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A MODIFIED SMOOTH HOMOTOPY ALGORITHM FOR SOLVING MATHEMATICAL PROGRAMS WITH BALL-CONSTRAINED VARIATIONAL INEQUALITIES

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Based on the Robinson's normal equation and Chen-Harker-Kanzow-Smale smooth function, a smoothing Homotopy equation for solving mathematical programs with ball-constrained variational inequalities(MPBVI), which is converted to an approximate standard optimization problem, is constructed. And the existence and global convergence of the homotopy path, for almost any interior point in the feasible region, is proved to convergent to the GKKT point of the approximate problems of the MPBVI. Finally, a numerical experiment is given to illustrate that the constructed homotopy method is feasible and effective.

Keywords: mathematical programs, homotopy method, KKT system, variational inequalities.

MSC2010: 90C30, 65H10.

1. Introduction

Given $f : R^{n+m} \to R, F : R^{n+m} \to R^m, g : R^{n+m} \to R, X_{ab} = \{(x, y) \in R^{n+m} : g(x, y) \leq 0\}, C(x) = \{y \in R^m : \|y\| \leq r, r > 0\}$.

In the paper, we will consider the following mathematical programs with ball-constrained variational inequalities (MPBVI):

$$\begin{aligned} &\min\ f(x,y)\\ &\text{s.t. } x \in X_{ab},\\ &F(x,y)^T(z-y) \geq 0,\ \forall z \in C(x). \end{aligned} \tag{1}$$

Throughout the paper, we suppose that the mappings $f(\cdot), F(\cdot)$ and $g(\cdot)$ are three continuously differentiable.

Problem (1) is a kind of mathematical programs with equilibrium constraints (MPEC), which plays very important role in many fields such as in designing of transportation networks, economic modelling, fixed point problems and shape optimization, see references, e.g., [11], [21]-[34] and [38]. Because of the nonconvex and nonsmooth of the feasible region of the MPEC, It is harder to solve this nonlinear programming problems. Many results on the research and applications of MPEC over the last couple of decades have been appeared. Upon the success methods for linear and nonlinear programming (NLP), many algorithms were extend to solve the MPEC, such as interior-point methods and so on, for more details see, e.g., [6, 9, 12, 13, 17, 19, 26].

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The difficulty of solving MPEC lies in the variational inequalities in the lower level. Discussed in some studies [3, 4, 7, 14], [33]-[37] and [42, 43], the variational inequalities were converted to a KKT system. However, this caused extra multiplier variables and the convergence proof required mapping F with strong monotonicity. In [41], to avoid to add extra multipliers, we exploited the Robinson's normal equation to deal with box-constrained variational inequalities in the mathematical program and obtained the existence and convergence of the smooth homotopy pathway. In this paper, we will use similar approach to solve the ball-constrained variational inequalities (1).

In this work, we will propose a homotopy method for solving MPBVI base on Robinson's equation and Chen-Harker-Kanzow-Smale smoothing functions. Homotopy method is a common algorithm for solving nonlinear problem. It has the advantage of global convergence than other interior point algorithms like Newton's method, that is, for almost all the initial points in the feasible region, the algorithm is convergent. It has been applied to solve all kinds of problem, such as zeros and fixed point of mappings and so on [8, 28, 30, 39, 40, 44]. For a good introduction and a complete survey about this method, one can see the literatures [1, 10].

An outline of this paper is as follows. In section 2, some definitions and properties are introduced. In section 3, the variational inequality problem (1) is coverted to an equivalent nonsmooth Robinson's normal equation. In order to select the initial point easily for a equation, a homotopy equation was constructed for the smooth shape of the Robison's equation. Then, we get an approximate standard optimization problem. In section 4, for the above approximate optimization, a combined homotopy equation is constructed and the existence and convergence of the homotopy path of the approximate optimization for almost any interior feasible point to the GKKT point of MPBVI are proved. In section 5, numerical test is presented to show its the effectiveness and feasibility of the proposed homotopy method.

