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MODULE JOHNSON AMENABILITY OF CERTAIN BANACH ALGEBRAS

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In this paper, we introduce the new notion module Johnson amenabil-ity for a Banach algebra which is a Banach module over another Banach algebra with compatible actions. We study the relations between this new notion and other various notions of module amenability. We characterize the module Johnson amenability of $\ell^1(S)$ as an $\ell^1(E)$-module, for an inverse semigroup S with subsemigroup E of idempotents. We investigate the module Johnson amenability of $\ell^1(S)$, whenever S is a Brandt semigroup or bicyclic semigroup or $\mathbb{N}$ with maximum as its product. As application we show that for every non-empty set Λ, $\mathbb{M}_\Lambda(\mathbb{C})$ as an $\mathfrak{A}$-module is module Johnson amenable if and only if Λ is finite, where $\mathfrak{A} = \left\{[a_{i,j}] \in \mathbb{M}_\Lambda(\mathbb{C}) \mid \forall i \neq j, a_{i,j} = 0\right\}$.

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1. Introduction and preliminaries

Amini introduced the concept of module amenability for a class of Banach algebras [1]. He showed that for an inverse semigroup S, $\ell^1(S)$ is module amenable if and only if S is amenable, where E is the set of idempotents [1, Theorem 3.1]. Some new generalizations of module amenability like module pseudo amenability, module pseudo-contractibility and module approximately amenability have been introduced, see [5], [11]. Bodaghi *et al.* showed that for an inverse semigroup S, $\ell^1(S)$ as an $\ell^1(E)$-module is module pseudo-amenable if and only if S is amenable [5, Theorem 3.13(i)]. Also the same result holds for the module approximately amenability [11, Theorem 3.9].

The notion of Johnson pseudo-contractibility for a Banach algebra was introduced by the second and third authors, which is a weaker notion than amenability and pseudo-contractibility but it is stronger than pseudo-amenability [14]. A Banach algebra $\mathcal{A}$ is called Johnson pseudo-contractible, if there exists a not necessarily bounded net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi^{**}_{\mathcal{A}}(m_\alpha)a \to a$ for

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every $a \in \mathcal{A}$. They also showed that for a locally compact group G, $M(G)$ is Johnson pseudo-contractible if and only if G is discrete and amenable [14, Proposition 3.3]. They characterized the Johnson pseudo-contractibility of $\ell^1(S)$, where S is a uniformly locally finite inverse semigroup [13, Theorem 2.3]. They showed that for a Brandt semigroup $S = M^0(G, I)$ over a non-empty set I, $\ell^1(S)$ is Johnson pseudo-contractible if and only if G is amenable and I is finite [13, Theorem 2.4]. By considering these notions, we generalize the concept of Johnson pseudo contractibility for a class of Banach algebras that are modules over another Banach algebra with compatible actions.

In part two of this paper, we define the module Johnson amenability for a Banach algebra $\mathcal{A}$ which is a Banach $\mathfrak{A}$-module. First we show that the module Johnson amenability is a stronger notion than module pseudo-amenability and module approximately amenability and it is a weaker notion than module pseudo-contractibility. Next for an inverse semigroup S with subsemigroup E of idempotents, we characterize the module Johnson amenability of $\ell^1(S)$ as an $\ell^1(E)$-module with amenability of S.

In part three, we provide some examples to distinguish our new notion with the Johnson pseudo-contractibility. Finally, as an application, we show that for every non-empty set Λ, the Banach algebra of $\Lambda \times \Lambda$-matrices over $\mathbb{C}$, $\mathbb{M}_\Lambda(\mathbb{C})$ as an $\mathfrak{A}$-module under this new notion is forced to have a finite index, where $\mathfrak{A} = \Big\{[a_{i,j}] \in \mathbb{M}_\Lambda(\mathbb{C}) \mid \forall i \neq j, a_{i,j} = 0\Big\}$.

2. **Module Johnson amenability**

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with the compatible actions:

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha),$$

for every $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$ and $x \in X$ and similarly for the right or two side actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for every $a \in \mathcal{A}$, $x \in X$, then X is called a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. If X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as follows:

$$\langle \alpha \cdot f, x\rangle = \langle f, x \cdot \alpha\rangle, \quad \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad (\alpha \in \mathfrak{A}, a \in \mathcal{A}, x \in X, f \in X^*),$$

and similarly for the right actions. Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D : \mathcal{A} \to X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

When X is commutative, every $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}),$$

These are called inner module derivations. $\mathcal{A}$ is called module amenable as an $\mathfrak{A}$-module, if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, every module derivation $D : \mathcal{A} \to X^*$ is inner [1, Definition 2.1].

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and let $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be the projective module tensor product of $\mathcal{A}$ and $\mathcal{A}$, which is isomorphic to the quotient space $\frac{\mathcal{A}\hat{\otimes}\mathcal{A}}{\mathcal{I}_{\mathcal{A}}}$, where $\mathcal{I}_{\mathcal{A}}$ is the closed linear span of

$$\Big\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b \mid \alpha \in \mathfrak{A}, a, b \in \mathcal{A}\Big\}$$

in $\mathcal{A}\hat{\otimes}\mathcal{A}$. Also consider the closed ideal $\mathcal{J}_{\mathcal{A}}$ of $\mathcal{A}$ generated by

$$\Big\{(a \cdot \alpha)b - a(\alpha \cdot b) \mid \alpha \in \mathfrak{A}, a, b \in \mathcal{A}\Big\}.$$

We denote $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{A}}$ by $\mathcal{I}$ and $\mathcal{J}$ repectively, unless otherwise specified. So $\mathcal{I}$ is an $\mathcal{A}$-submodule and a $\mathfrak{A}$-submodule of $\mathcal{A}\hat{\otimes}\mathcal{A}$, $\mathcal{J}$ is an $\mathcal{A}$-submodule and a $\mathfrak{A}$-submodule of $\mathcal{A}$, and both of the quotients $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ and $\frac{\mathcal{A}}{\mathcal{J}}$ are $\mathcal{A}$-module and $\mathfrak{A}$-module. Consider the product map $\omega_{\mathcal{A}} : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ defined by $a \otimes b \mapsto ab$ for every $a, b \in \mathcal{A}$ and let $\tilde{\omega}_{\mathcal{A}} : \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A} \to \frac{\mathcal{A}}{\mathcal{J}}$ be its induced product map defined by $\tilde{\omega}_{\mathcal{A}}(a \otimes b + \mathcal{I}) = ab + \mathcal{J}$. It is clear that $\tilde{\omega}_{\mathcal{A}}$ and $\tilde{\omega}_{\mathcal{A}}^{**} : \frac{(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}}{\mathcal{I}_{\mathcal{A}}^{\perp\perp}} \to \frac{\mathcal{A}^{**}}{\mathcal{J}^{\perp\perp}}$ are both $\mathcal{A}$-module morphisms and $\mathfrak{A}$-module morphisms.

Now we introduce the new notion of this work:

Definition 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. Then $\mathcal{A}$ is called module Johnson amenable, if there exists a not necessarily bounded net $(\tilde{m}_\alpha)$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ such that for every $a \in \mathcal{A}$*

(i) $a \cdot \tilde{m}_\alpha = \tilde{m}_\alpha \cdot a$,

(ii) $\tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha) \cdot a + \mathcal{J} \to a + \mathcal{J}^{\perp\perp}$ *in* $\frac{\mathcal{A}^{**}}{\mathcal{J}^{\perp\perp}}$.

