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CONSTRUCTION AND ANALYSIS OF ITERATIVE METHODS FOR SOLVING FIXED POINTS AND PSEUDOMONOTONE VARIATIONAL INEQUALITIES

Tzu-Chien Yin[1], Naseer Shahzad[2]

Fixed point problems and pseudomonotone variational inequalities have been studied extensively. An additional assumption "weak sequential continuity" imposed on pseudomonotone operators is used. This paper devotes to construct an iterative algorithm for finding a common point of fixed point problems and pseudomonotone variational inequalities under a weaker assumption than weak sequential continuity imposed on pseudomonotone operators. Strong convergence result of the proposed algorithm is shown.

Keywords: Fixed point, variational inequality, pseudomonotone operator, pseudocontractive operators, projection.

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1. Introduction

Let H be a real Hilbert space endowed with inner product and induced norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\emptyset \neq C \subset H$ be a closed and convex set.

Let $\varphi : C \to C$ be a nonlinear operator. Recall that the classical variational inequality is to seek a point $u^\dagger \in C$ such that

$$\langle \varphi(u^\dagger), u - u^\dagger \rangle \geq 0, \ \forall u \in C. \tag{1}$$

The solution set of variational inequality (1) is denoted by $VI(C, \varphi)$.

Variational inequality theory was introduced by Stampacchia ([19]) as a tool for the study of partial differential equations with applications principally drawn from mechanics. Such variational inequality unveiled its methodology for the study of problems in econom-ics, operations research and engineering, see [[1, 12, 13, 30, 33, 41]. Variational inequality theory provides us with algorithms with accompanying convergence analysis for computational purposes. It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems ([32, 37, 45]), complementarity prob-lems and fixed point problems ([8, 20, 25, 26, 28, 35, 39, 40]). For more information, the reader can refer to [4, 7, 21, 22, 27, 31, 36, 42].

One of the most important algorithms for solving VI (1) is projection algorithm ([1, 10, 11]) which generates a sequence $\{x_n\}$ by the following rule

$$x_{n+1} = P_C[x_n - \tau_n \varphi(x_n)], \ n \geq 0, \tag{2}$$

where $P_C : H \to C$ is the Orthogonal projection and $\{\tau_n\}$ is a candidated stepsize sequence.

[1]Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, e-mail: yintzuchien@mail.cmuh.org.tw

[2]Corresponding author, Department of Mathematics, King Abdulaziz University, P. O. B. 80203, Jeddah 21589, Saudi Arabia, e-mail: nshahzad@kau.edu.sa

In algorithm (2), operator φ must be Lipschitz continuous and strongly monotone or inverse strongly monotone. To weaken these assumptions, Korpelevich [19] proposed extragradient method. Extragradient algorithm and its variant form have been studied for solving monotone variational inequalities. Please refer to [1, 12, 13, 30, 33, 41]. Especially, Ceng, Teboulle and Yao [5] demonstrated the convergence analysis of extragradient algorithm for solving the pseudomonotone variational inequality and fixed point problems. In order to achieve the weak convergence result, in [5], an additional condition "sequentially weak-to-strong continuity" was imposed on pseudomonotone operator φ. However, this additional hypothesis is not satisfied even for the identity operator. Subsequently, Vuong [23] weaken this hypothesis imposed on φ to a weaker condition "sequentially weak-to-weak continuity".

On the other hand, in order to solve variational inequality (1), the Lipschitz constant of φ may be difficult to estimate, even if the underlying mapping is linear. For solving this difficulty, some self-adaptive methods for solving variational inequality problems have been developed. The advantage of self-adaptive method lies in the fact that prior information on Lipschitz constant of φ is not required, and convergence is still guaranteed, see [14, 15, 16, 18].

Motivated by the work in this field, the purpose of this paper is to investigate the problem of fixed point of pseudocontractive operator and pseudomonotone variational inequality. We construct an iterative algorithm for finding a common point of fixed point problems and pseudomonotone variational inequalities under a weaker assumption than weak sequential continuity imposed on φ. Strong convergence result of the proposed algorithm is shown.