2. **Preliminaries**

$$\begin{aligned} &\min\ f(u) \\ &\text{s.t. } g(u) \le 0, \\ &h(u) = 0, \end{aligned} \tag{2}$$

where $f : R^n \to R,\ g :\ R^n \to R^s,\ h :\ R^n \to R^m$ are locally Lipschitz continuous.

Definition 2.1 (see [15])**.** *The point u^* is said to be a generalized stationary point of (2) if there exists a Karush-Kuhn-Tucker (KKT) mutliplier vector $(\alpha, \beta) \in R^{s+m}$ such that the following generalized Karush-Kuhn-Tucker (GKKT) conditions hold:*

$$\begin{aligned} 0 \in \partial f(u^*) \quad + \quad &\partial g(u^*)^T\alpha + \partial h(u^*)^T\beta, \\ &\alpha \ge 0,\ Ag(u^*) \le 0, \\ &h(u^*) = 0, \end{aligned} \tag{3}$$

where ∂ denotes the Clarke generalized gradient for a scalar function and the Clarke generalized Jacobian for a vector-valued function, ([20]*), $A = \operatorname{diag}(\alpha)$.*

When $f,\ g$ and h are smooth at u^*, GKKT conditions happens to be usual Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned} \nabla f(u^*) \quad + \quad &\nabla g(u^*)\alpha + \nabla h(u^*)\beta = 0, \\ &\alpha \ge 0,\ Ag(u^*) \le 0, \\ &h(u^*) = 0. \end{aligned} \tag{4}$$

Under the circumstances,u^* is called a stationary point or a KKT point of (2).

For convenience, we may assume that in problem (2), the first $s_1(s_1 \leq s)$ inequality constraints are active and the rest are inactive at u^*, i.e.,

$$\begin{gathered} g_i(u^*) = 0,\ i = 1,\ \ldots,\ s_1,\ 1 \leq i \leq s_i, \\ g_i(u^*) > 0,\ i > s_i \end{gathered} \tag{5}$$

Set

$$G(u^*) = (g_1(u^*), \ldots, g_{s_1}(u^*),\, h_1(u^*), \ldots, h_m(u^*))^T. \tag{6}$$

In the sequence, some well-known regularity conditions associated with the problem (2) will be recalled, see [15].

Generalized Linear Independence Constraint Qualification (GLICQ): Each element of the generalized Jacobian $\partial G(u^*)$ has full row rank.

Definition 2.2. *Let $U \subset R^n$ be an open set, and let $\phi : U \to R^p$ be a smooth mapping. If Range $[\partial\phi(x)/\partial x] = R^p$ for all $x \in \phi^{-1}(y)$, then $y \in R^p$ is a regular value and $x \in R^n$ is a regular point.*

Lemma 2.1 (see [1])**.** *Let Q, N, P be smooth manifolds of dimensions q, m, p. Respectively, let $\phi : Q \times N \to P$ be a C^r map, where $r > max\{0, m - p\}$. If $0 \in P$ is a regular value of ϕ, then for almost all $\alpha \in Q$, 0 is a regular value of $\phi(\alpha, \cdot)$.*

Lemma 2.2 (inverse image theorem; see[16])**.** *If 0 is a regular value of the mapping $\phi_\alpha(\cdot) \to \phi(\alpha, \cdot)$, then $\phi_\alpha^{-1}(0)$ consists of some smooth manifolds.*

Lemma 2.3 (classification theorem of one-dimensional manifold; see [16])**.** *A one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

3. **Equivalent Reformulations of MPBVI**

It is well known that if C(x) is a closed convex subset of R^m, variational inequalities in problem (1)

$$F(x, y)^T(z - y) \geq 0,\ \forall z \in C(x) \tag{7}$$

is equivalent to solving the following Robinson's normal equation:

$$E_{c(x)}(v) = F(x, \textstyle\prod_{c(x)}(v)) + v - \prod_{c(x)}(v) = 0, \tag{8}$$

where for all $v \in R^m, \prod_{c(x)}(v)$ denote the projection of v onto $C(x)$. In the sense of above, if $v^* \in R^m$ is the solution of (7), then $y^* = \prod_{c(x)}(v^*)$ is the solution of (8); conversely, if y^* is the solution of (8), then $v^* = y^* - F(x, y^*)$ is the solution of (7), which is defined in [18].