Lemma 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. If $\mathcal{A}$ is Johnson pseudo-contractible, then $\mathcal{A}$ is module Johnson amenable.*

Proof. Since $\mathcal{A}$ is Johnson pseudo-contractible, there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that for every $a \in \mathcal{A}$

$$m_\alpha \cdot a = a \cdot m_\alpha \quad \text{and} \quad \omega_{\mathcal{A}}^{**}(m_\alpha) \cdot a \to a.$$

Let $\tilde{m}_\alpha = m_\alpha + \mathcal{I}^{\perp\perp}$ for every α. It is clear that $a \cdot \tilde{m}_\alpha = \tilde{m}_\alpha \cdot a$ for every $a \in \mathcal{A}$. By Goldstein's theorem one can see that $\tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha) = \omega_{\mathcal{A}}^{**}(m_\alpha) + \mathcal{J}^{\perp\perp}$. So

$$\tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha) \cdot (a + \mathcal{J}) = \omega_{\mathcal{A}}^{**}(m_\alpha) \cdot a + \mathcal{J}^{\perp\perp} \to a + \mathcal{J}^{\perp\perp}.$$

Thus the proof is complete. □

An element $\tilde{M} \in (\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ is called a module virtual diagonal if $\tilde{\omega}_{\mathcal{A}}^{**}(\tilde{M}) \cdot a = a + \mathcal{J}^{\perp\perp}$ and $\tilde{M} \cdot a = a \cdot \tilde{M}$ for every $a \in \mathcal{A}$ [12].

Lemma 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. If $\mathcal{A}$ has a module virtual diagonal, then $\mathcal{A}$ is module Johnson amenable.*

Proof. If $\mathcal{A}$ has a module virtual diagonal, then there exists an element $\tilde{M}$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ such that for every $a \in \mathcal{A}$

$$\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{M}) \cdot a = a + \mathfrak{J}^{\perp\perp} \quad \text{and} \quad \tilde{M} \cdot a = a \cdot \tilde{M}.$$

There exists an element M in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\tilde{M} = M + \mathcal{I}^{\perp\perp}$. By Goldstein's theorem one can see that $\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{M}) = \tilde{\omega}^{**}_{\mathcal{A}}(M + \mathcal{I}^{\perp\perp}) = \omega^{**}_{\mathcal{A}}(M) + \mathfrak{J}^{\perp\perp}$. By canonical module action for every $a \in \mathcal{A}$ we have

$$\begin{aligned}\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{M}) \cdot a + \mathfrak{J} &= (\omega^{**}_{\mathcal{A}}(M) + \mathfrak{J}^{\perp\perp}) \cdot a + \mathfrak{J} = \omega^{**}_{\mathcal{A}}(M) \cdot a + \mathfrak{J}^{\perp\perp} \\ &= (\omega^{**}_{\mathcal{A}}(M) + \mathfrak{J}^{\perp\perp}) \cdot a = \tilde{\omega}^{**}_{\mathcal{A}}(\tilde{M}) \cdot a = a + \mathfrak{J}^{\perp\perp}.\end{aligned}$$

It follows that $\mathcal{A}$ is module Johnson amenable. □

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. A Banach algebra $\mathcal{A}$ is said to be module pseudo-amenable (module pseudo-contractible), if there exists a net $(\tilde{u}_j)$ in $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ such that $\tilde{\omega}_{\mathcal{A}}(\tilde{u}_j) \cdot a + \mathfrak{J} \to a + \mathfrak{J}$ and $\tilde{u}_j \cdot a - a \cdot \tilde{u}_j \to 0$ ($\tilde{u}_j \cdot a - a \cdot \tilde{u}_j = 0$) for every $a \in \mathcal{A}$ [5, Definition 2.1], [5, Definition 2.2].

Proposition 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. If $\mathcal{A}$ is module Johnson amenable, then $\mathcal{A}$ is module pseudo amenable.*

Proof. Since $\mathcal{A}$ is module Johnson amenable, there exists a net $(\tilde{m}_\alpha)_{\alpha\in I}$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ such that $a\cdot\tilde{m}_\alpha = \tilde{m}_\alpha\cdot a$ and $\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\cdot a+\mathfrak{J} \to a+\mathfrak{J}^{\perp\perp}$ for every $a \in \mathcal{A}$. By Goldstein's theorem for every α there exists a net $(u^\alpha_\beta)_{\beta\in\Omega}$ in $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ such that $wk^*\text{-}\lim_\beta \hat{u}^\alpha_\beta = \tilde{m}_\alpha$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$. So

$$wk^*\text{-}\lim_\alpha wk^*\text{-}\lim_\beta a \cdot \hat{u}^\alpha_\beta - \hat{u}^\alpha_\beta \cdot a = 0 \quad \text{in} \quad (\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}, \tag{1}$$

and since $\tilde{\omega}^{**}_{\mathcal{A}}$ is a wk^*-continuous map,

$$wk^*\text{-}\lim_\alpha wk^*\text{-}\lim_\beta (\tilde{\omega}^{**}_{\mathcal{A}}(\hat{u}^\alpha_\beta) \cdot a + \mathfrak{J}) = a + \mathfrak{J}^{\perp\perp} \quad \text{in} \quad \frac{\mathcal{A}^{**}}{\mathfrak{J}^{\perp\perp}}. \tag{2}$$

Let $X = I \times \Omega^I$ be a directed set with the product ordering defined by

$$(\alpha, \beta) \leq_X (\alpha', \beta') \Leftrightarrow \alpha \leq_I \alpha' \text{ and } \beta \leq_{\Omega^I} \beta' \qquad (\alpha, \alpha' \in I, \quad \beta, \beta' \in \Omega^I),$$

where Ω^I is the set of all functions from I into Ω and $\beta \leq_{\Omega^I} \beta'$ means that $\beta(d) \leq_\Omega \beta'(d)$ for every $d \in I$. Suppose that $\gamma = (\alpha, \beta_\alpha)$ and $y_\gamma = u^\alpha_\beta$. Iterated limit theorem [10, Page 69], (1) and (2) imply that for every $a \in \mathcal{A}$

$$wk^*\text{-}\lim_\gamma a \cdot \hat{y}_\gamma - \hat{y}_\gamma \cdot a = 0 \quad \text{and} \quad wk^*\text{-}\lim_\gamma (\tilde{\omega}^{**}_{\mathcal{A}}(\hat{y}_\gamma) \cdot a + \mathfrak{J}) = a + \mathfrak{J}^{\perp\perp} = \widehat{a + \mathfrak{J}}.$$

So

$$wk\text{-}\lim_\gamma a \cdot y_\gamma - y_\gamma \cdot a = 0 \quad \text{and} \quad wk\text{-}\lim_\gamma (\tilde{\omega}_{\mathcal{A}}(y_\gamma) \cdot a + \mathfrak{J}) = a + \mathfrak{J} \quad (a \in \mathcal{A}).$$

By Mazur's lemma one can see that for every $a \in \mathcal{A}$

$$\lim_\gamma a \cdot y_\gamma - y_\gamma \cdot a = 0 \quad \text{and} \quad \lim_\gamma (\tilde{\omega}_{\mathcal{A}}(y_\gamma) \cdot a + \mathfrak{J}) = a + \mathfrak{J}.$$

It follows that $\mathcal{A}$ is module pseudo-amenable. □

Let $\mathcal{A}$ be a Banach algebra and an $\mathfrak{A}$-bimodule with compatible actions. Then $\mathcal{A}$ is module approximately amenable (as an $\mathfrak{A}$-module) if for every commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, every module derivation $D : \mathcal{A} \to X^*$ is approximately inner [11, Definition2.1].

