ON α-LOCALLY DOUBLY DIAGONALLY DOMINANT MATRICES

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In the paper, the authors introduce a new type of α-locally doubly diagonally dominant matrices, which is indeed a new subclass of H-matrices, present several new criteria for judging non-singular H-matrices according to the theory of the new type of α-locally doubly diagonally dominant matrices, and illustrate effectiveness and advantages of the proposed criteria by some numerical examples.

Keywords: non-singular H-matrix, α-locally doubly diagonally dominant matrix, criteria, irreducibility, non-zero element chain, numerical example

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1. Introduction

It is well known that H-matrices play an important role in the matrix theory, computational mathematics, and mathematical physics. See [1, 2, 11]. In recent years, a lot of work devoted to giving direct and iterative criteria for judging whether a certain matrix A is a non-singular H-matrix or not. See [3, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 17, 18, 19, 20, 21]. In practice, if the coefficient matrix A of the linear system $Ax = b$ is an H-matrix, then many classical iterative methods are convergent. See [2, Chapters 7] and [4]. Therefore, it is significant to look for numerical methods of judging non-singular H-matrices and to provide effective and practical criteria.

In this paper, we will introduce a new type of α-locally doubly diagonally dominant matrices, present several new criteria for non-singular H-matrices, and illustrate effectiveness and advantages of the proposed criteria by numerical examples.

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2. Definitions and lemmas

We recall some notation, definitions, and lemmas.

Let $\mathbb{C}^{n\times n}$ be the set of all $n \times n$ complex matrices, $\mathbb{N}$ the set of all positive integers, and $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. For $i \in \mathbb{N}$, let

$$P_i(A) = \sum_{j\neq i} |a_{ij}|, \quad R_i(A) = \sum_{j\neq i} |a_{ji}|, \quad q_i = \frac{P_i(A)}{|a_{ii}|}.$$

Definition 2.1 ([17, p. 294]). Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. The comparison matrix of A, denoted by $\mu(A) = (m_{ij})_{n\times n}$, is defined by

$$m_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

A matrix A is said to be an M-matrix if $A = \mu(A)$ and the real parts of eigenvalues of A are positive. A matrix A is said to be an H-matrix if $\mu(A)$ is an M-matrix.

Remark 1. By Definition 2.1, it is easy to see that every H-matrix is non-singular. If A is an H-matrix such that $\mu(A)$ is non-singular, then all diagonal entries of A are non-zero. See [3, p. 2361]. Thus, in what follows, we always assume that $a_{ii} \neq 0$ for all $i \in \mathbb{N}$.

Definition 2.2 ([5, Definition 1.1], [14, p. 233], and [16, p. 120]). A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called to be (row) diagonally dominant, denoted by $A \in D_0$, if

$$|a_{ii}| \geq P_i(A), \quad i \in \mathbb{N}. \tag{2.1}$$

A matrix A is said to be strictly diagonally dominant, denoted by $A \in D$, if all the inequalities in (2.1) are strict. A matrix A is said to be generalized strictly diagonally dominant, if there exists a positive diagonal matrix X such that AX is strictly diagonally dominant.

Remark 2. It was said in [6, p. 318] and [16, p. 120] that A is a non-singular H-matrix if and only if A is a generalized strictly diagonally dominant matrix.

Definition 2.3 ([5, Definition 1.1]). If $\mathbb{N}_1$ and $\mathbb{N}_2$ are non-empty, proper, and disjoint subsets of $\mathbb{N}$ such that $\mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N}$, then we say that $(\mathbb{N}_1, \mathbb{N}_2)$ is a separation of $\mathbb{N}$.

Definition 2.4 ([19, p. 3]). A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is said to be irreducibly diagonally dominant if it is irreducible and at least one of the inequalities in (2.1) strictly holds.

Definition 2.5 ([6, Definition 2]). A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called diagonally dominant with non-zero element chain, if the inequality (2.1) is valid for all $i \in \mathbb{N}$, where at least one strictly inequality holds and, for every i with $|a_{ii}| = P_i(A)$, there exists a non-zero elements chain $a_{ij_1}a_{j_1j_2}\dots a_{j_{k-1}j_k} \neq 0$ such that $|a_{j_kj_k}| > P_{j_k}(A)$.