Then, the mathematical programming (1) can be reformed as follows:

$$\begin{aligned} &\min\ f(x, \textstyle\prod_{c(x)}(v)) \\ &\text{s.t. } x \in X_{ab}, \\ &F(x, \textstyle\prod_{c(x)}(v)) + v - \prod_{c(x)}(v) = 0. \end{aligned} \tag{9}$$

If $v \in C(x)$, then the Euclidean projection of v onto $C(x)$ becomes

$$\prod_{c(x)}(v) = \begin{cases} \frac{rv}{\|v\|},\ if\ \ \| v \| > r, \\ v,\ if\ \ \| v \| \leq r. \end{cases} \tag{10}$$

and it can be reformed as following:

$$\prod_{c(x)}(v) = \frac{rv}{max\{r, \| v \|\}} = \frac{rv}{r + max\{0, \| v \| - r\}}.$$

For solving the (9) via smooth method, we choose modified Chen-Harker-Knzow-Smale function [5]:

$$f(z,\mu)=\frac{z+\sqrt{z^2+4\mu}}{2},(z,\mu)\in R\times R_{++}.$$

Then, the projection function $\prod_{C(x)}$ can be approximated as follows:

$$p(v,\mu)=\frac{rv}{h(v,\mu)},\ (v,\mu)\in R^m\times R_{++}$$

where

$$h(v,\mu)=r+f(\parallel v\parallel -r,\mu)=\frac{1}{2}(r+\parallel v\parallel +\sqrt{(\parallel v\parallel -r)^2+4\mu}).$$

It is obvious that for any $\mu>0$, the function $p(v,\mu)$ is continuously differentiable.

We are in a position to verify that the function $p(v,\mu)$ is a smooth approximation of the projection function $\prod_{C(x)}$. Since

$$(z)_+=\frac{z+|z|}{2}<f(z,\mu)=\frac{z+\sqrt{z^2+4\mu}}{2}\leq\frac{z+|z|+2\sqrt{\mu}}{2}=(z)_++\sqrt{\mu},$$

we have

$$h(v,\mu)=r+f(\sqrt{\parallel v\parallel -r},\mu)\leq r+(\parallel v\parallel -r)_++\sqrt{\mu}\leq max\{r,\parallel v\parallel\}+2\sqrt{\mu},$$

and

$$h(v,\mu)=r+f(\parallel v\parallel^2 -r,\mu)\geq r+(\parallel v\parallel -r)_+\geq max\{r,\parallel v\parallel\}.$$

From the above derivation, we yield

$$|h(v,\mu)-max\{r,\parallel v\parallel\}|\leq 2\sqrt{\mu}.$$

Hence,

$$\begin{aligned}\parallel p(v,\mu)-\textstyle\prod_{C(x)}(v)\parallel &= \tfrac{r\|v\||h(v,\mu)-max\{r,\|v\|\}|}{h(v,\mu)max\{r,\|v\|\}}\\ &\leq \|h(v,\mu)-max\{r,\parallel v\parallel\}\|\leq 2\sqrt{\mu}.\end{aligned}$$

The above result implies that $p(v,\mu)\to\prod_{C(x)}(v)$ when $\mu\to 0$.

