GRAPHICAL UNIFORM CONVEXITY, GRAPHICAL NORMAL STRUCTURE AND THE FIXED POINT PROPERTY OF GRAPHICAL METRIC SPACES

Tzu-Chien Yin[1], Eskandar Naraghirad[2]

In this paper we first introduce a new class of metric spaces called graphical uniformly convex spaces and investigate that a graphical complete metric space (X, d_G) has the fixed point property if every group of isometric automorphisms of X with a bounded orbit has a fixed point in X. We then prove that if (X, d_G) is graphical uniformly convex then the family of graphical admissible subsets of X possesses graphical uniformly normal structure and if so then it has the fixed point property. We also show that from other weaker assumptions than graphical uniform convexity, the fixed point property follows. Our formulation of graphical uniform convexity and its generalization can be applied not only to geodesic metric spaces. In contrast with the other previous research works, our results are valid in topological (not necessarily metric) spaces.

Keywords: graphical uniform convexity, graphical normal structure, isometric action, bounded orbit, fixed point property.

MSC2020: 47H10, 37C25.

1. Introduction

Fixed points of nonlinear operators is an important research filed in nonlinear functional analysis. It plays an important in differential equations, differential inclusions, integral equations, optimal control, variational inequalities, equilibrium problems and so on; see, e.g, [1]-[64] and the references therein. Both the existence and approximation of fixed points are attracting attentions. For approximation methods of fixed points, they are efficient for various problems in signal processing, image recovery, machine learning and so on; see, e.g., [6, 19, 20, 22, 26] and the references therein. In addition to the approximation methods, the existence of fixed points has been extensively studied since it guarantes the existence of various nonlinear operator equations. The concepts of the convex structure and the convex metric space where first introduced by Takahashi [35] and some fixed point theorems were extracted for nonexpansive mappings in convex metric spaces. In recent decades, researchers have developed the Banach's contarction theorem to all kinds of generalized metric spaces [28–30]. In 2004, Ran and Reuring [30] extended Banach's Contraction Principle in the context of partially ordered set. In [13], Jachymski generalized these spaces, by replacing the previous partially ordered structure with the graph structure. Afterwards, many researchers extended and generalized kinds of fixed point theorems to the metric space endowed with a graph. In 2017, Shukla et al. [33] introduced a new class of the graphical metric spaces and obtained some new fixed point theorems.

In this work, the concepts of the graphical uniformly convex and graphical uniformly normal structure by means of the convex structure are introduced. We investigate certain

[1]Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, e-mail: yintzuchien@mail.cmuh.org.tw

[2]Corresponding author. Department of Mathematics, Yasouj University, Yasouj 75918, Iran, e-mail: esnaraghirad@yu.ac.ir

properties of graphical metric spaces which can be used in the geometric group theory. If a group acts on a graphical complete metric space isometrically, then it is reasonable to ask a question about whether it has a fixed point in it or not. Usually we are interested in what kind of groups satisfy this property when the graphical metric space in question is canonical such as a Hilbert space. An important concept in this theory is the normal structure for a certain family of admissible subsets in a metric space and it can be applicable for investigating the fixed point property of isometric group action. In Section 3, we verify that under a certain extra assumption the graphical normal structure implies the fixed point property (Theorem 3.1).

Throughout this paper, we denote the set of real numbers and the set of natural numbers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let X be a non-empty set and $\Delta := \{(x, y) \in X \times X : x = y\}$. Consider G as a directed graph with no parallel edges such that $G := (V(G), E(G))$, where $V(G)$ be the set of vertices of graph G and $E(G)$ be the set of edges of graph G, then we say that a graph G is associated with X if $V(G) = X$ and $\Delta \subset E(G)$. In a graph G, if there exists a sequence $\{v_j\}_{j=0}^{l}$ of $l{+}1$ vertices such that $v_0 = x$, $v_l = y$ and $(v_{j-1}, v_j) \in E(G)$ for $j = 1, 2, \ldots, l$, then we say that there is path from x to y of length l. If there is a path between any two vertices of a graph G, then G is called a connected graph. In a directed graph, vertices x and y are said to be connected if there is a path from x to y and a path from y to x. In a graph G, if there is a directed path from x to y then it can be written shortly as $(xPy)_G$. For $z \in X$, we say that $z \in (xPy)_G$ if there exists a directed path from x to y containing z. For $l \in \mathbb{N}$, $[x]_l^G := \{y \in X : (xPy)_G$ of length $l\}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $(x_n P x_{n+1})_G$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ is called a G-termwise connected sequence. A connected subgraph G_1 of a graph G is said to be a connected component of G if it is not connected to other vertices in the supergraph. Throughout this paper, we assume that the graphs under consideration are directed, with nonempty sets of vertices and edges.

Definition 1.1. *Let X be a non-empty set and G be a graph associated with X. A function $d_G : X \times X \to \mathbb{R}$ satisfies the following conditions:*
(i) $d_G(x, y) \geq 0$ for all $x, y \in X$;
(ii) $d_G(x, y) = 0$ if and only if $x = y$;
(iii) $d_G(x, y) = d_G(y, x)$ for all $x, y \in X$;
(iv) for all $x, y, z \in X$ with $(xPy)_G$ and $z \in (xPy)_G$, we have $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$.
Then the function d_G is called a graphical metric on X and (X, d_G) is called a graphical metric space.

Definition 1.2. *Let X be a nonempty set endowed with a graph G and $d_G : X \times X \to \mathbb{R}$ be a function satisfying the following conditions:*
(GM1) $d_G(x, y) \geq 0$ for all $x, y \in X$;
(GM2) $d_G(x, y) = 0$ if and only if $x = y$;
(GM3) $d_G(x, y) = d_G(y, x)$ for all x, $y \in X$;
(GM4) $(xPy)_G, z \in (xPy)_G$ implies $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$ for all $x, y, z \in X$.
Then, the mapping d_G is called a graphical metric on X, and the pair (X, d_G) is called a graphical metric space.

Remark 1.1. *It is clear that any metric d on a metric space X is a graphical metric on X, but the converse is not correct in general, see, for instance, Example 2.5 in [42]. It is well-known that a graphical metric space (X, d_G) need not be a metric space.*

In this paper we first introduce a new class of metric spaces called graphical uniformly convex spaces and investigate that a graphical complete metric space (X, d_G) has the fixed point property if every group of isometric automorphisms of X with a bounded orbit has a fixed point in X. We then prove that if (X, d_G) is graphical uniformly convex then the family of graphical admissible subsets of X possesses graphical uniformly normal structure and if so then it has the fixed point property. We also show that from other weaker assumptions than graphical uniform convexity, the fixed point property follows. Our formulation of graphical uniform convexity and its generalization can be applied not only to geodesic metric spaces. Our results improve and supplement some results in the current literature; see, for example, [21].

2. **Graphical uniformly convexity and graphical normal structure**

In this section, we first introduce some basic concepts and notations which are taken from the papers [13, 42]. Let $(V(G), E(G))$ be the graph associated with the nonempty set G, where $V(G)$ is the set of vertices and $E(G)$ is the binary relation on $V(G)$. Elements of $E(G)$ are called edges. By G^{-1} we denote the conversion of a directed graph G, i.e. the graph obtained from G by reversing the direction of edges. Thus we have $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Given a directed graph G, one may generate a graph $\tilde{G}$ where we ignore the directions and replace the resulting multiple edges by single edges. We define $E(\tilde{G}) = E(G) \cup E(G^{-1})$, then $\tilde{G}$ is a symmetric directed graph. The directed graph G is called reflexive, if the set $E(G)$ contains all loops, i.e. $(x, x) \in G$ for each $x \in V(G)$. Moreover, a directed graph G is called transitive whenever $(x, y) \in E(G)$ and $(y, z) \in E(G) \Longrightarrow (x, z) \in E(G)$ for all $x, y, z \in E(G)$.