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space H. $\rightharpoonup$ means the weak convergence and $\to$ means the strong convergence. Use $\omega_w(u_n)$ to denote the set of all weak cluster points of the sequence $\{u_n\}$, i.e., $\omega_w(u_n) = \{u^\dagger : \exists\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup u^\dagger$ as $i \to \infty\}$.

A bounded linear operator Φ is said to be $\hat{\kappa}$-strongly positive on H if there exists a constant $\hat{\kappa} > 0$ such that

$$\langle \Phi(x), x\rangle \geq \hat{\kappa}\|x\|^2, \ \forall x \in H.$$

An operator $\phi : C \to C$ is said to be

(i) pseudocontractive if

$$\|\phi(u) - \phi(u^\dagger)\|^2 \leq \|u - u^\dagger\|^2 + \|(I - \phi)u - (I - \phi)u^\dagger\|^2, \ \forall u, u^\dagger \in C.$$

(ii) β-Lipschitz if there exists a constant $\beta \geq 0$ such that

$$\|\phi(u) - \phi(u^\dagger)\| \leq \beta\|u - u^\dagger\|, \ \forall u, u^\dagger \in C.$$

If $\beta < 1$, then ϕ is said to be β-contractive.

Use $Fix(\phi)$ to mean the set of fixed points of ϕ.

An operator φ is said to be

(i) monotone on C if

$$\langle \varphi(u) - \varphi(u^\dagger), u - u^\dagger\rangle \geq 0, \ \forall u, u^\dagger \in C.$$

(ii) pseudomonotone on H if

$$\langle \varphi(\tilde{u}), u - \tilde{u}\rangle \geq 0 \Rightarrow \langle \varphi(u), u - \tilde{u}\rangle \geq 0, \ \forall u, \tilde{u} \in H.$$

(iii) weakly sequentially continuous, if for given sequence $\{x_n\} \subset C$ satisfying $x_n \rightharpoonup \tilde{x}$, we conclude that $\varphi(x_n) \rightharpoonup \varphi(\tilde{x})$.

For given $u^\dagger \in H$, there exists a unique point in C, denoted by $P_C[u^\dagger]$ such that

$$\|u^\dagger - P_C[u^\dagger]\| \leq \|x - u^\dagger\|, \ \forall x \in C.$$

It is known that P_C is firmly nonexpansive, that is, P_C satisfies

$$\|P_C[q^*] - P_C[q^\dagger]\|^2 \leq \langle P_C[q^*] - P_C[q^\dagger], q^* - q^\dagger \rangle, \ \forall q^*, q^\dagger \in H.$$

Moreover, P_C satisfies the following inequality

$$\langle q^* - P_C[q^*], q^\dagger - P_C[q^*] \rangle \leq 0, \ \forall q^* \in H, q^\dagger \in C. \tag{3}$$

Lemma 2.1 ([?]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \to C$ be a β_2-Lipschitz pseudocontractive operator. Let γ be a constant in $(0, \frac{1}{\sqrt{1+\beta_2^2}+1})$. Then,*

$$\|\phi[(1-\gamma)x + \gamma\phi(x)] - \hat{p}\|^2 \leq \|x - \hat{p}\|^2 + (1-\gamma)\|\phi[(1-\gamma)x + \gamma\phi(x)] - x\|^2,$$

for all $x \in C$ and $\hat{p} \in Fix(\phi)$.

Lemma 2.2 ([?]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \to C$ be a continuous pseudocontractive operator. Then ϕ is demi-closed, namely,*

$$\left.\begin{array}{c} \{u_n\}_{n=0}^\infty \subset C \\ u_n \rightharpoonup \tilde{u} \in C \\ \phi(u_n) \to u^\dagger \end{array}\right\} \Rightarrow \phi(\tilde{u}) = u^\dagger.$$

Lemma 2.3 ([?]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let φ be a continuous and pseudomonotone operator on H. Then $x^\dagger \in VI(C, \varphi)$ iff $x^\dagger$ solves the following variational inequality*

$$\langle \varphi(p^\dagger), p^\dagger - x^\dagger \rangle \geq 0, \ \forall p^\dagger \in C.$$