In conclusion, the problem (9) becomes the following form:

$$\begin{aligned}&\min\ f(x,p(y,u))\\ &\text{s.t. } x\in X_{ab},\\ &F(x,p(y,u))+y-p(y,u)=0,\end{aligned}\tag{11}$$

For convenience of getting initial points, for given $x\in X_{ad}$, we construct the following smooth equation:

$$q(x,y,\mu)=(1-\mu)[F(x,p(y,\mu))+y-p(y,\mu)]+\mu(y-y^{(0)})=0.\tag{12}$$

Meanwhile, we definite $f(x,y,\mu)=f(x,p(y,\mu))$. Then, we define the following optimization problem

$$\begin{aligned}&\min f(x,y,\mu)\\ &\text{s.t. }\ x\in X_{ab},\\ &q(x,y,\mu)=0.\end{aligned}\tag{13}$$

Problem (13) may be viewed as a perturbation of (9) with the parameter μ. For any $\mu\neq 0$, (13) is a smooth optimization problem. When $\mu=0$, (13) coincides with (9). We denote the feasible set of Problem (13) by E.

Lemma 3.1 ([8]). *Let $F \in C^2$. Then, for fixed $x \in X_{ad}$, for almost all $y^{(0)} \in C(x)$, the homotopy equation (12) determines a smooth curve $\Gamma \subset R^m \times (0,1]$, starting from $(y^{(0)}, 1)$ and approaches the hyperplane at $\mu = 0$. When $\mu \to 0$, the limit set $T \subset R^m \times \{0\}$ of Γ is nonempty, and the the y-componentwise y^* of any point $(y^*, 0) \in T$ is the solution of (12) and $\prod_{C(x)}(y^*)$is a solution of the variational inequalities problem (7).*

The proof is omitted here, for its rigorous proof, the reader is referred to see reference [8].

4. **Main Results**

Let $\theta = (x, y)$, then problem (13) is rewritten as follows:

$$\begin{aligned} &\min \ f(\theta) \\ &\text{s.t. } g(\theta) \le 0, \\ &\ h(\theta, \mu) = 0. \end{aligned} \tag{14}$$

For convenience, the notions are given as follows:

$$\begin{aligned} &\Omega_1(\mu) = \{\theta \in R^{n+m} : g(\theta) < 0,\ h(\theta, \mu) = 0\}, \\ &\Omega_2(\mu) = \{\theta \in R^{n+m} : g(\theta) \le 0,\ h(\theta, \mu) = 0\}, \\ &\partial\Omega_2(\mu) = \{\theta \in \Omega_2(\mu) : \prod_{i=1}^{s} g_i(\theta) = 0\}, \\ &I(\theta) = \{i \in \{1, \ldots, s\} : g_i(\theta) = 0\}. \end{aligned}$$

Assumptions

(A.1) For any $\mu \in [0, 1]$, $\Omega_1(\mu)$ is nonempty;
(A.2) For any $\theta \in \Omega_2(\mu), \mu \in (0, 1]$

$$\{\theta + \sum_{i \in I(\theta)} \alpha_i \nabla g_i(\theta) + \nabla h_\theta(\theta, 1)\beta : \alpha_i \ge 0, i \in I(\theta); \beta \in R^m\} \bigcap \Omega_2(1) = \{\theta\};$$

(A.3) For any $\mu \in (0, 1], \theta \in \Omega_2(\mu), (\nabla g_i(\theta), i \in I(\theta), \nabla_\theta h(\theta, \mu))$ is full column rank ;
(A.4) When $\mu = 0$, $\theta \in \Omega_2(\mu)$, $\{\nabla g_i(\theta), i \in I(\theta), \partial_\theta h(\theta, \mu)^T\}$ satisfy generalized linear independence constraint qualification.

Remark 4.1. *From a geometric point of view, (A.2) satisfies the normal cone condition.*

When $\mu = 0$, the problem (14) is nonsmooth optimization and its GKKT system can be written as follows:

$$\begin{aligned} &0 \in \partial f(\theta) + \nabla g(\theta)\alpha + \partial_\theta h(\theta, \mu)^T \beta, \\ &h(\theta, \mu) = 0, \\ &\alpha \ge 0, g(\theta) \le 0, Ag(\theta) = 0, \end{aligned} \tag{15}$$

where $\alpha \in R^s_+$, $\beta \in R^m$ are multipliers, $\partial_\theta h(\theta, \mu)$ is a Jacobian matrix of h at point θ, $A = \text{diag}(\alpha)$.

