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ON THE CATEGORY OF FUZZY TOLERANCE RELATIONS AND RELATED TOPICS

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This paper uses category theory to approach the topic of fuzzy tolerance relations and coverings, and it establishes the connection between them. The relations are described from a fuzzy set point of view.

Keywords: fuzzy sets, category theory, tolerance relation, covering.

1. **Introduction**

Lotfi Zadeh introduced the fuzzy sets in 1965. For a comprehensive introduction we refer the reader to [9], [10]. The theory of fuzzy sets allowed the study of new types of relations. A fuzzy relation R on a set X is a $X \times X$ fuzzy set ([11], [14]). If the relation is reflexive and symmetric then it is called a fuzzy tolerance relation. The tolerance relations were first studied by Zeeman and have been pursued in [2], [3], [4], [6], [12], [13]. An example of a tolerance relation is the approximation of real numbers: Every number $x \in \mathbb{R}$ is approximately equal to itself with degree 1 and x is approximately equal to y just as much as y is approximately equal to x.

In [8], we studied fuzzy coverings and coverages. Our aim is to go even further and focus on the category of fuzzy tolerance relations in connection with coverings ([2], [5]), with the use of category theory ([1], [7], [15]).

The paper is divided into four sections, the first being this introduction. In the second section, we recall some basic definition regarding fuzzy sets, fuzzy relations and coverings of fuzzy sets. In the third section, we tackle the case of normal coverings of fuzzy sets. In Proposition 3.2, we define the normal exponential of two coverings. In Theorem 3.1, we construct an involution function on the category of normal coverings. In Proposition 3.4, we describe several connections between a (normal) covering and its associated relation.

In the fourth section, we study Tol, the category of fuzzy tolerance relations, building on results of Belohlávek [3]. The study of Tol involves primarily the limits and colimits of the category. In Theorem 4.2, we establish some connections between Covering, the category of fuzzy coverings, and Tol. In Theorem 4.3, we construct an isomorphism between t-Covering, the category of tolerance coverings, which is a subcategory of Covering, and Tol.

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2. **Preliminaries**

Let X be a set.

Definition 2.1. *We say that A is a* fuzzy set*, or a* fuzzy subset of X*, if $A : X \to [0,1]$ is a function. $A(x)$ is the membership degree to which x belongs to A.*

Definition 2.2. *We say that $A : X \to [0,1]$ is a* normal fuzzy set *if there exists $x \in X$ such that $A(x) = 1$.*

Remark 2.1. A *crisp subset* A of X, i.e. a subset $A \subseteq X$, can be identified with its characteristic function $A : X \to \{0,1\}, A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$.

Definition 2.3. *The pair $(X, (A_i)_{i\in I})$ is called a* crisp covering *of X if:*
(a) $A_i \subseteq X$ *for all* $i \in I$.
(b) $\bigcup_{i\in I} A_i = X$.

Definition 2.4. *A crisp covering $(X, (A_i)_{i\in I})$ is called a* partition of X*, if the sets A_i's are non-empty and disjoint.*

Definition 2.5. *We say that $(X, (A_i)_{i\in I})$ is a* fuzzy covering*, or, simply, a* covering *of X, if $A_i : X \to [0,1]$ are fuzzy sets such that for all $x \in X$, there exists $i \in I$ with $A_i(x) = 1$.*

In this case, we can also say that X is covered *by the fuzzy sets A_i, $i \in I$.*

Definition 2.6. *If $A : X \to [0,1]$ is a fuzzy set then we define the crisp sets:*
(a) $A^{\uparrow} : X \to [0,1]$, $A^{\uparrow}(x) = \begin{cases} 1, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}$.
(b) $A^{\downarrow} : X \to [0,1]$, $A^{\downarrow}(x) = \begin{cases} 1, & A(x) = 1 \\ 0, & A(x) < 1 \end{cases}$.

Definition 2.7. *Let $\alpha \in [0,1]$. The* $\alpha-$cut *of the fuzzy set $A : X \to [0,1]$, denoted by A_α, is $A_\alpha : X \to [0,1]$, $A_\alpha(x) = \begin{cases} 1, & A(x) \geq \alpha \\ 0, & A(x) < \alpha \end{cases}$.*

Note that $A^{\uparrow} = \bigcup_\alpha A_\alpha$ and $A^{\downarrow} = A_1$.

Definition 2.8. *A* fuzzy relation *R between X and Y is a fuzzy set*

$$R : X \times Y \to [0,1].$$

The converse *of R is the fuzzy relation $R^{\smile} : Y \times X \to [0,1]$, where*

$$R^{\smile}(y,x) = R(x,y) \text{ for all } x \in X \text{ and } y \in X.$$

Definition 2.9. *The* composition *of the fuzzy relations $R : X \times Y \to [0,1]$ and $Q : Y \times Z \to [0,1]$ is the relation $R;Q : X \times Z \to [0,1]$ where*

$$(R;Q)(x,z) = \bigvee_{y\in Y} (R(x,y) \wedge Q(y,z)), \ (\forall) x \in X, z \in Z.$$

Definition 2.10. *Let* $(X,(A_i)_{i\in I})$ *be a covering. Its* associated fuzzy relation *is* $R : X\times I \to [0,1]$, $R(x,i) = A_i(x)$.

Definition 2.11. *Let* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(I,(B_y)_{y\in Y}\right)$ *be two coverings. Their* composition *is* $\left(X,(A_i)_{i\in I}\right);\left(I,(B_y)_{y\in Y}\right) = \left(X, \bigvee_{i\in I}\left(A_i(x)\wedge B_y(i)\right)_{y\in Y}\right)$.

Proposition 2.1. *The composition of two coverings is a covering.*

Proof. Let $\left(X,(A_i)_{i\in I}\right)$ and $\left(Y,(B_j)_{j\in J}\right)$ be two coverings. For any $x\in X$ there exists $i\in I$ with $A_i(x) = 1$. It follows that $(A_i(x)\wedge B_y(i)) = B_y(i)$. Since $\left(I,(B_y)_{y\in Y}\right)$ is a covering, there exists $y\in Y$ such that $B_y(i) = 1$. Therefore $\left(X,(A_i)_{i\in I}\right);\left(I,(B_y)_{y\in Y}\right)$ is a covering. □

Definition 2.12. *(*[8, Definition 3.1]*) Let* Covering *be the category which has:*

(1) $\mathrm{Ob}(\text{Covering}) = \left\{\left(X,(A_i)_{i\in I}\right) \mid \left(X,(A_i)_{i\in I}\right) \text{ is a fuzzy covering}\right\}$.