Throughout this paper, we always assume that the directed graph G with edge weights by assigning the distance between two vertices to each edge, is symmetric, reflexive and transitive.

Definition 2.1. *Let $u, v \in V(G)$, a path (or directed path) of length $l \in \mathbb{N}$ between u and v in G is defined as a sequence $\{x_j\}_{j=0}^{l}$ of vertices with $u = x_0$, $v = x_l$ and $(x_{j-1}, x_j) \in E(G)$ for $j = 1, 2, \ldots, l$.*

Set $[u]_G^l = \{v \in U(G) :$ there exists a path directing from u to v having lenth $l\}$.

Definition 2.2. *Let (X, d_G) be a graphical metric space. A relation R on X is such that $(uRv)_G$ if there exists a path directing from u to v in G and $w \in (uRv)_G$ if w is contained in the path $(uRv)_G$. We say that a sequence $\{x_n\}_{n\in\mathbb{N}} \in Y$ is G-termwise connected $(G-TWC)$ if $(x_n R x_{n+1})_G$ for all $n \in \mathbb{N}$.*

Let (X, d_G) be a graphical metric space. We define an open ball $B_G(x, \epsilon)$ with center $x \in X$ and radius $\epsilon > 0$ as $B_G(x, \epsilon) = \{y \in X : (xPy)_G, d_G(x, y) < \epsilon\}$. Since $\Delta \subset E(G)$, then we get $x \in B_G(x, \epsilon)$ and so $B_G(x, \epsilon) \neq \emptyset$ for all $x \in X$ and $\epsilon > 0$. The collection $\mathcal{B}_G = \{B_G(x, \epsilon) : x \in X, \epsilon > 0\}$ is a neighborhood system for the topology τ_G on X induced by the graphical metric d_G. A subset U of X is called open if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_G(x, \epsilon) \subset U$. Also, a subset C of X is called closed if its complement $X \setminus C$ is open.

Next, we define the concepts of convergence, Cauchy sequence and completeness in a graphical metric space.

Definition 2.3. *Let (X, d_G) be a graphical metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent and converges to $x \in X$ if, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_G(x_n, x) < \epsilon$ for all $n > n_0$. It is evident that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to x, if and only if $\lim_{n\to\infty} d_G(x_n, x) = 0$.*

Definition 2.4. *Let (X, d_G) be a graphical metric space A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a Cauchy sequence if, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_G(x_n, x_m) < \epsilon$ for all $n, m > n_0$. It can easily be shown that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if and only if $\lim_{n,m\to\infty} d_G(x_n, x_m) = 0$.*

Definition 2.5. *A graphical metric space (X, d_{G_1}) is said to be complete if every Cauchy sequence in X converges in X. Let G_2 be another graph such that $V(G_2) = X$, then (X, d_{G_1})*

is said to be G_2-complete if, every G_2-termwise connected Cauchy sequence in X converges in X.

Let us begin with defining properties of a metric space X which we are concerned with. The first one is a numerical generalization of the concept of uniform convexity for L_p spaces to metric spaces. For $p = 2$, this is called the NC-inequality, which was first introduced in [5]. The following definition, which is inspired by [3], is slightly different from the usual one. In particular, we do not have to assume that X is a geodesic metric space (i.e., for any $x, y \in X$ there is a geodesic segment connecting them).

Definition 2.6. *A graphical metric space (X, d_G) is called graphical (p, c)-uniformly convex for $p \in [1, \infty)$ and $c > 0$ if for any $x, y \in G$ with $(xRy)_G$ there is some $m \in (xRy)_G$ such that every $z \in G$ with $(zRm)_G$ and $z \in (xRy)_G$ satisfies that*

$$d_G(z,m)^p \le \frac{1}{2}\{d_G(z,x)^p + d_G(z,y)^p\} - c d_G(x,y)^p.$$

If we can find such p and c, then we simply say that $X = (X, d_G)$ is graphically uniformly convex.

Remark 2.1. *In the usual definition, it is assumed that X is a geodesic metric space and the above condition is replaced with a similar condition for each point m on every geodesic segment connecting x and y. See [23] and [25]. The usual definition for uniform convexity implies ours and in this case the constants should be restricted to $p > 1$ and $c \le \frac{1}{2p}$. However, when X is complete and when $p = 2$ and $c = \frac{1}{4}$, our condition automatically implies that X is a uniquely geodesic metric space (i.e., for any $x, y \in X$ there is a unique geodesic segment connecting them) and X is contractible. These facts were shown in [3]. Moreover, a complete metric space X is graphically $(2, \frac{1}{4})$-uniformly convex if and only if it is a $CAT(0)$-space.*

In the above definition, the freedom of the constant c has the benefit of generalizing the concept of graphical uniform convexity. This was already done in [25] for geodesic metric spaces with $p = 2$. Actually, a $CAT(1)$-space (X, d_G) with diameter not greater than $\frac{\pi}{2} - \epsilon$ for $\epsilon \in (0, \frac{\pi}{2})$ is $(2, c)$-uniformly convex for $c = (\frac{\pi}{2} - \epsilon)\frac{\sin(\epsilon)}{2\cos\epsilon}$. This in particular shows that the sphere with diameter less than $\frac{\pi}{2}$ is uniformly convex though its spherical distance function $d : X \times X \to [0, \infty)$ is not convex on geodesic segments. Furthermore, as an example below shows, there is a complete metric space X which is uniformly convex but is not a geodesic metric space. Note that a complete metric space (X, d_G) is geodesic if and only if it is metrically convex, that is, for any two points $x, y \in X$ there is a midpoint $m \in X$ satisfying $d_G(x, m) = d_G(y, m) = d_G(x, y)/2$ (see [3] and [16, Theorem 2.16]. Although the following example is rather artificial, our definition of graphically uniform convexity might have the advantage of treating a possible situation where a subspace X with relative distance d_G embedded in an infinite-dimensional Riemannian or Finsler manifold M is not known to be a geodesic metric space but (X, d_G) is graphical uniformly convex.

Example 2.1. *Let $S_\theta = \{w \in \mathbb{R}^2 : 0 \le argw \le \theta\}$ be the infinite circular sector with center at the origin 0 and angle $\theta \in (0, \pi/3)$. Then $X = X_\theta$ is given as the part of S_θ that is not contained in the open unit disk $\mathbb{D} = \{w \in \mathbb{R}^2 : |w| < 1\}$, that is, $X_\theta = S_\theta \setminus \mathbb{D}$. We provide X_θ with the restriction of the Euclidean distance d on $\mathbb{R}^2$. Clearly (X_θ, d) is not a geodesic metric space. However, we see that X_θ is graphically (p, c)-uniformly convex for $p = 2$ and $c = (2\cos\theta - 1)/4$. Its proof can be found in [21].*

Next, the relation between the graphical radius and the diameter of certain graphical admissible subsets gives another condition for a metric space. Here, for a subset A of a graphical metric space (X, d_G), we denote its diameter and graphical Chebyshev radius by

$$\mathrm{diam}_G(A) = \sup\{d_G(x,y) : x, y \in A\};\ \mathrm{rad}_G(A) = \inf\{r > 0 : A \subset B_G(z,r)\ (\text{for some } z \in A)\},$$

where $B_G(z,r) = \{x \in X : d_G(z,x) \leq r\}$ is the closed metric ball with center z and radius r. We regard a non-empty subset $A \subset X$ *graphical admissible* if it is the intersection of some closed metric balls $\{B_G(z_i, r_i)\}_{i\in I}$($I$ is an index set) of X. The family of all such non-empty bounded closed subsets A of X is denoted by $\mathcal{A}_G(X)$.

Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of a nonempty set Y.

Definition 2.7. *Let (X, d_G) be a graphical metric space. The family $\mathcal{A}_G(X)$ of graphically admissible subsets of a metric space (X, d_G) possesses graphical normal structure if every subset $A \in \mathcal{A}_G(X)$ with $diam_G(A) > 0$ satisfies $rad_G(A) < diam_G(A)$. Moreover, $\mathcal{A}_G(X)$ possesses graphical uniformly normal structure if there exists a positive constant $\alpha > 0$ such that this inequality is uniformly valid in the form $rad_G(A) \leq (1-\alpha)diam_G(A)$.*

Our first result shows the implication of the above two properties.

Theorem 2.1. *If a graphical metric space (X, d_G) is graphical (p, c)-uniformly convex, then $\mathcal{A}_G(X)$ has graphical uniformly normal structure. More precisely,*

$$\mathrm{rad}_G(A) \leq (1-c)^{\frac{1}{p}} \mathrm{diam}_G(A)$$

for every $A \in \mathcal{A}_G(X)$.

Proof. Let $A \in \mathcal{A}_G(X)$ be arbitrarily chosen with $d = \mathrm{diam}_G(A) > 0$. Let us select an arbitrary $\epsilon > 0$. Then there are $x, y \in A$ such that $d_G(x, y) \geq d - \epsilon$. For these x and y, the definition of graphical (p, c)-uniform convexity finds some point $m_\epsilon \in X$ that satisfies

$$d_G(z, m_\epsilon)^p \leq \frac{1}{2}[d_G(z, x)^p + d_G(z, y)^p] - c d_G(x, y)^p \tag{1}$$

for every $z \in X$. First we check that m_ϵ belongs to A. Suppose that A is the intersection of closed metric balls $\{B_G(z_i, r_i)\}_{i\in I}$(I is an index set) of X for all indices $i \in I$. From the assumption $x, y \in B_G(z_i, r_i)$, we conclude that $d_G(z_i, x) \leq r_i$ and $d_G(z_i, y) \leq r_i$ for each $i \in I$. This, together with (1) to $z = z_i$, amounts to

$$d_G(z_i, m_\epsilon)^p \leq \frac{1}{2}[d_G(z_i, x)^p + d_G(z_i, y)^p] \leq r_i^p. \tag{2}$$

This implies that $m_\epsilon \in B_G(z_i, r_i)$ and hence $m_\epsilon \in A$. Now, for an arbitrary $z \in A$ we are let to $d_G(z, x) \leq d$ and $d_G(z, y) \leq d$. Substituting these bounds and $d_G(x, y) \geq d - \epsilon$ to (2), we deduce that $d_G(z, m_\epsilon)^p \leq d^p - c(d-\epsilon)^p$. This entails to $d_G(z, m_\epsilon) \leq d(1 - c(1-\frac{\epsilon}{d})^p)^{\frac{1}{p}}$, and hence A is in the closed ball of center $m_\epsilon \in A$ and radius $d(1 - c(1-\frac{\epsilon}{d})^p)^{\frac{1}{p}}$. Since $\epsilon > 0$ is arbitrary, letting $\epsilon \to 0$, we see that $\mathrm{rad}_G(A) \leq (1-c)^{\frac{1}{p}} \mathrm{diam}_G(A)$. □

3. **The fixed point property**

In this section, we investigate the following property of a graphical metric space (X, d_G) concerning the action of its automorphism group. We denote by $\mathrm{Aut}(X, d_G)$ the group of isometric bijections of X onto itself with respect to graphical distance d_G.

Definition 3.1. *A graphical metric space (X, d_G) has the fixed point property if every subgroup $H \subset Aut(X, d_G)$ with a bounded orbit in X has a fixed point in X. We notice that if the orbit $H(x)$ of $x \in X$ is bounded then the orbit $H(x')$ for any other $x' \in X$ is also bounded. In particular, if H has a fixed point in X then H has a bounded orbit $H(x)$ for every $x \in X$.*

Now, we use the following notations: the family $\mathcal{A}_G(X)$ of graphical admissible subsets of X is compact if every totally ordered sub-family $\{A_i\}_{i\in I} \subset \mathcal{A}_G(X)$ with respect to the inclusion relation satisfies $\cap_{i\in I} A_i \neq \emptyset$ that is, $\cap_{i\in I} A_i \in \mathcal{A}_G(X)$.

We will verify that the properties introduced in the previous section imply the fixed point property. By a similar process to [16, Theorem 5.1], we can verify the following result, which has its origin in [17] and whose abstract formulation is due to [27]. We notice that if $\mathcal{A}_G(X)$ is compact then (X, d_G) is complete (see [16]).

Theorem 3.1. *If the graphical admissible family $\mathcal{A}_G(X)$ of a graphical metric space (X, d_G) possesses graphical normal structure and $\mathcal{A}_G(X)$ is compact, then (X, d_G) has the fixed point property.*

Proof. Let $H \subset \operatorname{Aut}(X, d_G)$ with a bounded orbit $H(x)$ $(x \in X)$. For a closed metric ball containing $H(x)$, we take the intersection $A_* \neq \emptyset$ of all its images under H. Then $A_* \in \mathcal{A}_G(X)$ is invariant under H. We consider the H-invariant sub-family of $\mathcal{A}_G(X)$:

$$\mathcal{A}_G^H(X) = \{A \in \mathcal{A}_G(X) | h(A) = A \ (\forall h \in H)\}.$$

Since $A_* \in \mathcal{A}_G(X)$, this is not an empty family. Also it is evident that if $A_i \in \mathcal{A}_G(X)$ for all $i \in I$ then $\cap_{i \in I} A_i$ is H-invariant. Then the compactness of $\mathcal{A}_G(X)$ implies that $\mathcal{A}_G(X)$ is inductive and Zorn's Lemma ensures the existence of a minimal element $A_0 \in \mathcal{A}_G(X)$ with respect to the inclusion relation. We will prove that A_0 consists of a single point $a \in X$. This shows that a is a fixed point of G. Suppose to the contrary that A_0 is not a single point set. Then $\operatorname{diam}_G(A_0) > 0$ and the graphical normal structure of $\mathcal{A}_G(X)$ implies that $\operatorname{rad}_G(A_0) < \operatorname{diam}_G(A_0)$. Choose a constant r with $\operatorname{rad}_G(A_0) < r < \operatorname{diam}_G(A_0)$ and set $C = \{x \in A_0 : A_0 \subset B_G(x, r)\}$. This is not empty since $\operatorname{rad}_G(A_0) < r$. Then we have $y \in (\cap_{y \in A_0} B_G(y, r)) \cap A_0$, which can be verified as follows. Take $x \in C \subset A_0$ arbitrarily. Since $A_0 \subset B_G(x, r)$ by the definition of C, every $y \in A_0$ satisfies $d_G(x, y) \leq r$. Hence x belongs to $\cap_{y \in A_0} B_G(y, r)$.