When $\mu \neq 0$, the GKKT system (15) is the following KKT system, i.e.,

$$\begin{aligned} &\nabla f(\theta) + \nabla g(\theta)\alpha + \nabla_\theta h(\theta, \mu)\beta = 0, \\ &h(\theta, \mu) = 0, \\ &\alpha \ge 0, g(\theta) \le 0, Ag(\theta) = 0. \end{aligned} \tag{16}$$

For solving the KKT system (16), we construct the following homotopy equation:

$$H(w, w^{(0)}, \mu) = \begin{pmatrix} (1-\mu)(\nabla f(\theta) + \nabla g(\theta)\alpha) + \nabla_\theta h(\theta, \mu)\beta + \mu(\theta - \theta^{(0)}) \\ h(\theta, \mu) \\ Ag(\theta) - \mu A^{(0)} g(\theta^{(0)}) \end{pmatrix} = 0, \tag{17}$$

where $w^{(0)} = (\theta^{(0)}, \alpha^{(0)}, \beta^{(0)})^T \in \Omega_1(1) \times R^s_{++} \times \{0\}$, $w = (\theta, \alpha, \beta)^T \in R^{n+m} \times R^s_+ \times R^m$, $A = \mathrm{diag}(\alpha), A^{(0)} = \mathrm{diag}(\alpha^{(0)}), \mu \in (0, 1]$.

When $\mu = 1$, it is obvious that homotopy equation (17)becomes

$$\begin{array}{c} \nabla_\theta h(\theta, 1)\beta + \theta - \theta^{(0)} = 0, \\ h(\theta, 1) = 0, \\ Ag(\theta) - A^{(0)} g(\theta^{(0)}) = 0. \end{array} \tag{18}$$

According to (A.2), it is known that$\theta = \theta^{(0)}$, $\beta = 0$. Because of $g(\theta^{(0)}) < 0$, it follows that $A = A^{(0)}$. Then, when $\mu = 1$, $w^{(0)} = (\theta^{(0)}, \alpha^{(0)}, 0)^T \in \Omega_1(1) \times R^s_{++} \times \{0\}$ is the unique solution of homotopy equation (17); When $\mu = 0$, the homotopy equation $H(w, w^{(0)}, \mu) = 0$ is coincide with the GKKT of problem (9).

We denote that $H_{w^{(0)}}(w, \mu) = H(w, w^{(0)}, \mu)$, and the zero-point set of H is $H^{-1}_{w^{(0)}} = \{(w, \mu) \in \Omega_1(\mu) \times R^s_+ \times R^m \times (0, 1] : H_{w^{(0)}}(w, \mu) = 0\}$.

By the reference [41], under the same assumptions (A.1)-(A.4) on the functions f, h and g, and the feasible set of homotopy equation (17) used in [41], we can get the following Theorem 4.1 and Theorem 4.2, whose proof is the same as reference [41]. Therefore, it is omit here.

Theorem 4.1. *Let $H_{w^{(0)}}(w, \mu)$ be defined as (17), Assumptions* (A.1)-(A.3) *hold, then for almost all initial points in the feasible set $w^{(0)} = (\theta^{(0)}, \alpha^{(0)},\ 0)^T \in \Omega_1(1) \times R^s_{++} \times R^m$, 0 is a regular value of H, and $H^{-1}_{w^{(0)}}(0)$ contains a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)}, 1)$.*

Theorem 4.2. *Let f, g, F be three times continuous differentiable, suppose assumptions (A.1)-(A.4) hold. For almost any initial point in the feasible set $w^{(0)} = (\theta^{(0)}, \alpha^{(0)}, \beta^{(0)})^T \in \Omega_1(1) \times R^s_{++} \times \{0\}$, if 0 is a regular value of H, then the curve $\Gamma_{w^{(0)}} \subset \Omega_1(\mu) \times R^s_+ \times R^m \times (0, 1]$ is bounded.*