(2) $\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Y,(B_j)_{j\in J}\right)\right) = \{(f,\rho) | f : X\to Y, \rho : I \to J,$ *such that* $A_i(x)\le B_{\rho(i)}(f(x)), (\forall)x\in X, (\forall)i\in I\}$.

(3) $(g,\theta)\circ(f,\rho) = (g\circ f, \theta\circ\rho)\in \mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Z,(C_k)_{k\in K}\right)\right)$, *for all* $(f,\rho)\in\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Y,(B_j)_{j\in J}\right)\right)$ *and* $(g,\theta)\in\mathrm{Hom}\left(\left(Y,(B_j)_{j\in J}\right),\left(Z,(C_k)_{k\in K}\right)\right)$.

(4) $id_{\left(X,(A_i)_{i\in I}\right)} = (id_X, id_I)$, *for all* $\left(X,(A_i)_{i\in I}\right)\in\mathrm{Ob}(\text{Covering})$.

3. Fuzzy relations

3.1. Normal coverings

Definition 3.1. *A* normal covering $\left(X,(A_i)_{i\in I}\right)$ *is a covering with the property that for all* $i\in I$*, there exists* $x\in X$ *such that* $A_i(x) = 1$.

Remark 3.1. A normal covering $\left(X,(A_i)_{i\in I}\right)$ can be regarded as a set of objects X associated with a set of attributes $(A_i)_{i\in I}$ so that every object has an attribute and every attribute has an element.

Remark 3.2. A partition $(X,(A_i)_{i\in I})$, in the sense of Definition 2.4, is a normal covering.

Proposition 3.1. *The composition of two normal coverings is a normal covering.*

Proof. Let $\left(X,(A_i)_{i\in I}\right)$ and $\left(Y,(B_j)_{j\in J}\right)$ be two normal coverings. According to Proposition 2.1, their composition is a covering. For any $y\in Y$ there exists $x\in X$ such that $A_i(x)\wedge B_y(i) = 1$. On the other hand, for any $y\in Y$ there exists $i\in I$ such that that $B_y(i) = 1$ and, also, for any $i\in I$ there exists $x\in X$ such that $A_i(x) = 1$. Hence $A_i(x)\wedge B_y(i) = 1$ and, therefore, $\left(X,(A_i)_{i\in I}\right);\left(I,(B_y)_{y\in Y}\right)$ is a normal covering. □

In [8, Theorem 3.2(g)] we introduced the exponential $\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}$ of two coverings $\left(X,(A_i)_{i\in I}\right)$ and $\left(Y,(B_j)_{j\in J}\right)$ and proved that it is a covering. In general,

$\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}$ is not a normal covering. For example we can pick a the covering (X,A_1) and (Y,B_1,B_2), where $A_1 = X$, $B_1 = \emptyset$ and $B_2 = Y$. We have two functions between the sets of indexes $\rho_1,\rho_2 : \{1\} \to \{1,2\}$, where $\rho_1(1) = 1$ and $\rho_2(1) = 2$. For all the functions $f : X \to Y$, for all $x \in X$ we have:

$$A_1(x) \to B_{\rho_1(1)}\left(f\left(x\right)\right) = 0$$

and

$$A_1(x) \to B_{\rho_2(1)}\left(f\left(x\right)\right) = 1$$

where $a \to b = \sup\{c \in [0,1] \ : \ a \wedge c \leq b\}$. This means that $\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}$ is not a normal covering. In the following proposition, we introduce an object in *Covering*, which is always normal.

Proposition 3.2. *Let* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(Y,(B_j)_{j\in J}\right)$ *be two coverings and*

$$\overline{Y^X} = U_1\left(\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Y,(B_j)_{j\in J}\right)\right)\right),$$
$$\overline{J^I} = U_2\left(\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Y,(B_j)_{j\in J}\right)\right)\right),$$

where $U_1, U_2 : \mathrm{Covering} \to \mathrm{Set}$ *are the forgetful functors with* $U_1(f,\rho) := f$ *and* $U_2(f,\rho) = \rho$, *for any morphism* (f,ρ) *in* Covering*. Let*

$$\overline{\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}} := \left(\overline{Y^X}, \left(\bigwedge_{(x,i)\in X\times I}\left(A_i(x) \to B_{\rho(i)}(f(x))\right)\right)_{\rho\in\overline{J^I}}\right)$$

Then $\overline{\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}}$ *is a normal covering and we call it, the* normal exponential *of* $\left(X,(A_i)_{i\in I}\right)$ *and* $\left(Y,(B_j)_{j\in J}\right)$*. The exponential in* Covering*, the functors* U_1 *and* U_2 *were discussed in* [8].

Proof. If $\rho \in \overline{J^I}$ then there exists $f \in \overline{Y^X}$ such that in Covering we have the morphism $(f,\rho) : \left(X,(A_i)_{i\in I}\right) \to \left(Y,(B_j)_{j\in J}\right)$. Then for all $x \in X$ and $i \in I$ we have $A_i(x) \leq B_{\rho(i)}\left(f\left(x\right)\right)$, which is equivalent to say that for all $\rho \in \overline{J^I}$ and all $x \in X$ there exists $f \in \overline{Y^X}$ such that $A_i(x) \to B_{\rho(i)}\left(f\left(x\right)\right) = 1$.

Therefore $\overline{\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}}$ is a normal covering. For more details refer to [8]. □

Definition 3.2. *A subcategory C of D is called* coreflective *if for every object* $Y \in \mathrm{Ob}(D)$ *there exists an object* $Y_C \in \mathrm{Ob}(C)$ *and a morphism* $g_C : Y_C \to Y$ *such that for all the objects* $X \in \mathrm{Ob}(C)$ *and for all morphisms* $f : X \to Y$ *there exists a unique morphism* $h : X \to Y_C$ *such that* $g_C \circ h = f$.

Definition 3.3. Let N-Covering be the category in which the objects are normal coverings and the morphisms between the normal coverings
$(f,\rho) : \left(X,(A_i)_{i\in I}\right) \to \left(Y,(B_j)_{j\in J}\right)$ have the property:

$$A_i(x) \leq B_{\rho(i)}(f(x)),\ \forall x \in X,\ \forall i \in I.$$

Proposition 3.3. N-Covering *is the full coreflective subcategory of* Covering.

Proof. From Definition 3.3, it is clear that N-Covering is a full subcategory of Covering. In order to prove that N-Covering is a coreflective subcategory of Covering, we consider a fuzzy covering $\left(Y, (B_j)_{j\in J}\right)$ and we define $\left(Y, (B_j)_{j\in J}\right)_{\text{N-Covering}} = \left(Y, (B_j)_{j\in J'}\right)$, where J' is the subset of J in which for all $j \in J'$ there exists $y \in Y$ such that $B_j(y) = 1$. We have that $\left(Y, (B_j)_{j\in J}\right)_{\text{N-Covering}} \in \text{Ob}\,(\text{N-Covering})$.