Lemma 2.4 ([?]). *Let $\{z_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\}$ be three real number sequences. If $z_{n+1} \leq (1-\alpha_n)z_n + s_n, \forall n \geq 0$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\limsup_{n\to\infty} s_n/\alpha_n \leq 0$ or $\sum_{n=1}^\infty |s_n| < \infty$, then $\lim_{n\to\infty} z_n = 0$.*

3. Main results

In this section, we first construct an iterative algorithm for finding a common point of fixed point of pseudocontractive operator ϕ and a solution of pseudomonotone variational inequality (1). Subsequently, we give the convergence analysis of the proposed algorithm.

Let C be a nonempty convex closed subset of a real Hilbert space H. Let $\psi : C \to C$ be a ρ-contractive operator. Let Φ be a $\hat{\kappa}$-strongly positive bounded linear operator on H. Let the operator φ be pseudomonotone on H and β_1-Lipschitz continuous on C. Let $\phi : C \to C$ be a β_2-Lipschitz pseudocontractive operator. Suppose that the operator φ possesses the property (WSC):

$$\left.\begin{array}{c} \{u_n\}_{n=0}^\infty \subset H \\ u_n \rightharpoonup u \in H \\ \liminf_{n\to\infty} \|\varphi(u_n)\| = 0 \end{array}\right\} \Rightarrow \varphi(u) = 0.$$

Remark 3.1. *It is obviously that if φ is sequentially weakly continuous, then φ satisfies the above property (WSC).*

Let $\{\vartheta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ be three real number sequences in $(0, 1)$. Let κ, ς, ϖ, ζ and β be five constants. Suppose that these iterative parameters satisfy the following conditions:

(C1): $\beta_1 > 0$, $\beta_2 > 1$ and $0 < \underline{\vartheta} < \vartheta_n < \overline{\vartheta} < \gamma_n < \overline{\gamma} < \frac{1}{\sqrt{1+\beta_2^2}+1} (\forall n \geq 0)$;

(C2): $\hat{\kappa} > 0$, $\rho \in (0, 1)$, $\kappa \in (0, 1)$, $\varsigma \in (0, 1)$, $\varpi \in (0, 1)$, $\zeta \in (0, 2)$ and $\beta\rho < \hat{\kappa}$;

(C3): $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

In the sequel, assume that $\Omega := Fix(\phi)\cap VI(C,\varphi) \neq \emptyset$. In this position, we state our algorithm below.

Algorithm 3.1. *Take any initial value $x_0 \in C$ and set $n = 0$.*

Step 1. Assume that x_n is known and calculate

$$v_n = (1-\vartheta_n)x_n + \vartheta_n\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)]. \tag{4}$$

Step 2. Finding the smallest nonnegative integer $int(v_n)$ such that

$$u_n = P_C[v_n - \kappa\varsigma^{int(v_n)}\varphi(v_n)], \tag{5}$$

and

$$\kappa\varsigma^{int(v_n)}\|\varphi(u_n)-\varphi(v_n)\| \leq \varpi\|u_n - v_n\|. \tag{6}$$

If $u_n = v_n$, then set $y_n = v_n$ and go to Step 3. Otherwise, calculate

$$y_n = P_C\Big[v_n + \zeta(1-\varpi)\|u_n - v_n\|^2\frac{u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2}\Big]. \tag{7}$$

Step 3. Calculate

$$x_{n+1} = P_C[\alpha_n\beta\psi(x_n) + (I-\alpha_n\Phi)y_n]. \tag{8}$$

Step 4. Set $n := n+1$ and return to Step 1.