Conversely, if we select $x \in (\cap_{y \in A_0} B_G(y, r)) \cap A_0$ arbitrarily, then for every $y \in A_0$ we deduce that $d_G(x, y) \leq r$. This ensures $A_0 \subset B_G(x, r)$ and the definition of C implies that x belongs to C. The above representation of C in particular implies that $C \in \mathcal{A}_G(X)$. Moreover, we will prove that $C \in \mathcal{A}_G(X)$, that is, $h(C) = C$ for every $h \in H$. It is enough to show that $h(C) \subset C$ for every $h \in G$ because this includes $h^{-1}(C) \subset C$ and hence $C \subset h(C)$. Take an arbitrary $x \in C$, which satisfies $d_G(x, y) \leq r$ for every $y \in A_0$. It follows that $d_G(h(x), h(y)) \leq r$ for every $h \in H$. This yields that $h(y) \in B_G(h(x), r)$ for every $y \in A_0$, that is, $A_0 = h(A_0) \subset B_G(h(x), r)$. On the other hand, we know that $h(x) \in A_0$ from $x \in C \subset A_0$. Therefore $h(x)$ belongs to C. This means that $h(C) \subset C$. However, we see that $\operatorname{diam}_G(C) \leq r$. Indeed, for any x and y in C, it holds that $d_G(x, y) \leq r$ because $x \in A_0 \subset B_G(y, r)$. Since $\operatorname{diam}_G(C) \leq r < \operatorname{diam}_G(A_0)$, we conclude $C \subset A_0$. This contradicts the minimality of A_0 in $\mathcal{A}_G(X)$. Thus we have proved that $A_0 = \{a\}$, which is fixed by H. This completes the proof. □

If (X, d) is complete and $\mathcal{A}_G(X)$ has uniformly normal structure then $\mathcal{A}_G(X)$ is compact, which was proved in [2] and [14] for a weaker condition of the compactness (countable compactness) and completed by the work of [18] (see Section 5). We may also consult [16, Theorem 5.4]. Consequently, we obtain the following result as a corollary to Theorem 3.1. The fact is that an argument in [14] can directly show this result without using the compactness.

Corollary 3.1. *If the graphical admissible family $\mathcal{A}_G(X)$ of a graphical complete metric space (X, d_G) has graphical uniformly normal structure, then (X, d_G) has the fixed point property.*

Also, Theorem 2.1 and Corollary 3.1 yield the following result.

Corollary 3.2. *If a graphical complete metric space (X, d_G) is graphical uniformly convex, then (X, d_G) has the fixed point property.*

Note that this result is already known and can be proved directly. Actually, every bounded subset A in a graphical uniformly convex complete metric space (X, d_G) has the unique graphical circumcenter, which is the center of a closed metric ball containing A with the minimum circumcenter radius attained. See below in the next section for precise definition. This fact will be also proved later in Theorem 4.2 under a weaker assumption. If we take A as the bounded orbit of $H \subset \mathrm{Aut}(X, d_G)$, then its unique circumcenter is a fixed point of H. This result is called the Bruhat-Tits theorem [5]. Its presentation using circumcenter circumcenter can be found in [4]. In the case of p in general, see [25, Lemma 2.3].

4. **Graphical uniform pseudo-convexity**

We extend the concept of graphical uniform convexity of a complete metric space so that it still holds the fixed point property.

Definition 4.1. *A graphical metric space (X, d_G) is graphical uniformly k-pseudo-convex for $k \in [0, 1)$ if there are some constants $p \geq 1$ and $c > 0$ such that for any $x, y \in X$ there is some $m \in X$ such that every $z \in X$ satisfies*

$$d_G(z, m)^p \leq \frac{1 + k^p c}{2}[d_G(z, x)^p + d_G(z, y)^p] - c d_G(x, y)^p.$$

If there is such k, then we simply say that (X, d_G) is graphical uniformly pseudo-convex.

We define the following points close to graphical circumcenter for each bounded subset A of a graphical metric space (X, d_G). For $A \subset X$ and $x \in X$, set $r_{x,G}(A) = \sup_{a \in A} d_G(x, a)$. Let us define the graphical circumradius of A by

$$r_{X,G}(A) = \inf\{x \in X : r_{x,G}(A) = \inf\{r > 0 : A \subset B_G(x, r)(\text{for some } \ x \in X)\},$$

which is not greater than the graphical Chebyshev radius $\mathrm{rad}_G(A)$. In general, we see that $r_{X,G}(A) \leq \mathrm{rad}_G(A) \leq \mathrm{diam}_G(A) \leq 2r_{G,X}(A)$. For every $\epsilon \geq 0$, we say that $x \in X$ is a graphical ϵ-circumcenter of a bounded subset $A \subset X$ if it satisfies $r_{x,G}(A) \leq r_{X,G}(A) + \epsilon$.

When $\epsilon = 0$, this is nothing but a graphical circumcenter of A. Clearly a graphical ϵ-circumcenter always exists for every bounded subset A and for every positive $\epsilon > 0$.

Lemma 4.1. *Let A be a bounded subset with $r_{X,G}(A) > 0$ (or $\mathrm{diam}_G(A) > 0$) in a graphical uniformly k-pseudo-convex metric space (X, d_G). Then, for any $\tilde{k} \in (k, 1)$, there is some $\epsilon > 0$ such that any graphical ϵ-circumcenters $x, y \in X$ of A satisfy $d_G(x, y) \leq \tilde{k} r_{X,G}(A)$.*

Proof. Let $\epsilon > 0$ be fixed and x, y in X be arbitrary graphical ϵ-circumcenters of A. Then we can select some $m \in X$ that satisfies the inequality of graphical uniform k-pseudo-convexity for some $p \geq 1$ and $c > 0$. Let z in A be such that $d_G(z, m) \geq r_{m,G}(A) - \epsilon$. By definition, $d_G(z, x) \leq r_{x,G}(A)$ and $d_G(z, y) \leq r_{y,G}(A)$. Substituting these three estimates to the graphical uniformly pseudo-convex inequality, we obtain

$$(r_{m,G}(A) - \epsilon)^p \leq \frac{1 + k^p c}{2}[r_{x,G}(A)^p + r_{y,G}(A)^p] - c d_G(x, y)^p.$$

Moreover, since $r_{x,G}(A) \leq r_{X,G}(A) + \epsilon$ and $r_{y,G}(A) \leq r_{X,G}(A) + \epsilon$, it follows that

$$(r_{X,G}(A) - \epsilon)^p \leq (r_{X,G}(A) + \epsilon)^p + k^p c(r_{X,G}(A) + \epsilon)^p - c d_G(x, y)^p.$$

Employing $(r_{X,G}(A) + \epsilon)^p (r_{X,G}(A) - \epsilon)^p \leq 4p\epsilon(r_{X,G}(A) + \epsilon)^{p-1}$ for $p \geq 1$, we conclude that

$$\begin{aligned} d_G(x, y)^p &\leq \frac{4p\epsilon}{c}(r_{X,G}(A) + \epsilon)^{p-1} + k^p(r_{X,G}(A) + \epsilon)^p \\ &= \frac{4p\epsilon}{c(r_{X,G}(A) + \epsilon) + k^p}\left[1 + \frac{\epsilon}{r_{X,G}(A)}\right]^p r_{X,G}(A)^p. \end{aligned}$$

Then, for any $\tilde{k} \in (k, 1)$, we can make the last term bounded by $\tilde{k} p r_{X,G}(A)^p$ if we choose a sufficiently small $\epsilon > 0$. □

The following result can be proved by the previous lemma.