Theorem 4.3. *Let f, g and F are three times continuously differentiable, suppose (A.1)-(A.4) hold. Then, when $\mu \to 0$, the KKT solution of the problem (14) exists, and for almost any initial point $w^{(0)} \in \Omega_1(1) \times R^s_{++} \times \{0\}$, $H^{-1}_{w^{(0)}}$ contains a smooth path starting from $(w^{(0)}, 1)$, which is defined by $\Gamma_{w^{(0)}}$. When $\mu \to 0$, the limit point set $\Xi \times \{0\} \subseteq \Omega_2(1) \times R^s_+ \times R^m \times \{0\}$ of $\Gamma_{w^{(0)}}$ is nonempty, and every point in Ξ is the solution of homotopy equation (17). If $\Gamma_{w^{(0)}}$ is bounded, $(w^*, 0)$ is the terminus of $\Gamma_{w^{(0)}}$, then w^* is a GKKT solution of the approximate problem (14). And $(x^*, \prod_{C(x)})$ is a GKKT solution of the MPBVI(1).*

Proof. According to Theorem 4.1, we know that, for almost any initial point $w^{(0)} = (\theta^{(0)}, \alpha^{(0)}, 0)^T \in \Omega_1(1) \times R^s_{++} \times R^m$, 0 is a regular value of H, and $H^{-1}_{w^{(0)}}$ contains a smooth curve starting from $(w^{(0)}, 1)$ defined by $\Gamma_{w^{(0)}}$. Because of Lemma 2.3, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit circle or unit interval. According to

$$\frac{\partial H_{w^{(0)}}(w, 1)}{\partial w}|_{w=w^{(0)}} = \begin{pmatrix} I_{m+n} & 0 & \nabla h(\theta^{(0)}, 1) \\ \nabla h(\theta^{(0)}, 1)^T & 0 & 0 \\ A^{(0)} \nabla g(\theta^{(0)})^T & diag(g(\theta^{(0)})) & 0 \end{pmatrix}$$

and $g(\theta^{(0)}) < 0$, it is easy to know that $\partial H_{w^{(0)}}(w, 1)/\partial w$ is nonsingular. It shows that $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit interval. Let (w^*, μ^*) be the limit of $\Gamma_{w^{(0)}}$, the following three cases are possible:

(i) $(w^*, \mu^*) \in \Omega_1(\mu^*) \times R^s_+ \times R^m \times \{1\}$;
(ii) $(w^*, \mu^*) \in \partial\Omega_2(\mu^*) \times R^s_+ \times R^m \times (0, 1]$;
(iii) $(w^*, \mu^*) \in \partial\Omega_2(\mu^*) \times R^s_+ \times R^m \times \{0\}$.

Since $H_{w^{(0)}}(w,1)=0$ has a unique solution $(w^{(0)},1)$, case (i) is not possible. Next, we shall exclude case(ii). When case (ii) is true, we have $A^{(0)}>0,\ g(\theta^{(0)})<0$. Then there exists a sequence $\{(w^{(k)},\mu_k)\}\subset\Gamma_{w^{(0)}}$ such that $1\le i\le s,\ g_i(\theta^{(k)})\to 0$. According to the third equation of (17), it is easy to know that $\|\alpha_i^{(k)}\|\to\infty$. It is a contradiction to Theorem 4.2, case (ii) will not happen. Then, case (iii) is the unique possible case and w^* is the solution of the GKKT system (15). $(x^*,\prod_{C(x)})$ is a GKKT solution of the MPBVI(1). The proof is completed. □

On the basis of Theorem 4.3, for almost all initial points in the feasible $w^{(0)}=(\theta^{(0)},\alpha^{(0)},0)^T\in\Omega_1(1)\times R^s_{++}\times R^m$, homotopy equation (17) produce a smooth curve, which is called homotopy path. When $\mu\to 0$, through homotopy path, we get a solution of GKKT system (15). For the sake of tracing homotopy path. Let s denote the arc length parameter of $\Gamma_{w^{(0)}}$, then there exists a smooth function $(w(s),\mu(s))$ such that

$$\begin{cases} H_{w^{(0)}}(w(s),\mu(s))=0,\\ w(0)=w^{(0)},\mu(0)=1.\end{cases}\tag{19}$$

Base on the differential of the first equation of (19) with respect to s, we can get limit point of $\Gamma_{w^{(0)}}$, which is a solution of the GKKT system (15). This is made precise in the following theorem.