We define the morphism $(1_Y, \iota_{J'}) : \left(Y, (B_j)_{j\in J'}\right) \to \left(Y, (B_j)_{j\in J}\right)$, where 1_Y is the identity function on Y and $\iota_{J'} : J' \to J$ where $\iota(j) = j$ for all $j \in J'$.

Let $\left(X, (A_i)_{i\in I}\right)$ be a normal covering and let $(f, \rho) : \left(X, (A_i)_{i\in I}\right) \to \left(Y, (B_j)_{j\in J}\right)$ be a morphism in *Covering*. But since $\left(X, (A_i)_{i\in I}\right)$ is a normal covering for all $i \in I$ there exists $x \in X$ such that $A_i(x) = 1$. Then:

$$1 = A_i(x) \le B_{\rho(i)}\left(f(x)\right), \text{ hence } Im(\rho) \subseteq J'.$$

Thus, we can define $\rho' : I \to J'$ such that $\rho'(i) = \rho(i)$. Then we have the unique morphism (f, ρ') with the property

$$(f, \rho) = (1_Y, \iota_{J'}) \circ (f, \rho),$$

as required. □

Theorem 3.1. *Let* $\Im$: N-Covering $\to$ N-Covering *be the functor defined:*

(a) *On objects,* $\Im\left(\left(X, (A_i)_{i\in I}\right)\right) = \left(I, (B_x)_{x\in X}\right)$*, where* $B_x(i) = A_i(x)$*, for all* $x \in X$ *and* $i \in I$*.*

(b) *On morphisms,* $\Im((f, \rho)) = (\rho, f)$*.*

Then $\Im$ *is an involution functor, i.e.* $\Im \circ \Im = 1_{\text{N-Covering}}$*.*

Proof. We have to prove that $\Im$ is correctly defined. If $\left(X, (A_i)_{i\in I}\right)$ is a normal covering then for all $i \in I$ there exists $x \in X$ such that $A_i(x) = B_x(i) = 1$. Then $\left(I, (B_x)_{x\in X}\right)$ is a normal covering.

If $(f, \rho) : \left(X, (A_i)_{i\in I}\right) \to \left(Y, (C_j)_{j\in J}\right)$ is a morphism in N-Covering then for all $x \in X$ and for all $i \in I$ we have $A_i(x) \le C_{\rho(i)}\left(f\left(x\right)\right)$.

If $\Im\left(\left(X, (A_i)_{i\in I}\right)\right) = \left(I, (B_x)_{x\in X}\right)$ and $\Im\left(\left(Y, (C_j)_{j\in J}\right)\right) = \left(J, (D_y)_{y\in Y}\right)$ then for all $i \in I$ and for all $x \in X$ we have $B_x(i) \le D_{f(y)}\left(\rho\left(i\right)\right)$.

From the above considerations, it follows that $\Im$ is well defined. Also, it is obvious that $\Im \circ \Im = 1_{\text{N-Covering}}$. □

3.2. Tolerance relations

Definition 3.4. *Let* $T, T' : X \times X \to [0, 1]$ *be two fuzzy relations on* X*. We say that:*

(a) T *is* smaller *than* T' *or* T' *is* greater *than* T *if* $T(x, y) \le T'(x, y)$*, for all* $x, y \in X$*. We write* $T \subseteq T'$ *(or* $(X, T) \subseteq (X, T')$*).*

(b) T *is* strictly smaller *than* T' *or* T' *is* strictly greater *than* T *if* T *is smaller than* T' *and there exists* $x, y \in X$ *such that* $T(x, y) < T'(x, y)$*. We write* $T \subset T'$ *(or* $(X, T) \subset (X, T')$*).*

Let $\mathbb{I}_X$ be the relation associated to the normal covering $\left(X, (\delta_{xy})_{x\in X}\right)$.

Definition 3.5. *([15]) Let $R : A \times B \to [0,1]$ be a fuzzy relation. We say that:*
(a) R *is* univalent *if* $R^{\smile}; R \subseteq \mathbb{I}_B$.
(b) R *is* total *if* $\mathbb{I}_A \subseteq R; R^{\smile}$.
(c) R *is a* map *if* R *is univalent and total.*
(d) R *is* injective *if* $R^{\smile}$ *is univalent.*
(e) R *is* surjective *if* $R^{\smile}$ *is total.*
(f) R *is an* isomorphism *if* R *and* $R^{\smile}$ *are maps.*

Proposition 3.4. *Let $\left(X, (A_i)_{i\in I}\right)$ be a normal covering, R its associated normal relation, i.e. $R : X \times I \to [0,1], R(x,i) = A_i(x)$ and $\left(X, (A_i)_{i\in I}\right)^{\smile} = \left(I, (B_x)_{x\in X}\right)$ its inversion, where $B_x(i) = A_i(x)$, for all $i \in I$ and for all $x \in X$. Then:*
(a) *R is total.*
(b) *R is a map if and only if there exists $J \subseteq I$ such that $\left(X, (A_i)_{i\in J}\right)$ is a partition and $A_i = \emptyset$ for all $i \in I \backslash J$.*
(c) *R is injective if and only if $\left|A_i^{\uparrow}\right| \le 1$ for all $i \in I$.*
(d) *R is surjective if and only if $\left(X, (A_i)_{i\in I}\right)$ is a normal covering.*
(e) *R is an isomorphism if and only if $\left(X, (A_i)_{i\in I}\right)$ is a partition with $\left|A_i^{\uparrow}\right| = 1$ for all $i \in I$.*

Proof. (a) Since $\left(X, (A_i)_{i\in I}\right)$ is a covering, for any $x \in X$, there exists $i \in I$ such that $A_i(x) = 1$. Therefore $\bigvee_{i\in I} (A_i(x) \wedge A_i(y)) = 1$ for all $x = y$. Hence R is total.

(b) Let $J \subset I$ with $A_i = \emptyset$ only if $i \in I \setminus J$. Then $\left(X, (A_i)_{i\in J}\right)$ is a partition if and only if $(R^{\smile}; R)(i,j) = \bigvee_{x\in X} (A_i(x) \wedge A_j(x)) = \begin{cases} 1, & i = j \in J \\ 0, & \text{otherwise} \end{cases}$, that is R is univalent. The conclusion follows from (a).

(c) If there exists $i \in I$ such that $\left|A_i^{\uparrow}\right| > 1$ then there exists $x, y \in X$, $x \ne y$ such that $A_i(x) \wedge A_i(y) = 1$, hence R is not injective. The converse is similar.