Corollary 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with bounded approximate identity. If $\mathcal{A}$ is module Johnson amenable, then $\mathcal{A}$ is module approximately amenable.*

Proof. If $\mathcal{A}$ is module Johnson amenable, then Proposition 2.1 and [5, Theorem 3.2] imply that $\mathcal{A}$ is module approximately amenable. □

Proposition 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. If $\mathcal{A}$ is module pseudo-contractible, then $\mathcal{A}$ is module Johnson amenable.*

Proof. It is clear. □

Following [4, §2], let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible action and let $\varphi \in \Delta_{\mathfrak{A}} \cup \{0\}$, where $\Delta_{\mathfrak{A}}$ is a character space of $\mathfrak{A}$. Consider a linear map $\phi : \mathcal{A} \to \mathfrak{A}$ such that for every $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$

$$\phi(ab) = \phi(a)\phi(b), \quad \phi(a \cdot \alpha) = \phi(\alpha \cdot a) = \varphi(\alpha)\phi(a).$$

A bounded linear functional $m : \mathcal{A}^* \to \mathbb{C}$ is called a module (ϕ, φ)-mean on $\mathcal{A}^*$ if $m(f \cdot a) = \varphi \circ \phi(a)m(f)$, $m(f \cdot \alpha) = \varphi(\alpha)m(f)$ and $m(\varphi \circ \phi) = 1$ for every $f \in \mathcal{A}^*$, $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. We say that $\mathcal{A}$ is module (ϕ, φ)-amenable if there exists a module (ϕ, φ)-mean on $\mathcal{A}^*$.

Proposition 2.3. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module such that $a \cdot \alpha = \varphi(\alpha)a$ for every $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, where φ, ϕ are as above. If $\mathcal{A}$ is module Johnson amenable and $\alpha \cdot \tilde{m}_\alpha = \tilde{m}_\alpha \cdot \alpha$ for every $\alpha \in \mathfrak{A}$ ($\tilde{m}_\alpha$ as in Definition 2.1), then $\mathcal{A}$ is module (ϕ, φ)-amenable.*

Proof. Since $\mathcal{A}$ is module Johnson amenable, there exists a net $(\tilde{m}_\alpha)_{\alpha \in I}$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ such that $a \cdot \tilde{m}_\alpha = \tilde{m}_\alpha \cdot a$ and $\tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha) \cdot a + \mathfrak{J} \to a + \mathfrak{J}^{\perp\perp}$ for every $a \in \mathcal{A}$. Define $T : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ by $T(a \otimes b) = \varphi \circ \phi(b)a$ for every $a, b \in \mathcal{A}$. It is easy to see that $T = 0$ on $\mathfrak{I}$. So T drops to $\tilde{T} : \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A} \to \mathcal{A}$. Then for every $a \in \mathcal{A}$ and $\tilde{u} \in \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$

$$\tilde{T}(a \cdot \tilde{u}) = a\tilde{T}(\tilde{u}), \qquad \tilde{T}(\tilde{u} \cdot a) = \varphi \circ \phi(a)\tilde{T}(\tilde{u}), \tag{3}$$

and also

$$\tilde{T}(\alpha \cdot \tilde{u}) = \alpha \cdot \tilde{T}(\tilde{u}), \qquad \tilde{T}(\tilde{u} \cdot \alpha) = \varphi(\alpha)\tilde{T}(\tilde{u}) = \tilde{T}(\tilde{u}) \cdot \alpha. \tag{4}$$

Since $\phi|_{\mathfrak{J}} = 0$, ϕ drops to $\tilde{\phi} : \frac{\mathcal{A}}{\mathfrak{J}} \to \mathfrak{A}$. For every $a, b \in \mathcal{A}$ we have

$$\begin{aligned}\langle \tilde{T}(a \otimes b + \mathfrak{I}), \varphi \circ \phi\rangle &= \langle T(a \otimes b), \varphi \circ \phi\rangle = \langle \varphi \circ \phi(b)a, \varphi \circ \phi\rangle = \langle ab, \varphi \circ \phi\rangle \\ &= \langle ab + \mathfrak{J}, \varphi \circ \tilde{\phi}\rangle = \langle \tilde{\omega}_{\mathcal{A}}(a \otimes b + \mathfrak{I}), \varphi \circ \tilde{\phi}\rangle.\end{aligned}$$

Then

$$\langle \tilde{T}(u), \varphi \circ \phi\rangle = \langle \tilde{\omega}_{\mathcal{A}}(u), \varphi \circ \tilde{\phi}\rangle \qquad (u \in \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}). \tag{5}$$

Consider the map $\tilde{T}^{**} : (\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**} \to \mathcal{A}^{**}$. Since $\tilde{T}^{**}$ is wk^*-continuous, Goldstein's theorem, (3), (4), and (5) imply that for every $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$ and $F \in (\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$

$$\tilde{T}^{**}(a \cdot F) = a \cdot \tilde{T}^{**}(F), \quad \tilde{T}^{**}(F \cdot a) = \varphi \circ \phi(a)\tilde{T}^{**}(F), \tag{6}$$

and

$$\tilde{T}^{**}(\alpha \cdot F) = \alpha \cdot \tilde{T}^{**}(F), \quad \tilde{T}^{**}(F \cdot \alpha) = \varphi(\alpha)\tilde{T}^{**}(F) = \tilde{T}^{**}(F) \cdot \alpha, \tag{7}$$

and also

$$\langle \varphi \circ \phi, \tilde{T}^{**}(F)\rangle = \langle \varphi \circ \tilde{\phi}, \tilde{\omega}^{**}_{\mathcal{A}}(F)\rangle. \tag{8}$$

By (6) for every $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$ we have

$$\langle f{\cdot}a, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle = \langle f, a{\cdot}\tilde{T}^{**}(\tilde{m}_\alpha)\rangle = \langle f, \tilde{T}^{**}(a{\cdot}\tilde{m}_\alpha)\rangle = \langle f, \tilde{T}^{**}(\tilde{m}_\alpha{\cdot}a)\rangle = \varphi{\circ}\phi(a)\langle f, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle.$$

(7) imply that for every $\alpha \in \mathfrak{A}$ and $f \in \mathcal{A}^*$

$$\langle f{\cdot}\alpha, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle = \langle f, \alpha{\cdot}\tilde{T}^{**}(\tilde{m}_\alpha)\rangle = \langle f, \tilde{T}^{**}(\alpha{\cdot}\tilde{m}_\alpha)\rangle = \langle f, \tilde{T}^{**}(\tilde{m}_\alpha{\cdot}\alpha)\rangle = \varphi(\alpha)\langle f, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle.$$

Since $\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\cdot a + \mathfrak{J} \to a + \mathfrak{J}^{\perp\perp}$ for every $a \in \mathcal{A}$,

$$\begin{aligned}
\lim_\alpha \langle \varphi \circ \tilde{\phi}, \tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\rangle\langle \varphi \circ \tilde{\phi}, a + \mathfrak{J}^{\perp\perp}\rangle &= \lim_\alpha \langle \varphi \circ \tilde{\phi}(a + \mathfrak{J})\varphi \circ \tilde{\phi}, \tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\rangle \\
&= \lim_\alpha \langle a + \mathfrak{J} \cdot \varphi \circ \tilde{\phi}, \tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\rangle \\
&= \lim_\alpha \langle \varphi \circ \tilde{\phi}, \tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\cdot a + \mathfrak{J}\rangle \\
&= \langle \varphi \circ \tilde{\phi}, a + \mathfrak{J}^{\perp\perp}\rangle.
\end{aligned}$$

So $\langle \varphi{\circ}\tilde{\phi}, \tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\rangle \to 1$ in $\mathbb{C}$. (8) leads to $\langle \varphi{\circ}\phi, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle \to 1$. For sufficiently large α, $\langle \varphi \circ \phi, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle$ stays away from zero. Replacing $\tilde{T}^{**}(\tilde{m}_\alpha)$ by $\dfrac{\tilde{T}^{**}(\tilde{m}_\alpha)}{\langle \varphi \circ \phi, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle}$, we may assume that $\langle f \cdot a, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle = \varphi \circ \phi(a)\langle f, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle$, $\langle f \cdot \alpha, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle = \varphi(\alpha)\langle f, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle$ and $\langle \varphi \circ \phi, \tilde{T}^{**}(\tilde{m}_\alpha)\rangle = 1$. Hence the proof is complete. □