Theorem 4.1. *If a graphical complete metric space (X, d_G) is graphical uniformly pseudo-convex, then it has the fixed point property.*

Proof. Suppose that (X, d_G) is graphical uniformly k-pseudo-convex for $k \in [0, 1)$ and that $H \subset \mathrm{Aut}(X, d_G)$ has a bounded orbit $H(x_0)$ for $x_0 \in X$. Choose any $\tilde{k} \in (k, 1)$ and fix it. First we apply Lemma 4.1 to $A_0 = H(x_0)$. We may assume that $\mathrm{diam}_G(A_0) > 0$ for otherwise we obtain a fixed point x_0 of H. Then there is some $\epsilon_1 \in (0, \mathrm{diam}_G(A_0))$ such that any graphical ϵ_1-circumcenters x_1 and y_1 of A_0 satisfy $d_G(x_1, y_1) \leq \tilde{k} r_{X,G}(A_0)$. Note that every point of the orbit $H(x_1)$ is a graphical ϵ_1-circumcenter of A_0. Hence, for $A_1 = H(x_1)$, the above inequality implies that

$$\mathrm{diam}_G(A_1) \leq \tilde{k} r_{X,G}(A_0) \leq \tilde{k}\mathrm{diam}_G(A_0).$$

Next we apply Lemma 4.1 to $A_1 = H(x_1)$, for which we may assume that $\mathrm{diam}_G(A_1) > 0$. Then there is some $\epsilon_2 \in (0, \mathrm{diam}_G(A_1))$ such that any graphical ϵ_2-circumcenters x_2 and y_2 of A_1 satisfy $d_G(x_2, y_2) \leq \tilde{k} r_{X,G}(A_1)$. For $A_2 = H(x_2)$, we have $\mathrm{diam}_G(A_2) \leq \tilde{k} r_{X,G}(A_1) \leq \tilde{k}\mathrm{diam}_G(A_1)$. Continuing this process, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that each x_n is a graphical ϵ-circumcenter of the orbit $A_{n-1} = H(x_{n-1})$ for some $\epsilon_n \in (0, \mathrm{diam}_G(A_{n-1}))$ and that the orbits satisfy $\mathrm{diam}_G(A_n) \leq \tilde{k}\mathrm{diam}_G(A_{n-1})$ for every $n \in \mathbb{N}$. Since $\tilde{k} < 1$, this implies that $\mathrm{diam}_G(A_n) \to 0$ as $n \to \infty$.

It is readily checked that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Indeed, since x_n is a graphical ϵ_n-circumcenter of $A_{n-1} = H(x_{n-1})$, $d_G(x_n, x_{n-1}) \leq r_{X,G}(A_{n-1}) + \epsilon_n \leq 2\mathrm{diam}_G(A_{n-1})$. Then

$$\sum_{n=0}^{\infty} d_G(x_n, x_{n-1}) \leq 2(\mathrm{diam}_G(A_0)) \sum_{n=0}^{\infty} \tilde{k}^n < \infty,$$

which shows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of (X, d_G) is complete, we deduce that the limit x_∞ of $\{x_n\}_{n\in\mathbb{N}}$ exists in X. For every $h \in H$, we deduce that

$$d_G(g(x_\infty), x_\infty) = \lim_{n\to\infty} d_G(g(x_n), x_n) \leq \lim_{n\to\infty} \mathrm{diam}_G(A_n) = 0.$$

Thus x_∞ is a fixed point of H which completes the proof. □

We notice that the graphical uniform pseudo-convexity is preserved under a bi-Lipschitz homeomorphism with a small Lipschitz constant.

Definition 4.2. *We say that a (surjective) homeomorphism $f : X_1 \to X_2$ between metric spaces (X_1, d_{G_1}) and (X_2, d_{G_2}) is λ-bi-Lipschitz for $\lambda \geq 1$ if*

$$\frac{1}{\lambda} d_{G_1}(x, y) \leq d_{G_2}(f(x), f(y)) \leq \lambda d_{G_1}(x, y)$$

is satisfied for any $x, y \in X_1$.

Theorem 4.2. *Let (X_1, d_{G_1}) be a graphical (p, c)-uniformly convex metric space. If $f : X_1 \to X_2$ is a λ-bi-Lipschitz homeomorphism onto another graphical metric space (X_2, d_{G_2}) with Lipschitz constant $\lambda < (1 + c)^{\frac{1}{2p}}$, then (X_2, d_{G_2}) is graphical uniformly pseudo-convex.*

Proof. Take any $x_2, y_2 \in X_2$. For $x_1, y_1 \in X_1$ with $f(x_1) = x_2$ and $f(y_1) = y_2$, we choose $m_1 \in X_1$ that satisfies the inequality for graphical (p, c)-uniform convexity of (X_1, d_{G_1}). If we set $m_2 = f(m_1) \in X_2$, then for every $z_2 \in X_2$, we get

$$\begin{aligned} d_{G_2}(z_2, m_2)^p + c d_{G_2}(x_2, y_2)^p &\leq \lambda p[d_G(z_1, m_1)^p + c d_G(x_1, y_1)^p] \\ &\leq \frac{\lambda^{2p}}{2}[d_{G_2}(z_2, x_2)^p + d_{G_2}(z_2, y_2)^p], \end{aligned}$$

where $z_1 \in X_1$ is taken as $f(z_1) = z_2$. Here, since $1 \leq \lambda^{2p} < 1 + c$, there is some $k \in [0, 1)$ such that $\lambda^{2p} = 1 + k^p c$. This shows that (X_2, d_{G_2}) is graphical uniformly pseudo-convex. $\square$

Similarly, we can prove that if (X_1, d_{G_1}) is uniformly pseudo-convex and $f : X_1 \to X_2$ is a λ-bi-Lipschitz homeomorphism onto another metric space (X_2, d_{G_2}) with $\lambda \geq 1$ sufficiently close to 1, then (X_{G_2}, d_{G_2}) is also uniformly pseudo-convex. If there is a bi-Lipschitz homeomorphism $f : (X_1, d_{G_1}) \to (X_2, d_{G_2})$, then the conjugate $G_1' = fG_1f^{-1}$ for an isometry group $G \subset \mathrm{Aut}(X_1, d_{G_1})$ acts on (X_2, d_{G_2}) as uniformly bi-Lipschitz homeomorphisms, meaning that the Lipschitz constants λ are uniformly bounded for all elements of G'. In this situation, the existence of a fixed point of G_1 is equivalent to that of G'. In [15, Theorem 3.1], a certain fixed point property of a uniformly Lipschitz map is investigated.

The following example shows that neither uniform convexity nor normal structure of the admissible family are invariant under bi-Lipschitz homeomorphisms.

Example 4.1. *Let $ds(x, y) = \sqrt{dx^2 + dy^2}$ be the Euclidean metric on $\mathbb{R}^2$. Define a new metric $\tilde{ds}(x, y)$ on $\mathbb{R}^2$ by $d\tilde{s}(x, y) = ds(x, y)$ if $(x, y) \in \mathbb{R}^2 \setminus \mathbb{D}$ and $d\tilde{s}(x, y) = \lambda ds(x, y)$ if $(x, y) \in \mathbb{D}$, where $\mathbb{D}$ is the unit disk and λ is a constant with $1 < \lambda < (\frac{5}{4})^{\frac{1}{4}}$. Then the identity map is a λ-bi-Lipschitz homeomorphism between $(\mathbb{R}^2, ds)$ and $(\mathbb{R}^2, d\tilde{s})$. Since $(\mathbb{R}, ds)$ is $(2, 1/4)$-uniformly convex, Theorem 4.2 shows that $(\mathbb{R}^2, d\tilde{s})$ is uniformly pseudo-convex. On the other hand, the family of admissible subsets of $(\mathbb{R}, d\tilde{s})$ does not have normal structure. Indeed, by taking two closed balls with large radii and centers far away from the origin, which are put in a symmetric position with respect to the origin, we can make an admissible subset A as the intersection of these balls that consists of exactly two points. Actually, the distance between $(\mathbb{R} + 1, 0)$ and the origin with respect to $d\tilde{s}$ is $R + \lambda$ whereas the distance between $(\mathbb{R} + 1, 0)$ and $(0, 1)$ is less than $(\mathbb{R} + 1)\cos\theta - \sin\theta + 2\theta(\tan\theta = (\mathbb{R} + 1) - 1)$. This verifies that $rad_G(A) = diam_G(A)$, which implies that the admissible family of $(\mathbb{R}, d\tilde{s})$ does not have normal structure. In fact, although $(\mathbb{R}, d\tilde{s})$ is a geodesic metric space, the distance function is not convex on geodesic segments.*

5. **Graphical uniform convexity in the wider sense**

This section is added in revision. We will see here that our fashion of defining graphical uniform convexity also works for more general graphical uniform convexity, which is the generalization of uniform convexity of Banach spaces to metric spaces. See [15] for a recent account of such usual definition.