(d) R is surjective if and only if $|A_i^{\uparrow}| \ge 1$ for all $i \in I$. Note that $|A_i^{\uparrow}| \ge 1$ for all $i \in I$ if and only if $\left(X, (A_i)_{i\in I}\right)$ is a normal covering.

(e) Follows from (c) and (d). □

4. The category of fuzzy tolerance relations

Definition 4.1. *(Belohlávek et al. [3]) Let $T : X \times X \to [0,1]$ be a fuzzy relation. We say that:*
(a) T *is* reflexive *if* $T(x,x) = 1$*, for all* $x \in X$.
(b) T *is* symmetric *if* $T(x,y) = T(y,x)$*, for all* $x, y \in X$.
(c) T *is a* fuzzy tolerance relation *if it is reflexive and symmetric.*

Definition 4.2. *Let* Tol *be the category of fuzzy tolerance relations which has:*
(1) $\mathrm{Ob(Tol)} = \{(X,T) | T : X \times X \to [0,1]$ *is a fuzzy tolerance relation*$\}$.
(2) $\mathrm{Hom}\left((X,T), (Y,S)\right) = \{f : X \to Y | T(x,y) \le S\left(f(x), f(y)\right), (\forall) x, y \in X\}$.
(3) *If $f : (X,T) \to (Y,S)$ and $g : (Y,S) \to (Z,Q)$ then their composition is $g \circ f : (X,T) \to (Z,Q)$, $(g \circ f)(x) = g(f(x))$.*

(4) $id_{(X,T)} = id_X$, *for all* $(X, T) \in \mathrm{Ob}(\mathrm{Tol})$.

Remark 4.1. There exists a faithful functor F : Tol $\to$ N-Covering where on objects $F(X,T) = \left(X, (T_x)_{x\in X}\right)$ where $T_x : X \to [0,1]$, $T_x(y) = T(x,y)$ for all $x, y \in X$ and on morphisms $F(f) = (f, f)$.

Proof. Let $f : (X,T) \to (Y,S)$ be a morphism in Tol. We have
$T(x,y) \le S(f(x), f(y))$ for all $x, y \in X$, which can be rewritten as

$$T_x(y) \le S_{f(x)}(f(y))$$

for all $x, y \in X$. Which means that (f, f) is a morphism in *Covering* and F is well defined faithful functor. □

Remark 4.2. The set of all fuzzy tolerance relations on X forms a bounded lattice with the partial order given by $\subseteq$, introduced in Definition 3.4.

The smallest fuzzy tolerance relation is $0 = (X, \mathbb{I}_X)$ and the greatest fuzzy tolerance relation is $1 = (X, X \times X)$.

In the following Theorem, we present basic constructions in the Tol category.

Theorem 4.1. *In the* Tol *category, we have the following:*

(a) *The* initial object *is* $(\emptyset, \mathbb{I}_\emptyset)$ *and the* terminal object *is* $\left(\{*\}, \mathbb{I}_{\{*\}}\right)$.

(b) *The* product *of two fuzzy tolerance relations* (X,T) *and* (Y,S) *is the fuzzy tolerance relation* $(X \times Y, T \times S)$, *where for all* $x_1, x_2 \in X$, *and for all* $y_1, y_2 \in Y$ *we have* $(T \times S)((x_1,y_1),(x_2,y_2)) = T(x_1,x_2) \wedge S(y_1,y_2)$.

(c) *The* coproduct *of two fuzzy relations* (X,T) *and* (Y,S) *is the fuzzy tolerance relation* $(X \coprod Y, T \coprod S)$, *where* $(T \coprod S)(x,y) = \begin{cases} T(x,y), & x, y \in X \\ S(x,y), & x, y \in Y \end{cases}$.

(d) *The* equalizer *of* $f, g : (X,T) \to (Y,S)$ *is the object* (X_0, T_0) *where* $X_0 = \{x \in X | f(x) = g(x)\}$ *and* $T_0 : X_0 \times X_0 \to [0,1]$, $T_0(x,y) = T(x,y)$.

(e) *The* coequalizer *of* $f, g : (X,T) \to (Y,S)$ *is the object* $\left(Y/R, \widehat{S}\right)$ *where:*
   (i) R *is the equivalence relation on* Y *generated by* $f(x)Rg(x)$.
   (ii) $\widehat{S}(\widehat{x}, \widehat{y}) = \bigvee_{x\in\widehat{x}} (\vee_{y\in\widehat{y}} S(x,y))$, *for all* $\widehat{x}, \widehat{y} \in Y \backslash R$.

(f) *The* pullback *of* $f : (X,T) \to (Z,U)$ *and* $g : (Y,S) \to (Z,U)$ *is the fuzzy tolerance relation* $(X \times_Z Y, T \times_U S)$, *where* $X \times_Z Y = \{(x,y) | f(x) = g(y)\}$ *and for all* $(x_1,y_1),(x_2,y_2) \in X \times_Z Y$ *we have:*
$(T \times_U S)((x_1,y_1),(x_2,y_2)) = T(x_1,x_2) \wedge S(y_1,y_2)$.

(g) *The* pushout *of* $f : (X,T) \to (Y,S)$ *and* $g : (X,T) \to (Z,U)$ *is the fuzzy tolerance relation* $(Y \coprod_X Z, S \coprod_T U)$, $(Y \coprod_X Z) = \{(f(x), g(x)) \, | x \in X\}$, *and on* $Y \coprod_X Z$ *we have the equivalence relation* R *generated by* $f(x)Rg(x)$ *and for all* $\widehat{(y_1,z_1)}, \widehat{(y_2,z_2)} \in Y \coprod_X Z$ *we have:*

$$\left(S \coprod_T U\right)\left(\widehat{(y_1,z_1)}, \widehat{(y_2,z_2)}\right) = \left(\bigvee_{(y_1,y_2)\in(\widehat{y_1},\widehat{y_2})} S(y_1,y_2)\right) \vee \left(\bigvee_{(z_1,z_2)\in(\widehat{z_1},\widehat{z_2})} U(z_1,z_2)\right).$$

(h) *The* exponential *of the objects* (X,T) *and* (Y,S) *is the tolerance relation* $(Y,S)^{(X,T)} = \left(Y^X, S^T\right)$*, where* $Y^X = \mathrm{Hom}\left((X,T),(Y,S)\right)$ *and*

$$S^T(f,g) = \bigwedge_{x,y\in X} \left(T\left(x,y\right) \to S\left(f(x),g(y)\right)\right).$$

Proof. (a) For every tolerance relation (X,T) there exists a single morphism from $(\emptyset, \mathbb{I}_\emptyset)$ to (X,T) and a single morphism from (X,T) to $\left(\{*\}, \mathbb{I}_{\{*\}}\right)$.