Remark 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. It is clear that $\mathcal{A}^{**}$ is a Banach $\mathfrak{A}$-module with dual action and also $\mathfrak{J}_{\mathcal{A}} \subseteq \mathfrak{J}_{\mathcal{A}^{**}}$. By Goldstein's theorem and applying* [6, Theorem 2.6.15(ii)], *one can see that $\mathfrak{J}_{\mathcal{A}^{**}} \subseteq \mathfrak{J}^{\perp\perp}_{\mathcal{A}}$. Consider $\frac{\mathcal{A}^{**}}{\mathfrak{J}_{\mathcal{A}^{**}}}$ and $\frac{\mathcal{A}^{**}}{\mathfrak{J}^{\perp\perp}_{\mathcal{A}}}$ as Banach $\mathcal{A}$-module with natural actions. Since $\mathfrak{J}_{\mathcal{A}}\mathcal{A}^{**} \subseteq \mathfrak{J}_{\mathcal{A}^{**}}$ (similarly to the left action) and $\mathfrak{J}_{\mathcal{A}}\mathcal{A}^{**} \subseteq \mathfrak{J}^{\perp\perp}_{\mathcal{A}}$ (similarly to the left action), $\frac{\mathcal{A}^{**}}{\mathfrak{J}_{\mathcal{A}^{**}}}$ and $\frac{\mathcal{A}^{**}}{\mathfrak{J}^{\perp\perp}_{\mathcal{A}}}$ are Banach $\frac{\mathcal{A}}{\mathfrak{J}_{\mathcal{A}}}$-modules with natural actions* [6, Example 2.6.2 (iv)].

Remark 2.2. *By the discussion before Definition 3.5 in* [2], *there is a continuous linear map $\Omega_{\mathfrak{A}} : \frac{\mathcal{A}^{**}\hat{\otimes}\mathcal{A}^{**}}{\mathfrak{I}_{\mathcal{A}^{**}}} \longrightarrow \frac{(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}}{\mathfrak{I}^{\perp\perp}_{\mathcal{A}}}$ such that for every $a, b, x \in \mathcal{A}$ and $u \in \mathcal{A}^{**}\hat{\otimes}\mathcal{A}^{**}$ the following equalities hold:*

(i) $\Omega_{\mathfrak{A}}(a \otimes b + \mathfrak{I}_{\mathcal{A}^{**}}) = a \otimes b + \mathfrak{I}^{\perp\perp}_{\mathcal{A}}$;
(ii) $\Omega_{\mathfrak{A}}(u + \mathfrak{I}_{\mathcal{A}^{**}}) \cdot x = \Omega_{\mathfrak{A}}(u \cdot x + \mathfrak{I}_{\mathcal{A}^{**}})$;
(iii) $x \cdot \Omega_{\mathfrak{A}}(u + \mathfrak{I}_{\mathcal{A}^{**}}) = \Omega_{\mathfrak{A}}(x \cdot u + \mathfrak{I}_{\mathcal{A}^{**}})$;
(iv) $\tilde{\omega}^{**}_{\mathcal{A}}(\Omega_{\mathfrak{A}}(u + \mathfrak{I}_{\mathcal{A}^{**}})) = \lambda \circ \tilde{\omega}_{\mathcal{A}^{**}}(u + \mathfrak{I}_{\mathcal{A}^{**}})$,

where $\lambda : \frac{\mathcal{A}^{**}}{\mathfrak{J}_{\mathcal{A}^{**}}} \to \frac{\mathcal{A}^{**}}{\mathfrak{J}^{\perp\perp}_{\mathcal{A}}}; F + \mathfrak{J}_{\mathcal{A}^{**}} \mapsto F + \mathfrak{J}^{\perp\perp}_{\mathcal{A}}$ is a well defined continuous map. Note that [2] *contains alternative definitions of $\mathfrak{I}_{\mathcal{A}}$, $\mathfrak{J}_{\mathcal{A}}$, $\mathfrak{I}_{\mathcal{A}^{**}}$ and $\mathfrak{J}_{\mathcal{A}^{**}}$, but we can adopt the map $\Omega_{\mathfrak{A}}$ with our own definitions here.*

Proposition 2.4. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module. If $\mathcal{A}^{**}$ is module Johnson amenable, then $\mathcal{A}$ is module Johnson amenable.*

Proof. Since $\mathcal{A}^{**}$ is module Johnson amenable, there exists a net $(\tilde{m}_\alpha)_{\alpha\in I}$ in $(\mathcal{A}^{**}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}^{**})^{**}$ such that $F\cdot\tilde{m}_\alpha = \tilde{m}_\alpha\cdot F$ and $\tilde{\omega}^{**}_{\mathcal{A}^{**}}(\tilde{m}_\alpha)\cdot F + \mathfrak{J}_{\mathcal{A}^{**}} \to F + \mathfrak{J}^{\perp\perp}_{\mathcal{A}^{**}}$ for every $F\in\mathcal{A}^{**}$. Consider the canonical embedding map $i : (\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^* \hookrightarrow (\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{***}$. Let $\tilde{n}_\alpha = i^*\circ\Omega^{**}_{\mathfrak{A}}(\tilde{m}_\alpha)$, where $\Omega_{\mathfrak{A}}$ is a continuous map as in Remark 2.2. Properties (ii) and (iii) in Remark 2.2 imply that for every $a\in\mathcal{A}$

$$a\cdot\tilde{n}_\alpha = a\cdot(i^*\circ\Omega^{**}_{\mathfrak{A}}(\tilde{m}_\alpha)) = i^*\circ\Omega^{**}_{\mathfrak{A}}(a\cdot\tilde{m}_\alpha) = i^*\circ\Omega^{**}_{\mathfrak{A}}(\tilde{m}_\alpha\cdot a) = (i^*\circ\Omega^{**}_{\mathfrak{A}}(\tilde{m}_\alpha))\cdot a = \tilde{n}_\alpha\cdot a.$$

By Goldstein's theorem for every α, there exists (x^α_β) in $\mathcal{A}^{**}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}^{**}$ such that $wk^*\text{-}\lim_\beta x^\alpha_\beta = \tilde{m}_\alpha$ in $(\mathcal{A}^{**}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}^{**})^{**}$. By the property (iv) in Remark 2.2 we have

$$\begin{aligned}\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{n}_\alpha)\cdot(a+\mathfrak{J}_{\mathcal{A}}) &= \tilde{\omega}^{**}_{\mathcal{A}}\circ i^*\circ\Omega^{**}_{\mathfrak{A}}(\tilde{m}_\alpha)\cdot(a+\mathfrak{J}_{\mathcal{A}})\\ &= wk^*\text{-}\lim_\beta(\tilde{\omega}^{**}_{\mathcal{A}}\circ i^*\circ\Omega^{**}_{\mathfrak{A}}(x^\alpha_\beta)\cdot(a+\mathfrak{J}_{\mathcal{A}}))\\ &= wk^*\text{-}\lim_\beta(\tilde{\omega}^{**}_{\mathcal{A}}\circ i^*(\widehat{\Omega_{\mathfrak{A}}(x^\alpha_\beta)})\cdot(a+\mathfrak{J}_{\mathcal{A}}))\\ &= wk^*\text{-}\lim_\beta(\tilde{\omega}^{**}_{\mathcal{A}}\circ\Omega_{\mathfrak{A}}(x^\alpha_\beta)\cdot(a+\mathfrak{J}_{\mathcal{A}}))\\ &= wk^*\text{-}\lim_\beta(\lambda\circ\tilde{\omega}_{\mathcal{A}^{**}}(x^\alpha_\beta)\cdot(a+\mathfrak{J}_{\mathcal{A}})) \qquad (a\in\mathcal{A}).\end{aligned}$$