Definition 2.6 ([9, Definition 1]). Let $\alpha \in [0,1]$, $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, and $(\mathbb{N}_1, \mathbb{N}_2)$ be a separation of $\mathbb{N}$. If, for all $i \in \mathbb{N}_1$ and $j \in \mathbb{N}_2$,

$$|a_{jj}| - \sum_{\substack{t\in\mathbb{N}_2 \\ t\neq j}} |a_{jt}| > 0$$

and

$$\left(P_i(A) - \sum_{\substack{t\in\mathbb{N}_1 \\ t\neq i}} |a_{it}|q_t\right)^{\alpha} \left(|a_{jj}| - \sum_{\substack{t\in\mathbb{N}_2 \\ t\neq j}} |a_{jt}|\right)^{1-\alpha} > \left(\sum_{t\in\mathbb{N}_1} |a_{jt}|q_t\right)^{1-\alpha} \left(\sum_{t\in\mathbb{N}_2} |a_{it}|\right)^{\alpha},$$

then A is said to be an α-*SGD* matrix.

We also need the following lemmas.

Lemma 2.1 ([19, Theorem 1]). *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is irreducibly diagonally dominant, then A is a non-singular H-matrix.*

Lemma 2.2 ([6, p. 322]). *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is a diagonally dominant matrix with non-zero element chain, then A is a non-singular H-matrix.*

Lemma 2.3 ([9, Theorem 1]). *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is an α-SGD matrix, then A is a non-singular H-matrix.*

Lemma 2.4 ([8, p. 241] and [18, Theorem 1]). *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is a non-singular H-matrix, then there exists at least one $i \in \mathbb{N}$ such that $|a_{ii}| > P_i(A)$.*

Lemma 2.5 ([6, Lemma 1]). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a matrix whose diagonal entries are non-zero, $\delta = \{i | i \in \mathbb{N}, P_i(A) > 0\}$, and $A(\delta)$, whose rows and columns are indexed by δ, a sub-matrix of A. Then A is a non-singular H-matrix if and only if $A(\delta)$ is a non-singular H-matrix.*

3. **Main results**

We first introduce a new type of α-locally doubly diagonally dominant matrices as follows.

Definition 3.1. Let $\alpha \in [0,1]$ and $(\mathbb{N}_1, \mathbb{N}_2)$ be a separation of $\mathbb{N}$. A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is said to be strictly α-locally doubly diagonally dominant, denoted by $A \in SLDD(\alpha)$, if

$$P_i(A) > \sum_{\substack{t\in\mathbb{N}_1 \\ t\neq i}} |a_{it}|q_t, \quad P_j(A) > \sum_{\substack{t\in\mathbb{N}_2 \\ t\neq j}} |a_{jt}|q_t,$$

and

$$\left(P_i(A)-\sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}}|a_{it}|q_t\right)^{\alpha}\left(P_j(A)-\sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}}|a_{jt}|q_t\right)^{1-\alpha} > \left(\sum_{t\in\mathbb{N}_2}|a_{it}|q_t\right)^{\alpha}\left(\sum_{t\in\mathbb{N}_1}|a_{jt}|q_t\right)^{1-\alpha} \tag{3.1}$$

hold for all $i\in\mathbb{N}_1$ and $j\in\mathbb{N}_2$.

Definition 3.2. An irreducible matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$ is said to be irreducibly α-locally doubly diagonally dominant if

(1) the inequalities in (3.1) hold with $\geq$ instead of $>$,
(2) at least one of the inequalities in (3.1) strictly holds, and
(3) A satisfies other conditions of Definition 3.1.

Definition 3.3. A matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$ is said to be α-locally doubly diagonally dominant with a non-zero element chain if