Definition 5.1. *A graphical metric space (X, d_G) is graphical uniformly convex in the wider sense if there is a function $\delta(r, t) : [\frac{t}{2}, \infty) \times [0, \infty) \to [0, 1]$, which is called the modulus of convexity, such that*
(1) $\delta(r, t) = 0$ if and only if $t = 0$;
(2) for each fixed r, $\delta(r, t)$ is increasing with respect to t;
(3) for each fixed t, $\delta(r, t)$ is decreasing with respect to r, and if for any $x, y \in X$ there is some $m \in X$ such that, for every $z \in X$,

$$d_G(z, m) \leq \max\{d_G(z, x), d_G(z, y)\}(1 - \delta(\max\{d_G(z, x), d_G(z, y)\}, d_G(x, y))).$$

It is easy to see that if (X, d_G) is graphical (p, c)-uniformly convex, then it is graphical uniformly convex in the wider sense for modulus of convexity $\delta(r, t) = \frac{ct^p}{pr^p}$. We will state two theorems which are closely related to the fixed point property of a graphical complete metric space uniformly convex in the wider sense. The first one is concerning the existence and uniqueness of a graphical circumcenter of every bounded subset. This property implies the fixed point property. Indeed, the circumcenter of a bounded orbit of the isometric action of G is fixed by G. This is well-known as the Bruhat-Tits theorem, which was mentioned at Corollary 3.1.

Theorem 5.1. *Let (X, d_G) be a graphical complete metric space that is graphical uniformly convex in the wider sense. Then every bounded subset $A \subset X$ has the unique graphical circumcenter.*

Proof. For any $x, y \in X$, there is some $m \in X$ that satisfies the inequality for the definition of uniform convexity in the wider sense. We consider $r_m(A)$ for this m. For every $\epsilon > 0$, there is $z_\epsilon \in A$ such that $d_G(z_\epsilon, m) \geq r_m(A) - \epsilon$. Then we have

$$r_{X,G}(A) - \epsilon \leq d_G(z_\epsilon, m) \leq \max\{d_G(z_\epsilon, x), d_G(z_\epsilon, y)\}(1 - \delta(\max d_G(z_\epsilon, x), d_G(z_\epsilon, y), d_G(x, y))).$$

Using property (3) of δ for $d_G(z_\epsilon, x) \leq r_{x,G}(A), r_{y,G}(A)$, and taking $\epsilon \to \infty$, we obtain an inequality

$$r_{X,G}(A) \leq \max\{r_{x,G}(A), r_{y,G}(A)\}(1 - \delta(\max\{r_{x,G}(A), r_{y,G}(A)\}, d_G(x, y))) \tag{3}$$

for any $x, y \in X$. The existence of a graphical circumcenter of A is proved as follows. We may assume that $r_{X,G}(A) > 0$. Take a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $r_{x_n,G}(A) \to r_{X,G}(A)$ as $n \to \infty$. For every $k \in \mathbb{N}$, there is n_k such that $n \geq n_k$ implies $r_{x_n,G}(A) < r_{X,G}(A) + \frac{1}{k}$. We apply inequality (3) for $x = x_n$ and $y = x_l$ with $n, l \geq n_k$. It turns out that

$$\begin{aligned} r_{X,G}(A) &\leq \max\{r_{x_n,G}(A), r_{x_l,G}(A)\}(1 - \delta(\max\{r_{x_n mG}(A), r_{x_l}(A)\}, d_G(x_n, x_l))) \\ &\leq (r_{X,G}(A) + \frac{1}{k})(1 - \delta(r_{X,G}(A) + \frac{1}{k}, d_G(x_n, x_l))) \\ &\leq r_{X,G}(A) + \frac{1}{k} - r_{X,G}(A)\delta(r_{X,G}(A) + 1, d_G(x_n, x_l)). \end{aligned}$$

Here, the latter two estimates come again from property (3) of δ. This implies that $\delta(r_{X,G}(A) + 1, d_G(x_n, x_l)) \leq \frac{1}{k} - r_{X,G}(A)$ for any $n, l \geq n_k$. Letting $k \to \infty$ and using properties (1) and (2) of δ, we see that $d_G(x_n, x_l) \to 0$ as $n, l \to \infty$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there is the limit $x_0 = \lim_{n\to\infty} x_n$ in X. By $r_{x_0,G}(A) = \lim_{n\to\infty} r_{x_n,G}(A) = r_{X,G}(A)$, we find that x_0 is a circumcenter of A. The uniqueness is already seen from the above argument. Or, if x and y are circumcenters of A, then the substitution of $r_{x,G}(A) = r_{X,G}(A)$ and $r_{y,G}(A) = r_{X,G}(A)$ to (3) gives $\delta(r_{X,G}(A), d_G(x, y)) = 0$. This is possible only when $x = y$. □

The second one is concerning graphical normal structure and compactness of the family of graphical admissible subsets. For the graphical normal structure, it is sufficient to modify the proof of Theorem 5.1. For the (countable) compactness, we refer to [15, Theorem 2.2] in the case of geodesic metric spaces.

Theorem 5.2. *If a metric space (X, d) is graphical uniformly convex in the wider sense, then the family $\mathcal{A}_G(X)$ of admissible subsets has graphical normal structure. If (X, d_G) is complete in addition, then $\mathcal{A}_G(X)$ is compact.*

Proof. Take an arbitrary $A \in \mathcal{A}_G(X)$ with $d = \mathrm{diam}_G((A)) > 0$. Then there are $x, y \in A$such that $d_G(x, y) \geq d - \epsilon$ for an arbitrary $\epsilon > 0$. For these x and y, there is $m \in X$that satisfies the inequality for uniform convexity in the wider sense. We will check that m belongs to A. Suppose that A is the intersection of closed metric balls $B_G(z_i, r_i)$ for all indices $i \in I$. Since $x, y \in A \subset B_G(z_i, r_i)$, we have $d_G(z_i, x) \leq r_i$ and $d_G(z_i, y) \leq r_i$ for each $i \in I$. It follows from the inequality that $d_G(z_i, m) \leq \max\{d_G(z_i, x), d_G(zi, y)\} \leq r_i$. This implies that $m \in B(z_i, r_i)$ and hence $m \in A$. Consider an arbitrary $z \in A$. Then $d_G(z, x) \leq d$ and $d_G(z, y) \leq d$. Substituting these bounds and $d_G(x, y) \geq d - \epsilon > 0$ to the inequality and using the properties of δ, we obtain

$$d_G(z, m) \leq d_G(1 - \delta(d, d - \epsilon)) < d.$$

Thus we have $\mathrm{rad}_G(A) < \mathrm{diam}_G((A))$.