Remark (2.1) implies that λ is an $\dfrac{\mathcal{A}}{\mathfrak{J}_{\mathcal{A}}}$-module morphism. So

$$\begin{aligned}\tilde{\omega}^{**}_{\mathcal{A}}(\widehat{\tilde{n}_\alpha)\cdot(a}+\mathfrak{J}_{\mathcal{A}}) &= wk^*\text{-}\lim_\beta(\lambda^{**}\circ\tilde{\omega}^{**}_{\mathcal{A}^{**}}(\hat{x}^\alpha_\beta)\cdot(a+\mathfrak{J}_{\mathcal{A}}))\\ &= \lambda^{**}\circ\tilde{\omega}^{**}_{\mathcal{A}^{**}}(\tilde{m}_\alpha)\cdot(a+\mathfrak{J}_{\mathcal{A}})\\ &= \lambda^{**}(\tilde{\omega}^{**}_{\mathcal{A}^{**}}(\tilde{m}_\alpha)\cdot(a+\mathfrak{J}_{\mathcal{A}})).\end{aligned} \tag{9}$$

There exists (T_α) in $\mathcal{A}^{****}$ such that $\tilde{\omega}^{**}_{\mathcal{A}^{**}}(\tilde{m}_\alpha) = T_\alpha + \mathfrak{J}^{\perp\perp}_{\mathcal{A}^{**}}$. One can see that

$$\tilde{\omega}^{**}_{\mathcal{A}^{**}}(\tilde{m}_\alpha)\cdot(a+\mathfrak{J}_{\mathcal{A}}) = T_\alpha\cdot a + \mathfrak{J}^{\perp\perp}_{\mathcal{A}^{**}} \to a + \mathfrak{J}^{\perp\perp}_{\mathcal{A}^{**}} = \widehat{a+\mathfrak{J}_{\mathcal{A}^{**}}} \quad (a\in\mathcal{A}).$$

By (9) we have

$$\tilde{\omega}^{**}_{\mathcal{A}}(\widehat{\tilde{n}_\alpha)\cdot(a}+\mathfrak{J}_{\mathcal{A}}) \to \widehat{a+\mathfrak{J}}^{\perp\perp}_{\mathcal{A}}.$$

So $\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{n}_\alpha)\cdot(a+\mathfrak{J}_{\mathcal{A}}) \to a+\mathfrak{J}^{\perp\perp}_{\mathcal{A}}$. Thus $\mathcal{A}$ is module Johnson amenable. □

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. Suppose that $\Psi:\mathcal{A}\to\mathcal{B}$ is a continuous epimorphism such that $\Psi(\alpha\cdot x) = \alpha\cdot\Psi(x)$ and $\Psi(x\cdot\alpha) = \Psi(x)\cdot\alpha$ for every $\alpha\in\mathfrak{A}$ and $x\in\mathcal{A}$. If $\mathcal{A}$ is module Johnson amenable, then $\mathcal{B}$ is module Johnson amenable.*

Proof. Since $\mathcal{A}$ is module Johnson amenable, there exists a net $(\tilde{m}_\alpha)_{\alpha\in I}$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ such that $a\cdot\tilde{m}_\alpha = \tilde{m}_\alpha\cdot a$ and $\tilde{\omega}^{**}_{\mathcal{A}}(\tilde{m}_\alpha)\cdot a + \mathfrak{J}_{\mathcal{A}} \to a + \mathfrak{J}^{\perp\perp}_{\mathcal{A}}$ for every $a\in\mathcal{A}$. Define $\Psi\otimes\Psi:\mathcal{A}\hat{\otimes}\mathcal{A}\to\mathcal{B}\hat{\otimes}\mathcal{B}$ by $\Psi\otimes\Psi(x\otimes y) = \Psi(x)\otimes\Psi(y)$, for every $x,y\in\mathcal{A}$. So $\Psi\otimes\Psi$ is a bounded linear map. It is easy to see that $\Psi\otimes\Psi(\mathcal{I}_{\mathcal{A}})\subseteq\mathcal{I}_{\mathcal{B}}$. Then we can define $\Theta:\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}\to\mathcal{B}\hat{\otimes}_{\mathfrak{A}}\mathcal{B}$ by $\Theta(x\otimes y+\mathcal{I}_{\mathcal{A}}) = \Psi\otimes\Psi(x\otimes y)+\mathcal{I}_{\mathcal{B}}$ for every $x,y\in\mathcal{A}$. For every $\tilde{u}\in\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$, $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ we have

$$\Theta(a\cdot\tilde{u}) = \Psi(a)\cdot\Theta(\tilde{u}), \quad \Theta(\tilde{u}\cdot a) = \Theta(\tilde{u})\cdot\Psi(a). \tag{10}$$

Also

$$\Theta(\alpha\cdot\tilde{u}) = \alpha\cdot\Theta(\tilde{u}), \quad \Theta(\tilde{u}\cdot\alpha) = \Theta(\tilde{u})\cdot\alpha \tag{11}$$

Since Θ^{**} is wk^*-continuous, Goldstein's theorem and (10) imply that

$$\Theta^{**}(a\cdot F)=\Psi(a)\cdot\Theta^{**}(F),\quad \Theta^{**}(F\cdot a)=\Theta^{**}(F)\cdot\Psi(a)\quad (a\in\mathcal{A},F\in(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}).$$

Let $\tilde{n}_\alpha=\Theta^{**}(\tilde{m}_\alpha)$. So for every $a\in\mathcal{A}$

$$\Psi(a)\cdot\tilde{n}_\alpha=\Psi(a)\cdot\Theta^{**}(\tilde{m}_\alpha)=\Theta^{**}(a\cdot\tilde{m}_\alpha)=\Theta^{**}(\tilde{m}_\alpha\cdot a)=\Theta^{**}(\tilde{m}_\alpha)\cdot\Psi(a)=\tilde{n}_\alpha\cdot\Psi(a).$$

One can see that $\Psi(\mathcal{J}_\mathcal{A})\subseteq\mathcal{J}_\mathcal{B}$. So we can define $\tilde{\Psi}:\frac{\mathcal{A}}{\mathcal{J}_\mathcal{A}}\to\frac{\mathcal{B}}{\mathcal{J}_\mathcal{B}}$ by $\tilde{\Psi}(a+\mathcal{J}_\mathcal{A})=\Psi(a)+\mathcal{J}_\mathcal{B}$ for every $a\in\mathcal{A}$. It is clear that $\tilde{\Psi}$ is an epimorphism map and also $\tilde{\Psi}(\tilde{x}\cdot\alpha)=\tilde{\Psi}(\tilde{x})\cdot\alpha$ and $\tilde{\Psi}(\alpha\cdot\tilde{x})=\alpha\cdot\tilde{\Psi}(\tilde{x})$ for every $\alpha\in\mathfrak{A}$ and $\tilde{x}\in\frac{\mathcal{A}}{\mathcal{J}_\mathcal{A}}$. Since $\tilde{\omega}^{**}_\mathcal{A}(\tilde{m}_\alpha)\cdot a+\mathcal{J}_\mathcal{A}\to a+\mathcal{J}_\mathcal{A}{}^{\perp\perp}$ and $\tilde{\Psi}^{**}$ is continuous, $\tilde{\Psi}^{**}(\tilde{\omega}^{**}_\mathcal{A}(\tilde{m}_\alpha)\cdot a+\mathcal{J}_\mathcal{A})\to\tilde{\Psi}^{**}(a+\mathcal{J}_\mathcal{A}{}^{\perp\perp})$. Since $\tilde{\Psi}$ is an epimorphism map, by Goldstein's theorem one can see that $\tilde{\Psi}^{**}(\tilde{\omega}^{**}_\mathcal{A}(\tilde{m}_\alpha)\cdot a+\mathcal{J}_\mathcal{A})=\tilde{\Psi}^{**}(\tilde{\omega}^{**}_\mathcal{A}(\tilde{m}_\alpha))\cdot\tilde{\Psi}(a+\mathcal{J}_\mathcal{A})$. So for every $a\in\mathcal{A}$