(1) the inequalities in (3.1) hold with $\geq$ instead of $>$,
(2) for all $i\in\{i_1,\dots,i_k\}\cup\{j_1,\dots,j_l\}$, there exist a non-zero element chain $a_{ir_1}a_{r_1r_2}\dots a_{r_ti^*}\neq 0$ such that $i^*\in(\mathbb{N}_1\setminus\{i_1,\dots,i_k\})\cup(\mathbb{N}_2\setminus\{j_1,\dots,j_l\})\neq\emptyset$ and

$$\left(P_i(A)-\sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}}|a_{it}|q_t\right)^{\alpha}\left(P_j(A)-\sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}}|a_{jt}|q_t\right)^{1-\alpha} = \left(\sum_{t\in\mathbb{N}_1}|a_{jt}|q_t\right)^{\alpha}\left(\sum_{t\in\mathbb{N}_2}|a_{it}|q_t\right)^{1-\alpha} \tag{3.2}$$

hold for all $i\in\{i_1,\dots,i_k\}\subset\mathbb{N}_1$ and $j\in\{j_1,\dots,j_l\}\subset\mathbb{N}_2$, and
(3) A satisfies other conditions of Definition 3.1.

Now we start off to provide some new criteria for non-singular H-matrices.

Theorem 3.1. *Let $A=(a_{ij})\in\mathbb{C}^{n\times n}$. If $A\in SLDD(\alpha)$, then A is a non-singular H-matrix.*

Proof. If $\sum_{t\in\mathbb{N}_2}|a_{it}|q_t\neq 0$ for $i\in\mathbb{N}_1$, since $A\in SLDD(\alpha)$, then there exists a positive number d such that

$$\min_{i\in\mathbb{N}_1}K_i > d > \max_{j\in\mathbb{N}_2}k_j, \tag{3.3}$$

where

$$K_i=\frac{P_i(A)-\sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}}|a_{it}|q_t}{\sum_{t\in\mathbb{N}_2}|a_{it}|q_t} \quad\text{and}\quad k_j=\frac{\sum_{t\in\mathbb{N}_1}|a_{jt}|q_t}{P_j(A)-\sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}}|a_{jt}|q_t}$$

for $i\in\mathbb{N}_1$ and $j\in\mathbb{N}_2$.

Let

$$x_t = \begin{cases} q_t, & t \in \mathbb{N}_1, \\ dq_t, & t \in \mathbb{N}_2, \end{cases} \qquad X = \mathrm{diag}(x_1, x_2, \ldots, x_n), \qquad B = AX = (b_{ij})_{n\times n}. \tag{3.4}$$

For $i \in \mathbb{N}_1$, the double inequality (3.3) implies that

$$\begin{aligned} |b_{ii}| - P_i(B) &= P_i(A) - \sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}} |a_{it}|q_t - d\sum_{t\in\mathbb{N}_2} |a_{it}|q_t \\ &> P_i(A) - \sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}} |a_{it}|q_t - K_i\sum_{t\in\mathbb{N}_2} |a_{it}|q_t = 0. \end{aligned}$$

For $j \in \mathbb{N}_2$, the double inequality (3.3) means that

$$\begin{aligned} |b_{jj}| - P_j(B) &= d\left(P_j(A) - \sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}} |a_{jt}|q_t\right) - \sum_{t\in\mathbb{N}_1} |a_{jt}|q_t \\ &> k_j\left(P_j(A) - \sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}} |a_{jt}|q_t\right) - \sum_{t\in\mathbb{N}_1} |a_{jt}|q_t = 0. \end{aligned}$$

Then $|b_{ii}| > P_i(B)$ for $i \in \mathbb{N}$. Hence, A is a non-singular H-matrix in this case.

If $\sum_{t\in\mathbb{N}_2} |a_{it}|q_t = 0$ for $i \in \mathbb{N}_1$, by Definition 3.1, we have $P_j(A) > \sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}} |a_{jt}|q_t$ for all $j \in \mathbb{N}_2$. Consequently, there exists $d > 0$ satisfying

$$d\left(P_j(A) - \sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}} |a_{jt}|q_t\right) > \sum_{t\in\mathbb{N}_1} |a_{jt}|q_t. \tag{3.5}$$

Making use of (3.4) to (3.5) yields $|b_{ii}| > P_i(B)$ for $i \in \mathbb{N}$. Therefore, A is a non-singular H-matrix. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is an irreducibly α-locally doubly diagonally dominant matrix, then A is a non-singular H-matrix.*

Proof. In view of the irreducibility of A, there exists $a_{jt} \neq 0$ for $t \in \mathbb{N}_1$ and $j \in \mathbb{N}_2$.