(b) We claim that $T\times S$ is a fuzzy tolerance relation for all T and S fuzzy tolerance relations. Indeed:

(i) For all $(x,y)\in X\times Y$, $(T\times S)\left((x,y),(x,y)\right) = T(x,x)\wedge S(y,y) = 1$, hence $T\times S$ is reflexive.

(ii) For all $(x_1,y_1),(x_2,y_2)\in X\times Y$, we have $(T\times S)\left((x_1,y_1),(x_2,y_2)\right) = T\left(x_1,x_2\right)\wedge S\left(y_1,y_2\right) = T\left(x_2,x_1\right)\wedge S\left(y_2,y_1\right) = (T\times S)\left(x_2,y_2\right),\left(x_1,y_1\right)$, hence $T\times S$ is symmetric.

The associated morphisms to the product are:

$$p_X : (X\times Y, T\times S) \to (X,T),\ p_X(x,y) = x,\ (\forall)(x,y)\in X\times Y, \text{ and}$$
$$p_Y : (X\times Y, T\times S) \to (Y,S),\ p_Y(x,y) = y,\ (\forall)(x,y)\in X\times Y.$$

Let $f:(Z,U)\to(X,T)$ and $g:(Z,U)\to(X,T)$ be two morphisms. Then there exists a unique morphism $h:(Z,U)\to(X\times Y, T\times S)$, where

$$h(x,y) = \left(f\left(x\right), g\left(y\right)\right), \text{ for all } (x,y)\in X\times Y,$$

such that $p_X\circ h = f$ and $p_Y\circ h = g$.

(c) The coproduct $(X\coprod Y, T\coprod S)$ is obviously a fuzzy tolerance relation. The morphisms associated to the coproduct are $i_X:(X,T)\to(X\coprod Y, T\coprod S)$ and $i_Y:(Y,S)\to(X\coprod Y, T\coprod S)$ where $i_X(x) = x$, for all $x\in X$ and
$i_Y(y) = y$, for all $y\in Y$.

We note that for all the morphisms $f:(X,T)\to(Z,T)$, $g:(Y,S)\to(Z,T)$ there exists a unique morphism $h:(X\coprod Y, T\coprod S)\to(Z,T)$ with the property $f = h\circ i_X$ and $g = h\circ i_Y$, which is defined by $h(x) = \begin{cases} f(x), & x\in X \\ g(x), & x\in Y \end{cases}$.

(d) Let $i:(X_0,T_0)\to(X,T)$, $i(x) = x$, for all $x\in X$. We note that $T_0\left(x_1,x_2\right) = T\left(i\left(x_1\right), i\left(x_2\right)\right)$ and $f\circ i = g\circ i$.

To prove the universal property we consider $v:(Z,U)\to(X,T)$, a morphism with the properties:

(i) $U\left(z_1,z_2\right)\le T\left(v(z_1),v(z_2)\right)$, $(\forall)z_1,z_2\in Z$.

(ii) $f\circ v = g\circ v$.

Since $f(v(z)) = g(v(z))$, for all $z\in Z$, then $v(z)\in X_0$, for all $z\in Z$, and we can define the unique morphism $h:(Z,U)\to(X_0,T_0)$, by setting $h(z) = v(z)$, for all $z\in Z$.

(e) Let $p:(Y,S)\to\left(Y/R,\widehat{S}\right)$, $p(y) = \widehat{y}$ for all $y\in Y$. Obviously, p is a morphism.

Let (Z,U) be a fuzzy tolerance relation and let $u:(Y,S)\to(Z,U)$ be a morphism in Tol. It follows that $S\left(y_1,y_2\right)\le U\left(u\left(y_1\right),u\left(y_1\right)\right)$, $(\forall)y_1,y_2\in Y$. But that means that $S\left(y_1,y_2\right)\le\widehat{S}\left(p\left(y_1\right),p\left(y_2\right)\right)\le U\left(\left(u\circ p\right)\left(y_1\right),\left(u\circ p\right)\left(y_2\right)\right)$.

Then the unique function $h : (Y/R, \widehat{S}) \to (Z, U)$ with the property $h \circ p = u$ is $h(\widehat{y}) = u(y)$, where $y \in \widehat{y}$, for all $y \in Y$.

(f) We note that $(X \times_Z Y, T \times_U S)$ is a fuzzy tolerance relation and the projection morphisms $p_X : (X \times_Z Y, T \times_U S) \to (X, T)$ and
$p_Y : (X \times_Z Y, T \times_U S) \to (Y, S)$ on X, respectively on Y, have the property $f \circ p_X = g \circ p_Y$.

Let (V, R) be a fuzzy tolerance relation and $h_1 : (V, R) \to (X, T)$ and $h_2 : (V, R) \to (Y, S)$ be two morphisms with the property $f \circ h_1 = g \circ h_2$. Then there is a unique morphism $h : (V, R) \to (X \times_Z Y, T \times_U S)$, where $h(v) = (h_1(v), h_2(v))$, $(\forall) v \in V$, which has the properties $p_X \circ h = h_1$ and $p_Y \circ h = h_2$. Hence, $(X \times_Z Y, T \times_U S)$ is the pullback of (X, T) and (Y, S).

(g) From the way it is defined it is obvious that $(Y \coprod_X Z, S \coprod_T U)$ is a fuzzy tolerance relation. Its associated morphisms are
$p_1 : (Z, U) \to (Y \coprod_X Z, S \coprod_T U)$ and $p_2 : (Y, S) \to (Y \coprod_X Z, S \coprod_T U)$, where $p_1(z) = (\widehat{y, z})$, for all $z \in Z$ and $p_2(y) = (\widehat{y, z})$, for all $y \in Y$. The morphisms have the property $p_1 \circ f = p_2 \circ g$.

To prove that they have the universal property we choose a tolerance relation (V, Q), together with two morphisms $u_1 : (Z, U) \to (V, R)$ and
$u_2 : (Y, S) \to (V, R)$ with the property $u_1 \circ f = u_2 \circ g$. Since u_1 and u_2 are morphisms in Tol it follows that

(i) $U(z_1, z_2) \leq R(u_1(z_1), u_1(z_2))$, for all $z_1, z_2 \in Z$.

(ii) $S(y_1, y_2) \leq R(u_2(y_1), u_2(y_2))$, for all $y_1, y_2 \in Y$.

Then, the unique morphism $u_1, u_2) : (Y \coprod_X Z, S \coprod_T U) \to (V, R)$, where $(u_1, u_2)(\widehat{y, z}) = (u_2(\widehat{y}), u_1(\widehat{z}))$, $(\forall) y \in Y, z \in Z$, satisfies the properties $(u_1, u_2) \circ p_1 = u_1$ and $(u_1, u_2) \circ p_2 = u_2$.