$$\tilde{\Psi}^{**}(\tilde{\omega}^{**}_\mathcal{A}(\tilde{m}_\alpha))\cdot(\Psi(a)+\mathcal{J}_\mathcal{B})\to\Psi(a)+\mathcal{J}_\mathcal{B}{}^{\perp\perp},\tag{12}$$

in $\frac{\mathcal{B}^{**}}{\mathcal{J}_\mathcal{B}{}^{\perp\perp}}$. We claim that $\tilde{\omega}_\mathcal{B}\circ\Theta=\tilde{\Psi}\circ\tilde{\omega}_\mathcal{A}$. To see this for every $a,b\in\mathcal{A}$

$$\langle a\otimes b+\mathcal{I}_\mathcal{A},\tilde{\omega}_\mathcal{B}\circ\Theta\rangle=\langle\Psi(a)\otimes\Psi(b)+\mathcal{I}_\mathcal{B},\tilde{\omega}_\mathcal{B}\rangle=\Psi(ab)+\mathcal{J}_\mathcal{B}=\tilde{\Psi}(ab+\mathcal{J}_\mathcal{A})=\langle a\otimes b+\mathcal{I}_\mathcal{A},\tilde{\Psi}\circ\tilde{\omega}_\mathcal{A}\rangle.$$

(12) implies that

$$\tilde{\omega}^{**}_\mathcal{B}\circ\Theta^{**}(\tilde{m}_\alpha)\cdot(\Psi(a)+\mathcal{J}_\mathcal{B})\to\Psi(a)+\mathcal{J}_\mathcal{B}{}^{\perp\perp}\quad(a\in\mathcal{A}).$$

Hence $\tilde{\omega}^{**}_\mathcal{B}(\tilde{n}_\alpha)\cdot b+\mathcal{J}_\mathcal{B}\to b+\mathcal{J}_\mathcal{B}{}^{\perp\perp}$ for every $b\in\mathcal{B}$. □

Corollary 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and let L be a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is module Johnson amenable, then $\mathcal{A}/L$ is module Johnson amenable.*

Proof. Since the quotient map $q:\mathcal{A}\to\mathcal{A}/L$ satisfies conditions in Proposition 2.5, $\mathcal{A}/L$ is module Johnson amenable. □

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and let $\mathcal{B}$ be a Banach $\mathfrak{B}$-module. One can see that the Banach algebra $\mathcal{A}\hat{\otimes}\mathcal{B}$ is a Banach $\mathfrak{A}\hat{\otimes}\mathfrak{B}$-module with following actions:

$$(\alpha\otimes\beta)(a\otimes b)=\alpha\cdot a\otimes\beta\cdot b\quad(a,b\in\mathcal{A},\alpha\in\mathfrak{A},\beta\in\mathfrak{B}),$$

and similarly for the right action.

Proposition 2.6. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and let $\mathcal{B}$ be a Banach $\mathcal{B}$-$\mathfrak{B}$-module. Suppose that $\mathcal{B}$ has a non-zero idempotent. If $\mathcal{A}\hat{\otimes}\mathcal{B}$ is module Johnson amenable (as Banach $\mathfrak{A}\hat{\otimes}\mathfrak{B}$-module), then $\mathcal{A}$ is module Johnson amenable.*

Proof. Since $\mathcal{A}\hat{\otimes}\mathcal{B}$ is module Johnson amenable, there exists a net $(\tilde{m}_\alpha)$ in $\frac{((\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B}))^{**}}{\mathcal{I}^{\perp\perp}_{\mathcal{A}\hat{\otimes}\mathcal{B}}}$ such that $u\cdot\tilde{m}_\alpha=\tilde{m}_\alpha\cdot u$ and $\tilde{\omega}^{**}_{\mathcal{A}\hat{\otimes}\mathcal{B}}(\tilde{m}_\alpha)\cdot u+\mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}\to u+\mathcal{J}^{\perp\perp}_{\mathcal{A}\hat{\otimes}\mathcal{B}}$ for every $u\in\mathcal{A}\hat{\otimes}\mathcal{B}$. Suppose that b_0 is a non-zero idempotent of $\mathcal{B}$. By similar argument as in [9, Proposition 3.5], there exists a non-zero $f\in\mathcal{B}^*$ such that $f(b_0)=1$ and $f(bb_0)=f(b_0b)$ for every $b\in\mathcal{B}$. Define $\phi,\theta:(\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B})\to\mathcal{A}\hat{\otimes}\mathcal{A}$ and $\psi:\mathcal{A}\hat{\otimes}\mathcal{B}\to\mathcal{A}$ by

$$\phi(a_1\otimes b_1\otimes a_2\otimes b_2)=f(b_1b_2)a_1\otimes a_2,$$
$$\theta(a_1\otimes b_1\otimes a_2\otimes b_2)=f(b_0b_1b_2)a_1\otimes a_2,$$
$$\psi(a_1\otimes b_1)=f(b_1)a_1\quad(a_1,a_2\in\mathcal{A},b_1,b_2\in\mathcal{B}).$$

So for every $d \in (\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B})$ and $a \in \mathcal{A}$ we have

$$a \cdot \theta(d) = \phi((a \otimes b_0) \cdot d) \quad \text{and} \quad \theta(d) \cdot a = \phi(d \cdot (a \otimes b_0)), \tag{13}$$

and also

$$\psi(\omega_{\mathcal{A}\hat{\otimes}\mathcal{B}}(d)(a \otimes b_0)) = (\omega_{\mathcal{A}} \circ \theta(d))a. \tag{14}$$

One can see that $\phi(\mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) \subseteq \mathcal{I}_{\mathcal{A}}$, $\theta(\mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) \subseteq \mathcal{I}_{\mathcal{A}}$ and $\psi(\mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) \subseteq \mathcal{J}_{\mathcal{A}}$. So define $\tilde{\phi}, \tilde{\theta} : \dfrac{(\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B})}{\mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}} \to \dfrac{\mathcal{A}\hat{\otimes}\mathcal{A}}{\mathcal{I}_{\mathcal{A}}}$ and $\tilde{\psi} : \dfrac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}} \to \dfrac{\mathcal{A}}{\mathcal{J}_{\mathcal{A}}}$ by $\tilde{\phi}(d + \mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) = \phi(d) + \mathcal{I}_{\mathcal{A}}$, $\tilde{\theta}(d+\mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) = \theta(d)+\mathcal{I}_{\mathcal{A}}$ and $\tilde{\psi}(u+\mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) = \psi(u)+\mathcal{J}_{\mathcal{A}}$. For every $\tilde{d} \in \dfrac{(\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B})}{\mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}}$, there exists an element $d \in (\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B})$ such that $\tilde{d} = d + \mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}$. (13) implies that

$$\begin{aligned} a \cdot \tilde{\theta}(\tilde{d}) &= a \cdot \tilde{\theta}(d + \mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) = a \cdot (\theta(d) + \mathcal{I}_{\mathcal{A}}) = (a \cdot \theta(d)) + \mathcal{I}_{\mathcal{A}} \\ &= (\phi(a \otimes b_0) \cdot d)) + \mathcal{I}_{\mathcal{A}} = \tilde{\phi}((a \otimes b_0) \cdot d + \mathcal{I}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) \\ &= \tilde{\phi}((a \otimes b_0) \cdot \tilde{d}) \qquad (a \in \mathcal{A}), \end{aligned}$$