Theorem 4.4. *We can confirm the homotopy path $\Gamma_{w^{(0)}}$ by the following initial value problem of the ordinary differential equation,*

$$\begin{cases} H'(w,w^{(0)},\mu)\begin{pmatrix}\dot w(s)\\ \dot\mu(s)\end{pmatrix}=0,\\ \qquad w(0)=w^{(0)},\mu(0)=1.\end{cases}\tag{20}$$

There exists s^* such that $\mu(s^*)=0$, and $(w(s^*),\mu(s^*))$ is a solution of GKKT system (15).

On the basis of the Theorem 4.3 and Theorem 4.4, taking account of (19) and (20), we use a predictor-corrector (Euler-Newton method), which is the same as [41], for numerically tracing homotopy path $\Gamma_{w^{(0)}}$. about the details of the predictor-corrector, one can refer to [2].

Remark 4.2. *The tangent vector of every point on the homotopy path $\Gamma_{w^{(0)}}$ has two direction. For tracing the homotopy path, $\Gamma_{w^{(0)}}$, positive direction will be used, and the sign of the tangent vector is determined by the following theorem.*

5. **Numerical experiments**

In this section, a constructed numerical test will be implemented. The numerical experiments were done by running MATLAB on a PC with CPU of 2.00 GHz and RAM 2.0GB, and four starting points are chosen as $(x^{(0)},y^{(0)})\in X_{ab}\times C(x)$, some parameters are set as $\varepsilon_1=10^{-6},\ \varepsilon_2=10^{-5}$,$\alpha^{(0)}=(1,1,1,1)^T$, $\beta^{(0)}=(0,0)^T$.

Example 5.1.

$$\begin{aligned}&n=1,m=1,l=1\\ &f(x,y)=x_1^2-2x_1+x_2^2-2x_2+y_1+y_2,\\ &X_{ab}=[0,2]\times[0,2],\\ &F(x,y)=\begin{pmatrix}2(y_1-x_1)\\ 2(y_2-x_2)\end{pmatrix},\\ &C(x)=\{y\in R^2: y_1^2+y_2^2\le 16,\ i=1,2\}.\end{aligned}$$

For Example 5.1, let (x^*, y^*) denote the KKT points of this problem. The results are presented in the Table 1.

TABLE 1. The numerical results of Example 5.1.

$(x_1^{(0)}, x_2^{(0)}, y_1^{(0)}, y_2^{(0)})$	$(x_1^*, x_2^*, y_1^*, y_2^*)$	f^*
(1,1,1,1)	(0.0245,0.0245,0.0478,0.0478)	(-0.0012)
(1.5,1,2,1)	(0.0262,0.0256,0.0512,0.0500)	(-0.0011)
(1.8,1.8,3,3)	(0.0252,0.0252,0.0493,0.0493)	(-9.2992e-04)
(0.2,0.5,1,1)	(0.0257,0.0271,0.0504,0.0530)	(-8.0510e-04)
(1.3,1,0.8,1)	(0.0248,0.0247,0.0483,0.0482)	(-0.0013)

6. Conclusion

In this paper, we study homotopy method for solving the mathematical problems with ball-constrained variational inequalities. MPBVI was transformed into a nonsmooth optimization by using few variables. For the approximate problem of the MPBVI, the existence and global convergence of a smooth homotopy path from almost any feasible point was proven under some much weaker conditions. Meanwhile, numerical experiments show that the method is feasible and effective.

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