(h) We firstly prove that $\left(Y^X, S^T\right)$ is a fuzzy tolerance relation:

(i) We have that $S^T(f, f) = 1$, for all $f \in Y^X$, since $T(x, y) \leq S(f(x), f(y))$, $(\forall) x, y \in X \Longleftrightarrow S^T(f, f) = 1$, $(\forall) f \in Y^X$.

(ii) We have that $S^T(f, g) = S^T(g, f)$, for all $f, g \in Y^X$, since T and S are symmetric.

Let ev : $\left(Y^X, S^T\right) \times (X, T) \to (Y, S)$ be the *evaluation morphism*, that is $\mathrm{ev}(f, x) = f(x)$ for all $f \in Y^X$ and $x \in X$. We prove that $\left(Y^X, S^T\right)$ has the universal property.

Let (Z, U) be a fuzzy tolerance relation and $g : (Z, U) \times (X, T) \to (Y, S)$ be a morphism.

Let $\lambda g : (Z, U) \to \left(Y^X, S^T\right)$, where $\lambda g(z) = g(z, -) : (X, T) \to (Y, S)$ is a function. The function λg is a morphism, since for all $z \in Z$ and $x_1, x_2 \in X$ we have $(U, T)\left((z, x_1), (z, x_2)\right) = T\left(x_1, x_2\right) \leq S\left(g(z, x_1), g(z, x_2)\right)$.

Then $\lambda g \in Y^X$ is the unique morphism such that $\mathrm{ev} \circ \left(\lambda g \times id_{(X,T)}\right) = g$, which completes the proof. □

Definition 4.3. *(Belohlávek et al.* [3]*) A normal fuzzy set $A : X \to [0, 1]$ is called a* preclass *of the fuzzy tolerance relation $T : X \times X \to [0, 1]$ if for all $x, y \in X$, it holds that $A(x) \wedge A(y) \leq T(x, y)$*

Definition 4.4. *(Belohlávek et al. [3]) A preclass $K : X \to [0,1]$ of the fuzzy tolerance relation T is called a* class *of T if for all the preclasses A of T with $A(x) \geq K(x)$, for all $x \in X$, we have $A = K$.*

Definition 4.5. *A* base *of the fuzzy tolerance relation T is a set of classes K_i of T such that $T(x,y) = \bigvee_{i\in I}(K_i(x) \wedge K_i(y))$, $(\forall)x, y \in X$, and the set of all classes K_i, for all $i \in I - \{i_0\}$ with $i_0 \in I$, is not a base.*

Remark 4.3. For any $x \in X$ the crisp set $A_x := \{x\}$ is a preclass of T. Indeed, $A_x(x) \wedge A_x(x) = T(x,x) = 1$ and $A_x(x) \wedge A_x(y) = 0 \leq T(x,y)$ for all $y \neq x$. If follows that $(X, (A_x)_{x\in X})$ is a covering of X with preclasses of T. For $x \in X$, let $\mathcal{F}_x := \{F$ preclass of T with $A_x(y) \leq F(y)$, $(\forall)y \in X\}$. By Zorn's Lemma, we can choose a preclass $K_x \in \mathcal{F}_x$, maximal with respect to inclusion. Then K_x is a class of T with $K_x(x) = 1$. It follows that $(X, (K_x)_{x\in X})$ is a covering. Hence, X can be covered by classes of T.

If T can be covered with a finite number of classes (for example, when X is finite), then, we can obtain a (finite) base by removing classes and checking whether the remaining collection still restores the fuzzy tolerance relation. Also, a tolerance relation could have more bases, see [6, Example 5]. Assume that T has an infinite base $(K_i)_{i\in I}$. Since $\bigvee_{i\in I} K_i(x) = 1$ does not imply that there exists some i with $K_i(x) = 1$, $(X, (K_i)_{i\in I})$ is not necessarily a covering.

Lemma 4.1. *Let $T : X \times X \to [0,1]$ be a fuzzy tolerance relation. If T has a finite base $H = \{K_i | i \in I\}$ then $\left(X, (K_i)_{i\in I}\right)$ is a covering.*

Proof. For $x \in X$ we have that $1 = T(x,x) = \bigvee_{i\in I}(K_i(x) \wedge K_i(x))$. Since I is a finite set, it follows that there exists $i \in I$ such that $K_i(x) = 1$. Hence $\left(X, (K_i)_{i\in I}\right)$ is a covering. □

Proposition 4.1. *Let $T : X \times X \to [0,1]$ be a fuzzy relation. Then:*

(1) *$(X, T) \in \mathrm{Ob(Tol)}$ if and only if $(X, T_\alpha) \in \mathrm{Ob(Tol)}$, for all $\alpha \in (0,1]$, where T_α is the $\alpha-$cut of the fuzzy set $T : X \times X \to [0,1]$.*

(2) *If T is a tolerance relation and $H = \{K_i | i \in I\}$ is a finite base of T, then $T_\alpha(x,y) = \bigvee_i (K_{i_\alpha}(x) \wedge K_{i_\alpha}(y))$, for all $\alpha \in [0,1]$.*

Proof. (1) The assertion follows from the facts:

(i) $T_\alpha(x,x) = 1$, $(\forall)x \in X$ and $(\forall)\alpha \in (0,1] \Leftrightarrow T(x,x) = 1$, $(\forall)x \in X$.

(ii) $T_\alpha(x,y) = T_\alpha(y,x)$, $(\forall)x, y \in X$ and $(\forall)\alpha \in (0,1] \Leftrightarrow T(y,x) = T(y,x)$, $(\forall)x, y \in X$.

(2) It follows from applying the $\alpha-$cut operation in the formula $T(x,y) = \bigvee_{i\in I}(K_i(x) \wedge K_i(y))$, from Definition 4.5. □

In the following theorem, we establish some connections between Covering and Tol.

Theorem 4.2. *Let F : Covering $\to$ Tol be defined:*

(i) *On objects: $F\left(X, (A_i)_{i\in I}\right) = \left(X, \bigvee_{i\in I}(A_i(x) \wedge A_i(y))\right)$, where $\left(X, (A_i)_{i\in I}\right)$ is a covering.*

(ii) *On morphisms $F(f, \rho) = f$.*

Let G : Tol $\to$ Covering be defined:

(i) *On objects: $G(X,T) = \left(X, (T(x,-))_{x\in X}\right)$, where $(X,T) \in \mathrm{Ob(Tol)}$.*

(ii) *On morphisms:* $G(f) = (f, f)$.