Remark 3.2. *(i) There exits $int(v_n)$ such that (5) and (6) are satisfied. (ii) $0 < \frac{\varsigma\varpi}{\kappa\beta_1} < \varsigma^{int(v_n)} \leq 1 (n \geq 0)$. (iii) If $v_n = P_C[v_n - \kappa\varsigma^{int(v_n)}\varphi(v_n)]$, then $v_n \in VI(C,\varphi)$.*

Proposition 3.1. *$u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n) \neq 0 (\forall n \geq 0)$ and furthermore, for any $\hat{p} \in VI(C,\varphi)$,*

$$\langle u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n), v_n - \hat{p}\rangle \leq -(1-\varpi)\|u_n - v_n\|^2 < 0. \tag{9}$$

Proof. From the property (3) of projection P_C and (5), we have

$$\langle u_n - v_n + \kappa\varsigma^{int(v_n)}\varphi(v_n), u_n - \hat{p}\rangle \leq 0. \tag{10}$$

Note that

$$\begin{aligned}
&\langle u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n), v_n - \hat{p}\rangle \\
&\quad= \langle u_n - v_n + \kappa\varsigma^{int(v_n)}\varphi(v_n), v_n - \hat{p}\rangle - \kappa\varsigma^{int(v_n)}\langle\varphi(v_n), v_n - \hat{p}\rangle \\
&\qquad - \kappa\varsigma^{int(v_n)}\langle\varphi(u_n), v_n - u_n\rangle - \kappa\varsigma^{int(v_n)}\langle\varphi(u_n), u_n - \hat{p}\rangle \\
&\quad= \langle u_n - v_n + \kappa\varsigma^{int(v_n)}(\varphi(v_n) - \varphi(u_n)), v_n - u_n\rangle - \kappa\varsigma^{int(v_n)}\langle\varphi(v_n), v_n - \hat{p}\rangle \\
&\qquad + \langle u_n - v_n + \kappa\varsigma^{int(v_n)}\varphi(v_n), u_n - \hat{p}\rangle - \kappa\varsigma^{int(v_n)}\langle\varphi(u_n), u_n - \hat{p}\rangle.
\end{aligned} \tag{11}$$

Since $\langle\varphi(\hat{p}), v_n - \hat{p}\rangle \geq 0$ and $\langle\varphi(\hat{p}), u_n - \hat{p}\rangle \geq 0$, it follows from the pseudomonotonicity of φ that $\langle\varphi(v_n), v_n - \hat{p}\rangle \geq 0$ and $\langle\varphi(u_n), u_n - \hat{p}\rangle \geq 0$. This together with (10) and (11) implies that

$$\begin{aligned}
\langle u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n), v_n - \hat{p}\rangle &\leq \langle u_n - v_n + \kappa\varsigma^{int(v_n)}(\varphi(v_n) - \varphi(u_n)), v_n - u_n\rangle \\
&\leq -\|u_n - v_n\|^2 + \kappa\varsigma^{int(v_n)}\|\varphi(v_n) - \varphi(u_n)\|\|v_n - u_n\|.
\end{aligned} \tag{12}$$

By (6) and (12), we can obtain the desired result (9). □

Next, we prove the convergence of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $q^\dagger = P_\Omega(I - \Phi + \beta\psi)q^\dagger$.*

Proof. Let $\hat{p} \in \Omega$. It is well known that in any real Hilbert space H, for any $u, v \in H$ and $\vartheta \in R$, we have the following equality

$$\|\vartheta u + (1-\vartheta)v\|^2 = \vartheta\|u\|^2 + (1-\vartheta)\|v\|^2 - \vartheta(1-\vartheta)\|u-v\|^2. \tag{13}$$

With the help of above equality (13), by (4), we obtain

$$\begin{aligned}\|v_n - \hat{p}\|^2 &= \|(1-\vartheta_n)(x_n - \hat{p}) + \vartheta_n(\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - \hat{p})\|^2 \\ &= (1-\vartheta_n)\|x_n - \hat{p}\|^2 + \vartheta_n\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - \hat{p}\|^2 \\ &\quad - \vartheta_n(1-\vartheta_n)\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - x_n\|^2.\end{aligned} \tag{14}$$

According to Lemma 2.1, we have

$$\begin{aligned}\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - \hat{p}\|^2 &\le (1-\gamma_n)\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - x_n\|^2 \\ &\quad + \|x_n - \hat{p}\|^2.\end{aligned} \tag{15}$$