and similarly for the right action, $\tilde{\theta}(\tilde{d}) \cdot a = \tilde{\phi}(\tilde{d} \cdot (a \otimes b_0))$. (14) implies that

$$\begin{aligned} \tilde{\psi}(\tilde{\omega}_{\mathcal{A}\hat{\otimes}\mathcal{B}}(\tilde{d})(a \otimes b_0) + \mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}})) &= \tilde{\psi}((\omega_{\mathcal{A}\hat{\otimes}\mathcal{B}}(d) + \mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}})((a \otimes b_0) + \mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}})) \\ &= \tilde{\psi}((\omega_{\mathcal{A}\hat{\otimes}\mathcal{B}}(d)(a \otimes b_0) + \mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) \\ &= \psi(\omega_{\mathcal{A}\hat{\otimes}\mathcal{B}}(d)(a \otimes b_0)) + \mathcal{J}_{\mathcal{A}} \\ &= (\omega_{\mathcal{A}} \circ \theta(d))a + \mathcal{J}_{\mathcal{A}} \\ &= ((\omega_{\mathcal{A}} \circ \theta(d)) + \mathcal{J}_{\mathcal{A}})(a + \mathcal{J}_{\mathcal{A}}) \\ &= \tilde{\omega}_{\mathcal{A}}(\theta(d) + \mathcal{I}_{\mathcal{A}})(a + \mathcal{J}_{\mathcal{A}}) \\ &= \tilde{\omega}_{\mathcal{A}} \circ \tilde{\theta}(\tilde{d})(a + \mathcal{J}_{\mathcal{A}}). \end{aligned}$$

Now let $n_\alpha = \tilde{\theta}^{**}(\tilde{m}_\alpha)$. By Goldstein's theorem for every $\tilde{F} \in \dfrac{((\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B}))^{**}}{\mathcal{I}^{\perp\perp}_{\mathcal{A}\hat{\otimes}\mathcal{B}}}$ we have

$$a \cdot \tilde{\theta}^{**}(\tilde{F}) = \tilde{\phi}^{**}((a \otimes b_0) \cdot \tilde{F}), \quad \tilde{\theta}^{**}(\tilde{F}) \cdot a = \tilde{\phi}^{**}(\tilde{F} \cdot (a \otimes b_0)) \quad (a \in \mathcal{A}) \tag{15}$$

and also

$$\tilde{\psi}^{**}(\tilde{\omega}^{**}_{\mathcal{A}\hat{\otimes}\mathcal{B}}(\tilde{F}) \cdot (a \otimes b_0) + \mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) = \tilde{\omega}^{**}_{\mathcal{A}} \circ \tilde{\theta}^{**}(\tilde{F}) \cdot (a + \mathcal{J}_{\mathcal{A}}) \quad (a \in \mathcal{A}). \tag{16}$$

(15) implies that for every $a \in \mathcal{A}$

$$a \cdot n_\alpha = a \cdot \tilde{\theta}^{**}(\tilde{m}_\alpha) = \tilde{\phi}^{**}((a \otimes b_0) \cdot \tilde{m}_\alpha) = \tilde{\phi}^{**}(\tilde{m}_\alpha \cdot (a \otimes b_0)) = \tilde{\theta}^{**}(\tilde{m}_\alpha) \cdot a = n_\alpha \cdot a.$$

implies that

$$\begin{aligned} \lim_\alpha \tilde{\omega}^{**}_{\mathcal{A}}(n_\alpha) \cdot a + \mathcal{J}_{\mathcal{A}} = \lim_\alpha \tilde{\psi}^{**}(\tilde{\omega}^{**}_{\mathcal{A}\hat{\otimes}\mathcal{B}}(\tilde{m}_\alpha) \cdot (a \otimes b_0) + \mathcal{J}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) &= \tilde{\psi}^{**}((a \otimes b_0) + \mathcal{J}^{\perp\perp}_{\mathcal{A}\hat{\otimes}\mathcal{B}}) \\ = f(b_0)a + \mathcal{J}^{\perp\perp}_{\mathcal{A}} = a + \mathcal{J}^{\perp\perp}_{\mathcal{A}} \qquad (a \in \mathcal{A}). \end{aligned}$$

Hence $\mathcal{A}$ is module Johnson amenable. $\square$

The semigroup S is called inverse semigroup, if for every $s \in S$ there exists an element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$ [8]. Let S be an inverse semigroup with the idempotent set E. Consider $\ell^1(S)$ as a Banach module over $\ell^1(E)$ with the multiplication right action and the trivial left action, that is

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} \quad (s \in S, e \in E).$$

Theorem 2.1. *With the above notation, $\ell^1(S)$ is module Johnson amenable if and only if S is amenable.*

Proof. if S is amenable, then $\ell^1(S)$ has a module virtual diagonal [12, theorem 2.9], [1, Theorem 3.1]. Lemma 2.2 implies that $\ell^1(S)$ is module Johnson amenable. Conversely if $\ell^1(S)$ is module Johnson amenable, then Proposition 2.1 implies that $\ell^1(S)$ is module pseudo-amenable. Applying [5, Theorem 3.13 (i)], S is amenable. □

3. Examples and applications

Example 3.1. *Let S be the set of natural numbers $\mathbb{N}$ with the binary operation $(m, n) \longmapsto \max\{m, n\}$, where m and n are in $\mathbb{N}$. It is clear that S is an inverse semigroup. Since S is an amenable group, Theorem 2.1 implies that $\ell^1(S)$ is module Johnson amenable as an $\ell^1(E)$-module, where $E(S) = \mathbb{N}$. But it is not Johnson pseudo-contractible* [3, Example 2.5].

Example 3.2. *Let S be the bicyclic semigroup. Then S is generated by p and q subject to $pq = e \neq qp$ for the unit element e, that is $S = \{p^m q^n : m, n \geq 0\}$. Following* [7], *$S$ is an inverse amenable semigroup and $(p^m q^n)^* = p^n q^m$. Theorem 2.1 implies that $\ell^1(S)$ is module Johnson amenable as an $\ell^1(E)$-module, where $E(S) = \{p^n q^n : n \geq 0\}$. But it is not Johnson pseudo-contractible* [3, Example 2.2].

Example 3.3. *Let G be an amenable group, I be an infinite set and let $S = M^0(G, I)$ be a Brandt semigroup, that is the collection of all $I \times I$ matrices $(g)_{i,j}$ with $g \in G$ in the (i, j)-th position and zero elsewhere with the following multiplication*

$$(g)_{i,j}(h)_{k,l} = \begin{cases} (gh)_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where $g, h \in G$ and $i, j, k, l \in I$. S is an inverse amenable semigroup [7]. *Theorem 2.1 implies that $\ell^1(S)$ is module Johnson amenable as an $\ell^1(E)$-module, where $E(S) = \{(e)_{ii} : i \in I\} \cup \{0\}$. But it is not Johnson pseudo-contractible* [13, Theorem 2.4].

The Banach algebra of $\Lambda \times \Lambda$-matrices over $\mathbb{C}$, with finite ℓ^1-norm and matrix multiplication is denoted by $\mathbb{M}_\Lambda(\mathbb{C})$, where Λ is an arbitrary set. Suppose that $\mathfrak{A} = \left\{[a_{i,j}] \in \mathbb{M}_\Lambda(\mathbb{C}) \mid \forall i \neq j, a_{i,j} = 0\right\}$ as a closed subalgebra of $\mathbb{M}_\Lambda(\mathbb{C})$. One can see that $\mathcal{A} = \mathbb{M}_\Lambda(\mathbb{C})$ is Banach $\mathfrak{A}$-bimodule with respect to matrix multiplication. Since $a(\alpha \cdot b) = (a \cdot \alpha)b$ for every $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$, $\mathcal{J} = 0$. So $\frac{\mathcal{A}}{\mathcal{J}} = \mathcal{A}$ and $\frac{\mathcal{A}^{**}}{\mathcal{J}^{\perp\perp}} = \mathcal{A}^{**}$.