If $\sum_{t\in\mathbb{N}_2} |a_{it}|q_t = 0$ for $i \in \mathbb{N}_1$, by Definition 3.2, it is easy to see that A is a non-singular H-matrix.

If $\sum_{t\in\mathbb{N}_2} |a_{it}|q_t \neq 0$ for $i \in \mathbb{N}_1$, by the same arguments as above, we have

$$d \triangleq \min_{i\in\mathbb{N}_1} K_i = \max_{j\in\mathbb{N}_2} k_j, \tag{3.6}$$

which means that $d > 0$.

From (3.4) and (3.6), it follows that

$$|b_{ii}| - P_i(B) = P_i(A) - \sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}} |a_{it}|q_t - d\sum_{t\in\mathbb{N}_2} |a_{it}|q_t$$

$$\geq P_i(A) - \sum_{\substack{t\in\mathbb{N}_1\\ t\neq i}} |a_{it}|q_t - K_i \sum_{t\in\mathbb{N}_2} |a_{it}|q_t = 0$$

for $i \in \mathbb{N}_1$ and that

$$|b_{jj}| - P_j(B) = d|a_{jj}|q_j - \sum_{t\in\mathbb{N}_1} |a_{jt}|q_t - d\sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}} |a_{jt}|q_t$$

$$= d\left(P_j(A) - \sum_{\substack{t\in\mathbb{N}_2\\ t\neq j}} |a_{jt}|q_t\right) - \sum_{t\in\mathbb{N}_1} |a_{jt}|q_t \geq 0$$

for $j \in \mathbb{N}_2$. Consequently, by Definition 3.2, it follows that $|b_{ii}| \geq P_i(B)$ for all $i \in \mathbb{N}$, in which at least one strict inequality is valid. Since A is irreducible, then B is an irreducibly diagonally dominant matrix. Thus, in light of Lemma 2.1, we obtain that B is a non-singular H-matrix, and so is A. The proof of Theorem 3.2 is completed. □

Theorem 3.3. *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is an α-locally doubly diagonally dominant matrix with a non-zero elements chain, then A is a non-singular H-matrix.*

Proof. If $\sum_{t\in\mathbb{N}_2} |a_{it}|q_t = 0$ for all $i \in \mathbb{N}_1$, then it is easy to see that A is a non-singular H-matrix. If $\sum_{t\in\mathbb{N}_2} |a_{it}|q_t \neq 0$ for $i \in \mathbb{N}_1$, by similar arguments as above, we obtain $d \triangleq \min_{i\in\mathbb{N}_1} K_i = \max_{j\in\mathbb{N}_2} k_j > 0$. Utilizing (3.2) and (3.4), by Definition 3.3, we obtain $|b_{ii}| - P_i(B) \geq 0$ for $i \in \{j_1,\dots,j_l\} \cup \{i_1,\dots,i_k\}$, that is, $|b_{ii}| \geq P_i(B)$ for $i \in (\mathbb{N}_1 \setminus \{i_1,\dots,i_k\}) \cup (\mathbb{N}_2 \setminus \{j_1,\dots,j_l\})$. This means that B is a diagonally dominant matrix with non-zero elements chain. Furthermore, by Lemma 2.2, we obtain that B is a non-singular H-matrix, and so is A. The proof of Theorem 3.3 is completed. □

Remark 3. From Theorem 3.1, we find that the type of α-locally doubly diagonally dominant matrices is a subclass of H-matrices. Hence, the introduction of α-locally doubly diagonally dominant matrices well extends the theory of H-matrices.

4. **Numerical Examples**

We now illustrate effectiveness and advantages of the above proposed criteria by several numerical examples.

Example 4.1. Let

$$A = \begin{bmatrix} 3 & 2 & 6 & 0\\ 1 & 6.8 & 3 & 4\\ 1 & 2 & 9 & 4\\ 0 & 2 & 2 & 8 \end{bmatrix}$$

and $(\mathbb{N}_1, \mathbb{N}_2)$ be a separation of $\mathbb{N} = \{1, 2, 3, 4\}$. If $\mathbb{N}_1$ and $\mathbb{N}_2$ satisfy one of the following cases:

$$\mathbb{N}_1 = \{1\},\quad \mathbb{N}_2 = \{2,3,4\};\quad \mathbb{N}_1 = \{2\},\quad \mathbb{N}_2 = \{1,3,4\};\quad \mathbb{N}_1 = \{3\},$$
$$\mathbb{N}_2 = \{1,2,4\};\quad \mathbb{N}_1 = \{4\},\quad \mathbb{N}_2 = \{1,2,3\};\quad \mathbb{N}_1 = \{1,2\},\quad \mathbb{N}_2 = \{3,4\};$$

$$\mathbb{N}_1 = \{1,3\}, \quad \mathbb{N}_2 = \{2,4\}; \quad \mathbb{N}_1 = \{1,4\}, \quad \mathbb{N}_2 = \{2,3\}; \quad \mathbb{N}_1 = \{2,3\}, \\ \mathbb{N}_2 = \{1,4\}; \quad \mathbb{N}_1 = \{2,4\}, \quad \mathbb{N}_2 = \{1,3\};$$

then A is not an α-SGD matrix and does not satisfy the corresponding conditions in [9, Theorem 1]. But, when $\mathbb{N}_1$ and $\mathbb{N}_2$ satisfy one of the above cases, by Theorem 3.1, it is easy to see that A is a non-singular H-matrix.

Example 4.2. Consider

$$A = \begin{bmatrix} 3 & 2 & 6 & 0 \\ 1 & 6.8 & 3 & 4 \\ 1 & 2 & 9 & 4 \\ 0 & 2 & 2 & 10 \end{bmatrix}.$$

By [15, Theorems 2.1 and 2.2], we have

$$r_1 = \min\left\{\frac{\alpha_3 - \gamma_3}{|a_{33}| - \gamma_3}, \frac{\alpha_4 - \gamma_4}{|a_{44}| - \gamma_4}\right\} = \min\left\{\frac{12}{37}, \frac{1}{19}\right\} = \frac{1}{19},$$
$$r_2 = \max\left\{\frac{\beta_3}{\alpha_3 + \beta_3 - \gamma_3}, \frac{\beta_4}{\alpha_4 + \beta_4 - \gamma_4}\right\} = \max\left\{\frac{5}{9}, \frac{9}{11}\right\} = \frac{9}{11}.$$

Since $|a_{11}| = 3 < |a_{13}|(\delta_3 - r_1) + |a_{14}|(\delta_4 - r_1) + \beta_1 = 6.350\ldots$, the matrix A does not satisfy the corresponding conditions in [15, Theorem 2.1]. Since $|a_{11}| = 3 < r_2(|a_{13}|\delta_3 + |a_{14}|\delta_4) + \beta_1 = 5.818\ldots$, the matrix A does not satisfy the corresponding conditions in [15, Theorem 2.2] yet.

In virtue of [13, Theorem 2], we obtain

$$M_1 = \{(2,3)\}, \quad M_2 = \{(1,2),(1,3),(1,4)\}, \\ M_3 = \{(2,4),(3,4)\}, \quad M_4 = M_5 = M_6 = \emptyset,$$

and

$$\frac{R_2(A)R_3(A) - |a_{22}a_{33}|}{R_2(A)R_3(A) - P_2(A)P_3(A)} = 0.48,$$
$$\frac{|a_{11}a_{22}| - R_1(A)R_2(A)}{P_1(A)P_2(A) - R_1(A)R_2(A)} = 0.161\ldots,$$
$$\frac{|a_{11}a_{33}| - R_1(A)R_3(A)}{P_1(A)P_3(A) - R_1(A)R_3(A)} = 0.147\ldots,$$
$$\frac{|a_{11}a_{44}| - R_1(A)R_4(A)}{P_1(A)P_4(A) - R_1(A)R_4(A)} = 0.875.$$

Obviously,

$$\max_{(s,t)\in M_1} \frac{R_s(A)R_t(A) - |a_{ss}||a_{tt}|}{R_s(A)R_t(A) - P_s(A)P_t(A)} = 0.48 \\ > \min_{(i,j)\in M_2} \frac{|a_{ii}||a_{jj}| - R_i(A)R_j(A)}{P_i(A)P_j(A) - R_i(A)R_j(A)} = 0.147\ldots$$

Therefore, the matrix A does not satisfy the conditions of [13, Theorem 2].