It follows from (14) and (15) that

$$\begin{aligned}\|v_n - \hat{p}\|^2 &\le (1-\vartheta_n)\|x_n - \hat{p}\|^2 - \vartheta_n(1-\vartheta_n)\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - x_n\|^2 \\ &\quad + \vartheta_n(\|x_n - \hat{p}\|^2 + (1-\gamma_n)\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - x_n\|^2) \\ &= \|x_n - \hat{p}\|^2 + \vartheta_n(\vartheta_n - \gamma_n)\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - x_n\|^2.\end{aligned} \tag{16}$$

Since P_C is nonexpansive, from (7), one has

$$\begin{aligned}\|y_n - \hat{p}\|^2 &= \left\|P_C\Big[v_n + \zeta(1-\varpi)\|u_n - v_n\|^2 \frac{u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2}\Big] - \hat{p}\right\|^2 \\ &\le \left\|v_n - \hat{p} + \zeta(1-\varpi)\|u_n - v_n\|^2 \frac{u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2}\right\|^2 \\ &= \|v_n - \hat{p}\|^2 + \frac{2\zeta(1-\varpi)\|u_n - v_n\|^2}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2}\langle u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n), v_n - \hat{p}\rangle \\ &\quad + \frac{\zeta^2(1-\varpi)^2\|u_n - v_n\|^4}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2},\end{aligned}$$

which together with (9) and (16) implies that

$$\begin{aligned}\|y_n - \hat{p}\|^2 &\le \|v_n - \hat{p}\|^2 + \frac{\zeta^2(1-\varpi)^2\|u_n - v_n\|^4}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2} - \frac{2\zeta(1-\varpi)^2\|u_n - v_n\|^4}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2} \\ &= \|v_n - \hat{p}\|^2 - \frac{(2-\zeta)\zeta(1-\varpi)^2\|u_n - v_n\|^4}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2} \\ &\le \|x_n - \hat{p}\|^2 + \vartheta_n(\vartheta_n - \gamma_n)\|\phi[(1-\gamma_n)x_n + \gamma_n\phi(x_n)] - x_n\|^2 \\ &\quad - \frac{(2-\zeta)\zeta(1-\varpi)^2\|u_n - v_n\|^4}{\|u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2} \\ &\le \|x_n - \hat{p}\|^2.\end{aligned} \tag{17}$$

Since Φ is bounded linear, we have

$$\begin{aligned}\|1 - \alpha_n\Phi\| &= \sup\{\langle(1-\alpha_n\Phi)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \alpha_n\langle\Phi(u), u\rangle : u \in H, \|u\| = 1\}.\end{aligned}$$

This together with the strong positivity of Φ implies that

$$\|1 - \alpha_n\Phi\| \le 1 - \alpha_n\hat{\kappa}.$$

From (8) and (17), we get

$$\begin{aligned}\|x_{n+1}-\hat{p}\| &= \|P_C[\alpha_n\beta\psi(x_n)+(I-\alpha_n\Phi)y_n]-\hat{p}\| \\ &\le (I-\alpha_n\Phi)\|y_n-\hat{p}\| + \alpha_n\beta\|\psi(x_n)-\psi(\hat{p})\| + \alpha_n\|\beta\psi(\hat{p})-\Phi(\hat{p})\| \\ &\le (1-\alpha_n\hat{\kappa})\|x_n-\hat{p}\| + \alpha_n\beta\rho\|x_n-\hat{p}\| + \alpha_n\|\beta\psi(\hat{p})-\Phi(\hat{p})\| \\ &= [1-(\hat{\kappa}-\beta\rho)\alpha_n]\|x_n-\hat{p}\| + \alpha_n\|\beta\psi(\hat{p})-\Phi(\hat{p})\|.\end{aligned}$$

Subsequently, by induction, we get that $\|x_n-\hat{p}\| \le \max\{\frac{\|\beta\psi(\hat{p})-\Phi(\hat{p})\|}{\hat{\kappa}-\beta\rho}, \|x_0-\hat{p}\|\}$. Thus, the sequences $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ are bounded.