Now we assume that (X, d_G) is complete and consider a decreasing sequence of admissible subsets $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_G(X)$. For a fixed point $z \in X \setminus A_1$, the distances $d_G(z, A_n)$ from z to A_n are bounded and increasing, so we have $R = \lim_{n\to\infty} d_G(z, A_n) \in (0, \infty)$. Also, we can choose a point $x_n \in A_n$ for each $n \in \mathbb{N}$ such that $\lim_{n\to\infty} d_G(z, x_n) = R$. We will show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Then there is the limit point $x_\infty \in X$ of $\{x_n\}_{n\in\mathbb{N}}$ since X is complete. Each A_n contains x_∞ because $x_{n_l} \in A_n$ for every $n_l \geq n$. Thus $x_\infty \in \cap_{n\in\mathbb{N}} A_n$, which shows that the intersection is not empty. Suppose to the contrary that $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Then there is some $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there are $n_1, n_2 \geq n$ with $d_G(x_{n_1}, x_{n_2}) \geq \epsilon$. We apply the inequality of graphical uniform convexity in the wider sense to x_{n_1} and x_{n_2} ; there is some $m_n \in X$ such that $d_G(z, m_n) \leq \max\{d_G(z, x_{n_1}), d_G(z, x_{n_2})\}(1 - \delta(\max\{d_G(z, x_{n_1}), d_G(z, x_{n_2})\}, d_G(x_{n_1}, x_{n_2})))$ for every $z \in X$. Let z be a fixed element satisfying the above inequality. Then we obtain $d_G(z, m_n) \leq R(1 - \delta(R, \epsilon)) < R$. Here m_n is contained in A_n by the same reason as in the first paragraph. Hence $d_G(z, A_n) \leq d_G(z, m_n)$. Taking the limit as $n \to \infty$, we have a contradiction. This proves that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. We have shown that any decreasing sequence of admissible subsets $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_G(X)$ as non-empty intersection. This property is called countably compact. By the similar arguments as in [18] (see also [16, Theorem 5.5]) we conclude that if $\mathcal{A}_G(X)$ has graphical normal structure then compactness and countable compactness of $\mathcal{A}_G(X)$ are equivalent. □

If we corporate this theorem with Theorem 5.1, then we conclude that a complete metric space that is uniformly convex in the wider sense has the fixed point property.

Remark 5.1. *We can modify the definition of uniformly convexity in the wider sense by changing the modulus of convexity* δ *to an increasing function* $\tilde{\delta} : [0, 2] \to [0, 1] (\tilde{\delta}(s) = 0 \iff s = 0)$ *and the inequality to*

$$d_G(z, m) \leq \max\{d_G(z, x), d_G(z, y)\} \left(1 - \tilde{\delta}\left(\frac{d_G(x, y)}{\max\{d_G(z, x), d_G(z, y)\}}\right)\right).$$

This condition is stronger than the previous one because $\delta(r, t) = \tilde{\delta}\left(\frac{t}{r}\right)$ *gives the implication. The above proof of Theorem 5.2 shows that if* X *is uniformly convex in this sense then* $\mathcal{A}_G(X)$ *has uniformly normal structure with the property* $rad_G(A) \leq (1 - \alpha)diam_G((A))$ *for* $\alpha = \lim_{s\to 0^+} \tilde{\delta}(s)$. *Note that if* (X, d) *is an ultrametric space where* $d_G(x, y) \leq \max\{d_G(z, x), d_G(z, y)\}$ *is always satisfied, then the modulus of convexity is uniformly bounded by* $\tilde{\delta}(1)$.

6. **Concluding Remarks**

In this paper we study the fixed point property in a graphical metric space (X, d_G). We say that a complete graphical metric space (X, d_G) has the fixed point property if every group of isometric automorphisms of (X, d_G) with a bounded orbit has a fixed point in X. We prove that if (X, d_G) is graphically uniformly convex then the family of admissible subsets of (X, d_G) possesses uniformly normal structure and if so then it has the fixed point property. We also show that from other weaker assumptions than uniform convexity, the fixed point property follows. Our formulation of graphical uniform convexity and its generalization can be applied not only to geodesic metric spaces.

REFERENCES

[1] *N. T. An*, Robust feature selection via nonconvex sparsity-based methods, J. Nonlinear Var. Anal., **5**(2021), 59-77.

[2] *S. Bae and M. S. Park*, Fixed point theorems in metric spaces with uniform normal structure, J. Korean Math. Soc., **30**(1993), 51-62.

[3] *W. Ballmann*, Lectures on Spaces of Nonpositive Curvature, DMV-Semin., vol. 25, Birkhuser, 1995.

[4] *K. Brown*, Buildings, Springer Monogr. Math., Springer, 1988.

[5] *F. Bruhat and, J. Tits*, Groupes réductifs sur un corps local. I. Données radicielles valuées, Publ. Math. Inst. Hautes etudes Sci., **41**(1972), 5-251.

[6] *L. C. Ceng, C. F. Wen, J. C. Yao and Y. Yao*, Hybrid viscosity approximation methods for systems of variational inequalities and hierarchical fixed point problems, Filomat, **34**(2020), 1927-1947.

[7] *E. Cohen, S. Sabach and M. Teboulle*, Non-Euclidean proximal methods for convex-concave saddle-point problems, J. Appl. Numer. Optim., **3**(2021), 43-60.

[8] *P. Cubiotti and J. C. Yao*, On the Cauchy problem for a class of differential inclusions with applications, Appl. Anal., **99**(2020), 2543-2554.

[9] *V. Dadashi and M. Postolache*, Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators, Arab. J. Math., **9**(2020), 89-99.

[10] *Q. L. Dong, L. Liu and Y. Yao*, Self-adaptive projection and contraction methods with alternated inertial terms for solving the split feasibility problem, J. Nonlinear Convex Anal., **23(3)**(2022), 591-605.

[11] *Q. L. Dong, Y. Peng and Y. Yao*, Alternated inertial projection methods for the split equality problem, J. Nonlinear Convex Anal., **22**(2021), 53-67.

[12] *S. He, Z. Wang, Q. Dong, Y. Yao and Y. Tang*, The dual randomized Kaczmarz algorithm, J. Nonlinear Convex Anal. in press.

[13] *J. Jachymski*, The contraction principle formappings on ametric space with a graph, Proc. Amer. Math. Soc., **136**(2008), 1359-1373.

[14] *M. A. Khamsi*, On metric spaces with uniform normal structure, Proc. Am. Math. Soc., **106**(1989), 723-726.

[15] *M. A. Khamsi and A. R. Khan*, Inequalities in metric spaces with applications, Nonlinear Anal., **74**(2011), 4036-4045.

[16] *M. A. Khamsi and W. A. Kirk*, An Introduction to Metric Spaces and Fixed Point Theory, Pure Appl. Math., Wiley and Sons, 2001.

[17] *W. A. Kirk*, A fixed point theorem for mappings which do not increase distances, Am. Math. Mon., **72**(1965), 1004-1006.

[18] *J. Kulesza and T.C. Lim*, On weak compactness and countable weak compactness in fixed point theory, Proc. Amer. Math. Soc., **124**(1996), 3345-3349.

[19] *L. Liu, S. Y. Cho and J. C. Yao*, Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications, J. Nonlinear Var. Anal., **5**(2021), 627-644.

[20] *L. Liu and X. Qin*, Strong convergence theorems for solving pseudomonotone variational inequality problems and applications, Optim., doi:10.1080/02331934.2021.190564.

[21] *K. Matsuzaki*, Uniform convexity, normal structure and the fixed point property of metric spaces, Topol. Appl., **196**(2015), 684-695.

[22] *A. Moudafi*, Byrne's extended CQ-algorithms in the light of Moreau-Yosida regularization, Appl. Set-Valued Anal. Optim., **3**(2021), 21-26.