Theorem 3.1. *With the above notation, let Λ be a non-empty set. Then $\mathbb{M}_\Lambda(\mathbb{C})$ is module Johnson amenable if and only if Λ is finite.*

Proof. Since $\mathcal{A} = \mathbb{M}_\Lambda(\mathbb{C})$ is module Johnson amenable, there exists a net $(\tilde{m}_\alpha)_{\alpha\in I}$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$ such that $a\cdot\tilde{m}_\alpha = \tilde{m}_\alpha\cdot a$ and $\tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha)\cdot a \to a$ in $\mathcal{A}^{**}$ for every $a\in\mathcal{A}$. Let a be a non-zero element of $\mathcal{A}$. By Hahn-Banach theorem there exists a bounded linear functional ψ in $\mathcal{A}^*$ such that $\psi(a)\neq 0$. Since $wk^*\text{-}\lim_\alpha \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha)\cdot a = a$, we have

$$\lim_\alpha\langle a\cdot\psi, \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha)\rangle = \lim_\alpha\langle \psi, \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha)\cdot a\rangle = \langle\psi, a\rangle \neq 0.$$

So $\langle a\cdot\psi, \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha)\rangle \neq 0$ for every α. By Goldstein's theorem, there is a bounded net (x_α^β) in $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ such that $wk^*\text{-}\lim_\beta x_\alpha^\beta = \tilde{m}_\alpha$ in $(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A})^{**}$. It follows that

$$wk^*\text{-}\lim_\beta a\cdot x_\alpha^\beta - x_\alpha^\beta\cdot a = a\cdot\tilde{m}_\alpha - \tilde{m}_\alpha\cdot a = 0 \qquad (a\in\mathcal{A}).$$

Since $\tilde{\omega}_{\mathcal{A}}^{**}$ is wk^*-continuous,

$$wk^*\text{-}\lim_\beta a\cdot\tilde{\omega}_{\mathcal{A}}^{**}(x_\alpha^\beta) - \tilde{\omega}_{\mathcal{A}}^{**}(x_\alpha^\beta)\cdot a = 0 \quad\text{and}\quad wk^*\text{-}\lim_\beta \tilde{\omega}_{\mathcal{A}}^{**}(x_\alpha^\beta) = \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha).$$

Since (x_α^β) is a net in $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$, $wk\text{-}\lim_\beta a\cdot\tilde{\omega}_{\mathcal{A}}(x_\alpha^\beta) - \tilde{\omega}_{\mathcal{A}}(x_\alpha^\beta)\cdot a = 0$. Let $y_\beta = \tilde{\omega}_{\mathcal{A}}(x_\alpha^\beta)$. So (y_β) is a bounded net in $\mathcal{A}$ such that for every $a\in\mathcal{A}$

$$wk\text{-}\lim_\beta ay_\beta - y_\beta a = 0 \quad\text{and}\quad wk^*\text{-}\lim_\beta \hat{y}_\beta = \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha).$$

By a similar argument as in [13, Lemma 2.1] we complete the proof. Suppose that $y_\beta = [y_\beta^{i,j}]$, where $y_\beta^{i,j}\in\mathbb{C}$ for every i,j. Fixed $i_0\in\Lambda$, for every $j\in\Lambda$ we have

$$\varepsilon_{i_0,j}y_\beta - y_\beta\varepsilon_{i_0,j} = \sum_{\substack{i\in\Lambda\\ i\neq j}} y_\beta^{j,i}\varepsilon_{i_0,i} + (y_\beta^{j,j} - y_\beta^{i_0,i_0})\varepsilon_{i_0,j} - \sum_{\substack{i\in\Lambda\\ i\neq i_0}} y_\beta^{i,i_0}\varepsilon_{i,j},$$

where $\varepsilon_{i,j}$ is a matrix which belongs to $\mathbb{M}_\Lambda(\mathbb{C})$ and whose (i,j)-th entry is 1 and others are zero. Since the product of the weak topology on $\mathbb{C}$ coincides with the weak topology on $\mathcal{A}$ [15, Theorem 4.3],

$$wk - \lim_\beta y_\beta^{j,j} - y_\beta^{i_0,i_0} = 0 \quad\text{and}\quad wk - \lim_\beta y_\beta^{j,i} = 0, \tag{17}$$

whenever $i\neq j$ in Λ. Since $\|y_\beta\|\leq\|m_\alpha\|$, $(y_\beta^{i_0,i_0})$ is a bounded net in $\mathbb{C}$. So it has a convergent subnet $(y_{\beta_k}^{i_0,i_0})$ in $\mathbb{C}$. We may assume that $\lim_{\beta_k} y_{\beta_k}^{i_0,i_0} = l$. Since $(y_\beta^{j,j} - y_\beta^{i_0,i_0})$ is a net in $\mathbb{C}$, (17) implies that $\lim_\beta y_\beta^{j,j} - y_\beta^{i_0,i_0} = 0$ with respect to $|\cdot|$. So $\lim_{\beta_k} y_{\beta_k}^{i_0,i_0} - y_{\beta_k}^{j,j} = 0$. It follows that $\lim_{\beta_k} y_{\beta_k}^{j,j} = l$ for every $j\in\Lambda$. If $l=0$, then by (17) for every $i,j\in\Lambda$, $\lim_{\beta_k} y_{\beta_k}^{i,j} = 0$ in $\mathbb{C}$. So $wk\text{-}\lim_{\beta_k} y_{\beta_k}^{i,j} = 0$, where $i,j\in\Lambda$. Applying [15, Theorem 4.3], $wk\text{-}\lim_{\beta_k} y_{\beta_k} = 0$ in $\mathcal{A}$. It follows that $\lim_{\beta_k}\langle y_{\beta_k}, a\cdot\psi\rangle = 0$. On the other hand

$$\lim_{\beta_k}\langle y_{\beta_k}, a\cdot\psi\rangle = \lim_{\beta_k}\langle a\cdot\psi, \hat{y}_{\beta_k}\rangle = \lim_\alpha\langle a\cdot\psi, \tilde{\omega}_{\mathcal{A}}^{**}(\tilde{m}_\alpha)\rangle \neq 0,$$

which is a contradiction. So $wk\text{-}\lim_{\beta_k} y^{j,j}_{\beta_k} = l \neq 0$ for every $j \in \Lambda$. Using (17) we have $wk\text{-}\lim_{\beta_k} y^{j,i}_{\beta_k} = 0$ whenever $j \neq i$ in Λ. Applying [15, Theorem 4.3] again, $wk\text{-}\lim_{\beta_k} y_{\beta_k} = y_0$, where y_0 is a matrix with l in the diagonal position and 0 elsewhere. Thus $y_0 \in \overline{Conv(y_{\beta_k})}^{wk} = \overline{Conv(y_{\beta_k})}^{\|\cdot\|}$. So $y_0 \in \mathcal{A}$. But

$$\infty = \sum_{j\in\Lambda} |l| = \sum_{j\in\Lambda} |y^{j,j}_0| = \|y_0\| < \infty,$$

which is a contradiction. So Λ must be finite.
Conversely if Λ is finite, then $\mathbb{M}_\Lambda(\mathbb{C})$ is Johnson pseudo-contractible [13, Lemma 2.1]. Lemma 2.1 implies that $\mathbb{M}_\Lambda(\mathbb{C})$ is module Johnson amenable. □

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