By virtue of (3) and (8), we achieve

$$\begin{aligned}\|x_{n+1}-\hat{p}\|^2 &= \|P_C[\alpha_n\beta\psi(x_n)+(I-\alpha_n\Phi)y_n]-P_C[\hat{p}]\|^2 \\ &\le \langle \alpha_n\beta\psi(x_n)+(I-\alpha_n\Phi)y_n-\hat{p}, x_{n+1}-\hat{p}\rangle \\ &= \beta\alpha_n\langle\psi(x_n)-\psi(\hat{p}), x_{n+1}-\hat{p}\rangle + \alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \\ &\quad + (I-\alpha_n\Phi)\langle y_n-\hat{p}, x_{n+1}-\hat{p}\rangle \\ &\le \beta\rho\alpha_n\|x_n-\hat{p}\|\|x_{n+1}-\hat{p}\| + \alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \\ &\quad + \|I-\alpha_n\Phi\|\|y_n-\hat{p}\|\|x_{n+1}-\hat{p}\| \\ &\le [\beta\rho\alpha_n\|x_n-\hat{p}\| + (1-\hat{\kappa}\alpha_n)\|y_n-\hat{p}\|)\|x_{n+1}-\hat{p}\| \\ &\quad + \alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \\ &\le \frac{[\beta\rho\alpha_n\|x_n-\hat{p}\| + (1-\hat{\kappa}\alpha_n)\|y_n-\hat{p}\|]^2}{2} + \frac{\|x_{n+1}-\hat{p}\|^2}{2} \\ &\quad + \alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-\hat{p}\|^2 &\le [\hat{\kappa}\alpha_n\frac{\beta\rho}{\hat{\kappa}}\|x_n-\hat{p}\| + (1-\hat{\kappa}\alpha_n)\|y_n-\hat{p}\|]^2 + 2\alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \\ &\le \beta\rho\alpha_n\|x_n-\hat{p}\|^2 + (1-\hat{\kappa}\alpha_n)\|y_n-\hat{p}\|^2 + 2\alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle.\end{aligned} \tag{18}$$

On account of (17) and (18), we have

$$\begin{aligned}\|x_{n+1}-\hat{p}\|^2 &\le [1-(\hat{\kappa}-\beta\rho)\alpha_n]\|x_n-\hat{p}\|^2 + 2\alpha_n\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \\ &\quad + (1-\hat{\kappa}\alpha_n)\vartheta_n(\vartheta_n-\gamma_n)\|\phi[(1-\gamma_n)x_n+\gamma_n\phi(x_n)]-x_n\|^2 \\ &\quad - \frac{(1-\hat{\kappa}\alpha_n)(2-\zeta)\zeta(1-\varpi)^2\|u_n-v_n\|^4}{\|u_n-v_n-\kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2} \\ &\le [1-(\hat{\kappa}-\beta\rho)\alpha_n]\|x_n-\hat{p}\|^2 + (\hat{\kappa}-\beta\rho)\alpha_n\Big\{\frac{(1-\hat{\kappa}\alpha_n)\vartheta_n(\vartheta_n-\gamma_n)}{(\hat{\kappa}-\beta\rho)\alpha_n} \\ &\quad \times\|\phi[(1-\gamma_n)x_n+\gamma_n\phi(x_n)]-x_n\|^2 + \frac{2\alpha_n}{(\hat{\kappa}-\beta\rho)\alpha_n}\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \\ &\quad - \frac{(1-\hat{\kappa}\alpha_n)(2-\zeta)\zeta(1-\varpi)^2}{(\hat{\kappa}-\beta\rho)\alpha_n}\frac{\|u_n-v_n\|^4}{\|u_n-v_n-\kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2}\Big\}.\end{aligned} \tag{19}$$