[23] *A. Naor and L. Silberman*, Poincare inequalities, embeddings, and wild groups, Compos. Math., **147**(2011), 1546-1572.

[24] *L. V. Nguyen and X. Qin*, The minimal time function associated with a collection of sets, ESAIM Control Optim. Calc. Var., **26**(2020), 93.

[25] *S. Ohta*, Convexities of metric spaces, Geom. Dedic., **125**(2007), 225-250.

[26] *M. A. Olona, T. O. Alakoya and O.T. Mewomo*, Inertial algorithm for solving equilibrium, variational inclusion and fixed point problems for an infinite family of strict pseudocontractive mappings, J. Nonlinear Funct. Anal.,**2021**(2021), 10.

[27] *J. P. Penot*, Fixed point theorem without convexity, Bull. Soc. Math. Fr., Mem., **60**(1979), 129-152.

[28] *A. Petrusel, G. Petrusel and B. Samet*, Coupled fixed point theorems for symmetric contarctions in b-metric spaces with applications to operators equation systems, Proc. Amer. Math. Soc., **17(2)**(2006), 457-476.

[29] *A. Petrusel, G. Petrusel and J. C. Yao*, A study of a system of operator inclusions via fixed point theory approach and applications to functional-differential inclusions, Carpath J. Math., **32(3)**(2006), 349-361.

[30] *A. Ran and M. A. Reurings*, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., **132(5)**(2004), 1435-1443.

[31] *D. R. Sahu, A. Pitea and M. Verma*, A new iteration technique for nonlinear ope-rators as concerns convex programming and feasibility problems, Numer. Algor., **83(2)**(2020), 421-449.

[32] *S. S. Santra, O. Bazighifan and M. Postolache*, New conditions for the oscillation of second-order differential equations with sublinear neutral terms, Mathematics, **9(11)**(2021), Article Number 1159.

[33] *S. Shukla and V. C. RadenoviS*, Graphical metric space: a generalized setting in fixed point theory. RACSM, **111(3)**(2017), 641-655.

[34] *W. Sintunavarat and A. Pitea*, On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis, J. Nonlinear Sci. Appl., **9** (2016), 2553-2562.

[35] *W. Takahashi*, A convexity in metric space and nonexpansive mappings, I. Kodai Math Seminar Rep., **22(2)**(1970), 142-149.

[36] *B. S. Thakur, D. Thakur and M. Postolache*, A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, Appl. Math. Comput., **275**(2016), 147-155.

[37] *B. S. Thakur, D. Thakur and M. Postolache*, A new iteration scheme for approximating fixed points of nonexpansive mappings, Filomat, **30**(2016), 2711-2720.

[38] *D. Thakur, B. S. Thakur and M. Postolache*, New iteration scheme for numerical reckoning fixed points of nonexpansive mappings, J. Inequal. Appl., **2014**(2014), Art. No. 328.

[39] *G. I. Usurelu*, Split feasibility handled by a single-projection three-step iteration with comparative analysis, J. Nonlinear Convex Anal. **22(3)**(2021), 544-558.

[40] *G. I. Usurelu, A. Bejenaru and M. Postolache*, Newton-like methods and polynomiographic visualization of modified Thakur processes, Int. J. Comput. Math., **98(5)**(2021), 1049-1068.

[41] *G. I. Usurelu and M. Postolache*, Algorithm for generalized hybrid operators with numerical analysis and applications, J. Nonlinear Var. Anal., in press.

[42] *M. Younis, D. Singh and A. Goyal*, A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable, J. Fixed Point Theory Appl., DOI:10.1007/s11784-020-00831-3.

[43] *Y. Yao, Olaniyi S. Iyiola and Y. Shehu*, Subgradient extragradient method with double inertial steps for variational inequalities, J. Sci. Comput., **90**(2022), Article number 71.

[44] *Y. Yao, L. Leng, M. Postolache and X. Zheng*, Mann-type iteration method for solving the split common fixed point problem, J. Nonlinear Convex Anal., **18**(2017), 875-882.

[45] *Y. Yao, H. Li and M. Postolache*, Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions, Optim., in press, DOI: 10.1080/02331934.2020.1857757.

[46] *Y. Yao, Y. C. Liou and M. Postolache*, Self-adaptive algorithms for the split problem of the demicontractive operators, Optim., **67**(2018), 1309-1319.

[47] *Y. Yao, Y. C. Liou and J. C. Yao*, Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction, Fixed Point Theory Appl., **2015**(2015), Art. No. 127.

[48] *Y. Yao, Y. C. Liou and J. C. Yao*, Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations, J. Nonlinear Sci. Appl., **10**(2017), 843-854.

[49] *Y. Yao, M. Postolache and Y. C. Liou*, Strong convergence of a self-adaptive method for the split feasibility problem, Fixed Point Theory Appl., **2013**(2013), Art. No. 201.

[50] *Y. Yao, M. Postolache, Y. C. Liou and Z. Yao*, Construction algorithms for a class of monotone variational inequalities, Optim. Lett., **10**(2016), 1519-1528.

[51] *Y. Yao, M. Postolache and J. C. Yao*, Iterative algorithms for the generalized variational inequalities, U.P.B. Sci. Bull., Series A, **81**(2019), 3-16.

[52] *Y. Yao, M. Postolache and J. C. Yao*, An iterative algorithm for solving the generalized variational inequalities and fixed points problems, Mathematics, **7**(2019), Art. No. 61.

[53] *Y. Yao, M. Postolache and J. C. Yao*, Strong convergence of an extragradient algorithm for variational inequality and fixed point problems, U.P.B. Sci. Bull., Series A, **82(1)**(2020), 3-12.

[54] *Y. Yao, M. Postolache and Z. Zhu*, Gradient methods with selection technique for the multiple-sets split feasibility problem, Optim., **69**(2020), 269-281.

[55] *Y. Yao, X. Qin and J. C. Yao*, Projection methods for firmly type nonexpansive operators, J. Nonlinear Convex Anal., **19**(2018), 407-415.

[56] *Y. Yao, N. Shahzad and J. C. Yao*, Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems, Carpathian J. Math., **37**(2021), 541-550.

[57] *Y. Yao, J. C. Yao, Y. C. Liou and M. Postolache*, Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms, Carpathian J. Math., **34**(2018), 459-466.

[58] *Y. Yao, J. C. Yao and M. Postolache*, An iterate for solving quasi-variational inclusions and nonmonotone equilibrium problems, J. Nonlinear Convex Anal., in press.

[59] *S. Zeng, S. Migorski, Z. Liu and J. C. Yao*, Convergence of a generalized penalty method for variational-hemivariational inequalities, Commun. Nonlinear Sci. Numer. Simul., **92**(2021), 105476.

[60] *C. Zhang, Z. Zhu, Y. Yao and Q. Liu*, Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities, Optim., **68**(2019), 2293-2312.

[61] *X. Zhao, M. A. Kobis, Y. Yao and J. C. Yao*, A projected subgradient method for nondifferentiable quasiconvex multiobjective optimization problem, J. Optim. Theory Appl., **190**(2021), 82-107.

[62] *X. Zhao, J. C. Yao and Y. Yao*, A proximal algorithm for solving split monotone variational inclusions, U.P.B. Sci. Bull., Series A, **82(3)**(2020), 43-52.

[63] *X. Zhao and Y. Yao*, Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems, Optim., **69**(2020), 1987-2002.

[64] *L. J. Zhu, Y. Yao and M. Postolache*, Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems, U.P.B. Sci. Bull., Series A, **83**(2021), 3-14.