Nevertheless, if we choose $\mathbb{N}_1 = \{1,2\}$, $\mathbb{N}_2 = \{3,4\}$, and $\alpha = \frac{1}{2}$, then

$$[P_1(A) - |a_{12}|q_2]^{\alpha}[P_3(A) - |a_{34}|q_4]^{1-\alpha} = 5.522\ldots,$$

$$[|a_{13}|q_3 + |a_{14}|q_4]^\alpha[|a_{31}|q_1 + |a_{32}|q_2]^{1-\alpha} = 4.839\dots,$$
$$[P_1(A) - |a_{12}|q_2]^\alpha[P_4(A) - |a_{43}|q_3]^{1-\alpha} = 3.715\dots,$$
$$[|a_{13}|q_3 + |a_{14}|q_4]^\alpha[|a_{41}|q_1 + |a_{42}|q_2]^{1-\alpha} = 3.313\dots,$$
$$[P_2(A) - |a_{21}|q_1]^\alpha[P_3(A) - |a_{34}|q_4]^{1-\alpha} = 5.366\dots,$$
$$[|a_{23}|q_3 + |a_{24}|q_4]^\alpha[|a_{31}|q_1 + |a_{32}|q_2]^{1-\alpha} = 4.443\dots,$$
$$[P_2(A) - |a_{21}|q_1]^\alpha[P_4(A) - |a_{43}|q_3]^{1-\alpha} = 3.610\dots,$$
$$[|a_{23}|q_3 + |a_{24}|q_4]^\alpha[|a_{41}|q_1 + |a_{42}|q_2]^{1-\alpha} = 3.042\dots$$

By Theorem 3.1, we derive that A is a non-singular H-matrix.

Example 4.3. Let

$$A = \begin{bmatrix} 4 & 0.5 & 6 \\ 2 & 12 & 8 \\ 2 & 10 & 14 \end{bmatrix}.$$

(1) In view of [10], we easily obtain the following conclusions:
   (a) When $k = 1$, we acquire $S_1 + S_2 + S_3 = 1 + \frac{5}{12} > 1$.
   (b) When $k = 2$, for a separation $(\mathbb{N}_1, \mathbb{N}_2)$ of $\mathbb{N} = \{1, 2, 3\}$,
      (i) if $\mathbb{N}_1 = \{1\}$ and $\mathbb{N}_2 = \{2, 3\}$, then $S_1 = \frac{3}{2} > 1$;
      (ii) if $\mathbb{N}_1 = \{2\}$ and $\mathbb{N}_2 = \{1, 3\}$, then $S_1 + S_3 = \frac{7}{5} > 1$;
      (iii) if $\mathbb{N}_1 = \{3\}$ and $\mathbb{N}_2 = \{1, 2\}$, then $S_1 + S_2 = \frac{191}{140} > 1$.
   (c) When $k = 3$, it is easy to see that $S_1 = \frac{13}{8} > 1$.

   Therefore, the matrix A does not satisfy the corresponding conditions in [10, Theorem 1].
(2) It is not difficult to realize that the matrix A does not satisfy the corresponding conditions in [7, Theorem 1] and [21, Theorems 1 and 2].

Nevertheless, if we choose $\mathbb{N}_1 = \{1, 2\}$, $\mathbb{N}_2 = \{3\}$, and $\alpha = \frac{1}{2}$, then

$$[P_1(A) - |a_{12}|q_2]^\alpha[P_3(A)]^{1-\alpha} = 8.544\dots,$$
$$[|a_{13}|q_3]^\alpha[|a_{31}|q_1 + |a_{32}|q_2]^{1-\alpha} = 7.718\dots,$$
$$[P_2(A) - |a_{21}|q_1]^\alpha[P_3(A)]^{1-\alpha} = 9,$$
$$[|a_{23}|q_3]^\alpha[|a_{31}|q_1 + |a_{32}|q_2]^{1-\alpha} = 8.912\dots$$

Using Theorem 3.1 gives that A is a non-singular H-matrix.

5. **Conclusions**

In conclusion, a new type of α-locally doubly diagonally dominant matrices are introduced, which is now known as a subclass of non-singular H-matrices. Moreover, the authors establish several new and practical criteria for judging non-singular H-matrices by involved matrices. Consequently, the criteria for identifying nonsingular H-matrices is well extended.

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