We have that:

(1) *The functors* F *and* G *are correctly defined.*

(2) F *is surjective on objects.*

(3) G *is injective on objects.*

(4) $(G \circ F \circ G)(X, T) = (X, T; T)$, *for all* $(X, T) \in \mathrm{Ob}(\mathrm{Tol})$.

Proof. (1) It is obvious that F is correctly defined on objects.

Let $(f, \rho) : \left(X, (A_i)_{i\in I}\right) \to \left(Y, (B_j)_{j\in J}\right)$ be a morphism in Covering.

Since $A_i(x) \wedge A_i(y) \le \left(B_{\rho(i)}(f(x)) \wedge B_{\rho(i)}(f(y))\right)$, $(\forall)x, y \in X$, $(\forall)i \in I$, it follows that $\bigvee_{i\in I}(A_i(x) \wedge A_i(y)) \le \bigvee_{i\in I}\left(B_{\rho(i)}(f(x)) \wedge B_{\rho(i)}(f(y))\right)$. Then $F(f, \rho) \in \mathrm{Hom}_{\mathrm{Tol}}\left(F\left(X, (A_i)_{i\in I}\right), F\left(Y, (B_j)_{j\in J}\right)\right)$.

In order to prove that G is correctly defined, we let $A_x(y) = T(x, y)$ for all x, $y \in X$ and $B_x(y) = S(x, y)$ for all x, $y \in Y$. We have that $G(X, T) = (X, (A_x)_{x\in X})$. Since T is reflexive, it follows that $A_x(x) = 1$, for all $x \in X$, thus $G(X, T) \in \mathrm{Ob}(\mathrm{Covering})$. Let $f \in \mathrm{Hom}\left((X, T), (Y, S)\right)$. Then $T(x, y) \le S(f(x), f(y)), (\forall)x, y \in X \Longrightarrow A_x(y) \le B_{f(x)}(f(y)), (\forall)x, y \in X$.

(2) Assume T has a finite base $H = \{K_i | i \in I\}$. From Lemma 4.1, it follows that $(X, (K_i)_{i\in I})$ is a covering. We have that

$$F\left(X, (K_i)_{i\in I}\right) = \left(X, \left(\bigvee_i (K_i(x) \wedge K_i(y))\right)_{x,y\in X}\right) = (X, T).$$

It remains to study the case when T does not have a finite base. For $x, y \in X$, we define a fuzzy set $A_{xy} : X \to [0, 1]$, by setting $A_{xy}(z) = \begin{cases} T(x, y), & z = x \\ T(x, y), & z = y \\ 0, & otherwise \end{cases}$.

Since T is reflexive and $\left(X, (A_{xx})_{x\in X}\right) \subseteq \left(X, (A_{xy})_{x,y\in X}\right)$ it follows that $\left(X, (A_{xy})_{x,y\in X}\right)$ is a covering. We have that:

$$F\left(X, (A_{xy})_{x,y\in X}\right) = \left(X, \left(\bigvee_{x,y\in X} \left(A_{xy}(x') \wedge A_{xy}(y')\right)\right)_{x',y'\in X}\right) = (X, T)$$

(3) If $G(X, T) = G(X, S)$, then $T(x, y) = S(x, y)$ for all $x, y \in X$. Hence, G is injective.

(4) For any $(X, T) \in \mathrm{Ob}(\mathrm{Tol})$, we have that:

$$(G \circ F \circ G)(X, T) = (G \circ F)\left(X, (T(-, x))_{x\in X}\right) =$$

$$= \left(X, \bigvee_{y\in X} (T(x, y) \wedge T(y, z))\right) = (X, T; T),$$

hence we are done. □

Definition 4.6. *We say that* $\left(X, (A_i)_{i\in I}, \alpha\right)$ *is a* fuzzy tolerance covering *if* $\left(X, (A_i)_{i\in I}\right)$ *is a covering and* $\alpha : I \to \mathcal{P}^*(X)$, *where* $\mathcal{P}^*(X)$ *is the set of non-empty subsets of* X, *has the following properties:*

(a) *For all* $x \in X$*, there exists a unique* $i \in I$ *such that* $x \in \alpha(i)$.

(b) $A_i(x) = 1$*, for all* $x \in \alpha(i)$.

(c) $A_i(y) = A_j(x)$*, for all* $x \in \alpha(i)$ *and* $y \in \alpha(j)$.

(d) $A_i(x) = A_j(x)$ *for all* $x \in X \Leftrightarrow i = j$.

Example 4.1. To clarify the definition above we will construct a fuzzy tolerance covering. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $I = \{1, 2, 3\}$. We define $\alpha : I \to \mathcal{P}^*(X)$ such that the condition (a) is fulfilled, by setting $\alpha(1) = \{x_1, x_2\}, \alpha(2) = \{x_3, x_4, x_5\}, \alpha(3) = \{x_6\}$ Condition (b) implies: $A_1(x_1) = A_1(x_2) = 1, A_2(x_3) = A_2(x_4) = A_2(x_5) = 1, A_3(x_6) = 1$

		A_1		A_2			A_3
		x_1	x_2	x_3	x_4	x_5	x_6
A_1	x_1	1	1	$A_1(x_3)$	$A_1(x_3)$	$A_1(x_3)$	$A_1(x_6)$
	x_2	1	1	$A_1(x_3)$	$A_1(x_3)$	$A_1(x_3)$	$A_1(x_6)$
	x_3	$A_1(x_3)$	$A_1(x_3)$	1	1	1	$A_2(x_6)$
A_2	x_4	$A_1(x_3)$	$A_1(x_3)$	1	1	1	$A_2(x_6)$
	x_5	$A_1(x_3)$	$A_1(x_3)$	1	1	1	$A_2(x_6)$
A_3	x_6	$A_1(x_6)$	$A_1(x_6)$	$A_2(x_6)$	$A_2(x_6)$	$A_2(x_6)$	1

Condition (c) implies:

- $A_1(x_3) = A_1(x_4) = A_1(x_5) = A_2(x_1) = A_2(x_2)$ because $x_1, x_2 \in \alpha(1)$ and $x_3, x_4, x_5 \in \alpha(2)$.
- $A_1(x_6) = A_3(x_1) = A_3(x_2)$ because $x_1, x_2 \in \alpha(1)$ and $x_6 \in \alpha(3)$.
- $A_2(x_6) = A_3(x_3) = A_3(x_4) = A_3(x_5)$ because $x_3, x_4, x_5 \in \alpha(2)$ and $x_6 \in \alpha(3)$.

