of M. Then there exists $n \in \mathbb{N}$ such that $N =< \frac{1}{p^n} + \mathbb{Z} >$. Since $p(\frac{1}{p^{n+1}} + \mathbb{Z}) \in N$ and N is t-prime, $\frac{1}{p^{n+1}} + \mathbb{Z} \in N$ or $p \in \sqrt[t]{(N :_R M)}$ this implies $\frac{1}{p^{n+1}} + \mathbb{Z} = p^t(\frac{1}{p^{n+t+1}} + \mathbb{Z}) \in N$, which is a contradiction. Therefore $M = \mathbb{Z}_{p^\infty}$ does not have a t-prime submodule.*

Remark 3.1. *(i) By Proposition 2.2(i), every t-prime submodule of a module is a primary submodule. However, the converse is not true in general. For example; if* $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ *and* $N = 8\mathbb{Z} \oplus 4\mathbb{Z}$, *then as* $\sqrt{(N :_R M)} = 2\mathbb{Z}$, N *is a primary submodule of* M, *but it is not a 2-prime submodule, because* $2(4, 2) \in N$, *but* $2 \notin \sqrt[2]{(N :_R M)}$ *and* $(4, 2) \notin N$.
(ii) By Proposition 2.2(iii), every t-prime submodule of a module is a (t+1)-prime submodule. However, the converse is not true in general. For example; $N = 8\mathbb{Z} \oplus 4\mathbb{Z}$ *in part (i) is a 3-prime submodule; but it is not a 2-prime submodule.*
(iii) For prime number p, $p^t\mathbb{Z}$ *is a t-prime submodule of* $\mathbb{Z}$*-module* $\mathbb{Z}$*; but it is not a prime nor a s-prime submodule, for* $s < t$.

By Proposition 2.4(ii), for t-prime submodule N of M, $\sqrt{(N :_R M)}$ is a prime ideal. The next example shows that the converse is not true in general.

Example 3.4. *Let* $M = \mathbb{Z} \oplus \mathbb{Z}$ *and* $R = \mathbb{Z}$. *Consider the submodule* $N = 8\mathbb{Z} \oplus 4\mathbb{Z}$. *By Remark 3.1(i),* N *is not a 2-prime submodule; but* $\sqrt{(N :_R M)} = 2\mathbb{Z}$ *is a maximal ideal.*

The following example shows that the sum and intersection of two t-prime submodules is not a t-prime submodule in general.

Example 3.5. *(i) Let* $M = \mathbb{Z} \oplus \mathbb{Z}$ *and* $R = \mathbb{Z}$. *We consider the submodules* $N = 0 \oplus \mathbb{Z}$ *and* $K = \mathbb{Z} \oplus 0$. *It is easy to see that* K *and* N *are t-prime submodules. Since* $N + K = M$, $N + K = \mathbb{Z} \oplus \mathbb{Z}$ *is not a t-prime submodule of* M. *(It is not a proper submodule.)*
(ii) Let $M = R = \mathbb{Z}$. *Take* $N = 4\mathbb{Z}$ *and* $K = 9\mathbb{Z}$. *It is easy to see that* K *and* N *are t-prime submodules. Now* $N \cap K = 36\mathbb{Z}$ *is not a primary submodule and so, by Proposition 2.2(i), it is not a t-prime submodule of* M.

Example 3.6. $\mathbb{Q}$ *as a* $\mathbb{Z}$*-module has only one t-prime submodule. Indeed, it is clear that, the zero submodule is a prime submodule and so by Lemma 2.1(ii), it is a t-prime submodule of* $\mathbb{Q}$. *Let* N *be a non-zero t-prime submodule. It follows that* $(N :_{\mathbb{Z}} \mathbb{Q}) = 0$. *Then for* $0 \neq \frac{a}{b} \in N$ *and* $\frac{x}{y} \in \mathbb{Q} \backslash N$, $ay\frac{x}{by} = x\frac{a}{b} \in N$, *which is a contradiction.*

Example 3.7. *(i)* $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *as a* $R = \mathbb{Z}$*-module is a 2-torsion free module, but it is not a torsion free R-module.*
(ii) $M = \mathbb{Z} \oplus \mathbb{Z}_2$ *as a* $R = \mathbb{Z}$*-module is not a t-torsion free module, for any* $t \in \mathbb{N}$ *(Consider* $2(0, \overline{1}) = (0, \overline{0})$*).*

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