Condition (d) implies that the fuzzy sets A_i's are distinct, for all $i \in I$. We can represent $\left(X, (A_i)_{i \in I}, \alpha\right)$ in the table:

Remark 4.4. If $\left(X, (A_i)_{i \in I}, \alpha\right)$ is a fuzzy tolerance covering, then for all $i \in I$ the fuzzy set A_i can be written in terms of all the other fuzzy sets of the fuzzy tolerance covering i.e. for all $i \in I$ we have:

$$A_i(x) = \begin{cases} 1, & x \in \alpha(i) \\ A_j(y), & x \in \alpha(j), y \in \alpha(i) \end{cases}$$

Proposition 4.2. *If* $\left(X, (A_i)_{i \in I}, \alpha\right)$ *is a fuzzy tolerance covering, then:*

(1) $\left(X, (\alpha(i))_{i \in I}\right)$ *is a partition.*

(2) $\left(X, (\alpha(i))_{i \in I}\right) \subseteq \left(X, (A_i)_{i \in I}\right)$.

Proof. (1) According to Definition 4.6(a) for all $x \in \alpha(i)$ these exists a unique $i \in I$ such that $x \in X$. Hence $\left(X, (\alpha(i))_{i \in I}\right)$ is a partition.

(2) We have $A_i(x) = 1$, for all $x \in \alpha(i)$ therefore $\alpha(i) \subseteq A_i(x)$. □

Definition 4.7. *Let* t-Covering *be the category which has:*

(a) *Objects: The tolerance coverings.*

(b) *Morphisms: The functions* $f : X \to Y$ *such that for all* $x \in X$ *and* $i \in I$*, there exists* $j \in J$ *such that* $A_i(x) \leq B_j(f(x))$.

(c) *Composition is the usual function composition.*

Remark 4.5. Let $\left(X,(A_i)_{i\in I},\alpha\right)$ be a fuzzy tolerance covering and let

$$H=\{f:X\to X\mid (\forall)x\in X, i\in I,\ \text{if } x\in\alpha(i),\ \text{then } f(x)\in\alpha(i)\}.$$

Then $H\subseteq \mathrm{Hom}_{\text{t-Covering}}\left(\left(X,(A_i)_{i\in I},\alpha\right),\left(X,(A_i)_{i\in I},\alpha\right)\right)$. Indeed, it is enough to note that $A_i(x)=A_i\left(f(x)\right)$, for all $i\in I$ and $x\in X$.

Theorem 4.3. *The categories* Tol *and* t-Covering *are isomorphic.*

Proof. Let $T:X\times X\to[0,1]$ be a fuzzy tolerance relation. On X, we consider the equivalence relation $\sim$, defined by $x\sim x'$ if and only if $T(x,-)=T(x',-)$. We choose $(x_i)_{i\in I}$ a complete set of class representatives. We let

$$\alpha:I\to\mathcal{P}^*(X),\ \alpha(i)=\{x\in X\ |T(x,-)=T(x_i,-)\}.$$

We define $F:\mathrm{Tol}\to\text{t-Covering}$ as follows:
(a) We let $F(T)=\left(X,(A_i)_{i\in I},\alpha\right)$, where $A_i(y)=T(x,y)$, for all $x\in\alpha(i)$, where $\alpha(i)$ was defined above.
(b) For all $f:X\to Y$ functions and $T:X\times X\to[0,1]$, $S:Y\times Y\to[0,1]$ tolerance relations such that $T(x,y)\leq S\left(f(x),f(y)\right)$ let $F(f)=f$.
We check that $\left(X,(A_i)_{i\in I},\alpha\right)$ is a tolerance covering, i.e.:
(a) For all $x\in X$ there exists a unique $i\in I$ such that $x\in\alpha(i)$. Indeed, this is clear from the fact that $\sim$ is an equivalence relation.
(b) For all $x\in\alpha(i)$ we have $A_i(x)=1$. Indeed, since $A_i(y)=T(x,y)$ for all $y\in X$ and T is reflexive, it follows that $A_i(x)=T(x,x)=1$.
(c) For all $x\in\alpha(i)$ and $y\in\alpha(j)$ we have $A_i(y)=A_j(x)$. Indeed, since T is symmetric, it follows that $A_i(y)=T(x,y)=T(y,x)=A_j(x)$.
(d) $A_i(x)=A_j(x)$ for all $x\in X$ if and only if $i=j$.
We know that $A_i(x)=T(z,x)$ for all $z\in\alpha(i)$ and $x\in X$ and
$A_j(x)=T(z,x)$ for all $z\in\alpha(i)$ and $x\in X$. Then $z\in\alpha(i)\cap\alpha(j)$. From (a) it follows that $i=j$.
Hence, F is correctly defined.

We define a functor $G:\text{t-Covering}\to\mathrm{Tol}$, by:
(a) For all $\left(X,(A_i)_{i\in I},\alpha\right)$ tolerance coverings we let $G\left(X,(A_i)_{i\in I},\alpha\right)=(X,T)$ where $T(x,y)=A_i(y)$ for all $x\in\alpha(i)$.
(b) For all $f:\left(X,(A_i)_{i\in I},\alpha\right)\to\left(Y,(B_j)_{j\in j},\beta\right)$ let $G(f)=f$.
We prove that G is correctly defined. From Definition 4.6(a), it follows that for all $x\in X$ there exists a unique $i\in I$ such that $x\in\alpha(i)$, hence $T(x,y)=A_i(y)$ is correctly defined. We prove that T is a tolerance relation:
(a) T is reflexive. For all $x\in X$ there exists a unique $i\in I$ such that $x\in\alpha(i)$. From Definition 4.6(b) it follows that $T(x,x)=A_i(x)=1$ for all $x\in X$.
(b) T is symmetrical. For all $x,y\in X$ there exist unique $i,j\in I$ such that $x\in\alpha(i)$ and $y\in\alpha(j)$. Since $T(x,y)=A_i(y)$ and $T(y,x)=A_j(x)$, from Definition 4.6(c) we get that $T(x,y)=T(y,x)$.
We note that:

(i) $(G \circ F)(X,T) = (X,T)$ for all $(X,T) \in \mathrm{Ob}(\mathrm{Tol})$, and $(G \circ F)(f) = f$ for any morphism $f : X \to Y$.
(ii) $(F \circ G)\big(X, (A_i)_{i \in I}, \alpha\big) = \big(X, (A_i)_{i \in I}, \alpha\big)$ for all $\big(X, (A_i)_{i \in I}, \alpha\big) \in \mathrm{Ob}(\text{t-Covering})$.

It follows that the functors F and G are inverse to each other. Hence Tol is isomorphic with t-Covering.

□

5. Conclusions

In summary, we study the category Tol of fuzzy tolerance relations and we establish connections with the category Covering of fuzzy coverings. Besides that, we construct an isomorphism between Tol and t-Covering, the category of tolerance coverings, which is a subcategory of Covering.

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