For any $n \ge 0$, set $z_n = \|x_n-\hat{p}\|^2$ and

$$\begin{aligned}w_n &= \frac{(1-\hat{\kappa}\alpha_n)\vartheta_n(\vartheta_n-\gamma_n)}{(\hat{\kappa}-\beta\rho)\alpha_n}\|\phi[(1-\gamma_n)x_n+\gamma_n\phi(x_n)]-x_n\|^2 \\ &\quad - \frac{(1-\hat{\kappa}\alpha_n)(2-\zeta)\zeta(1-\varpi)^2}{(\hat{\kappa}-\beta\rho)\alpha_n}\frac{\|u_n-v_n\|^4}{\|u_n-v_n-\kappa\varsigma^{int(v_n)}\varphi(u_n)\|^2} \\ &\quad + \frac{2}{\hat{\kappa}-\beta\rho}\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n+1}-\hat{p}\rangle.\end{aligned} \tag{20}$$

Inequality (19) can be rewritten as

$$z_{n+1} \leq [1-(\hat{\kappa}-\beta\rho)\alpha_n]z_n + (\hat{\kappa}-\beta\rho)\alpha_n w_n, \ \forall n \geq 0. \tag{21}$$

Based on (20), we deduce

$$w_n \leq \frac{2}{\hat{\kappa}-\beta\rho}\langle \beta\psi(\hat{p}) - \Phi(\hat{p}), x_{n+1}-\hat{p}\rangle \leq \frac{2}{\hat{\kappa}-\beta\rho}\|\psi(\hat{p}) - \Phi(\hat{p})\|\|x_{n+1}-\hat{p}\|.$$

It follows from the boundedness of $\{x_n\}$ that $\limsup_{n\to\infty} w_n < +\infty$.

Next, we show that $\limsup_{n\to\infty} w_n \geq -1$. If $\limsup_{n\to\infty} w_n < -1$, then there exists m such that $w_n < -1$ when $n \geq m$. Hence, for all $n \geq m$, from (21), we obtain

$$\begin{aligned} z_{n+1} &\leq [1-(\hat{\kappa}-\beta\rho)\alpha_n]z_n - (\hat{\kappa}-\beta\rho)\alpha_n \\ &\leq z_n - (\hat{\kappa}-\beta\rho)\alpha_n. \end{aligned}$$

So,

$$z_{n+1} \leq z_m - (\hat{\kappa}-\beta\rho)\sum_{k=m}^{n}\alpha_k,$$

which implies that

$$\limsup_{n\to\infty} z_n \leq z_m - (\hat{\kappa}-\beta\rho)\limsup_{n\to\infty}\sum_{k=m}^{n}\alpha_k = -\infty.$$

This yields a contradiction. Therefore, $-1 \leq \limsup_{n\to\infty} w_n < +\infty$. At the same time, noting that $\{x_n\}$ is bounded, pick up any $x^\dagger \in \omega_w(x_n)$. We can choose a subsequence $\{n_i\}$ of $\{n\}$ such that $x_{n_i} \rightharpoonup x^\dagger \in C$ and

$$\begin{aligned} \limsup_{n\to\infty} w_n = \lim_{i\to\infty} w_{n_i} = \lim_{i\to\infty}\Big[&\frac{\vartheta_{n_i}(\vartheta_{n_i}-\gamma_{n_i})}{(\hat{\kappa}-\beta\rho)}\frac{\|\phi[(1-\gamma_{n_i})x_{n_i}+\gamma_{n_i}\phi(x_{n_i})]-x_{n_i}\|^2}{\alpha_{n_i}} \\ &- \frac{(2-\zeta)\zeta(1-\varpi)^2}{(\hat{\kappa}-\beta\rho)}\frac{\|u_{n_i}-v_{n_i}\|^4}{\|u_{n_i}-v_{n_i}-\kappa\varsigma^{int(v_{n_i})}\varphi(u_{n_i})\|^2\alpha_{n_i}} \\ &+ \frac{2}{\hat{\kappa}-\beta\rho}\langle\beta\psi(\hat{p})-\Phi(\hat{p}), x_{n_i+1}-\hat{p}\rangle\Big]. \end{aligned} \tag{22}$$

Since $\{x_{n_i+1}\}$ is bounded, without loss of generality, we assume that $\lim_{i\to\infty}\frac{2}{\hat{\kappa}-\beta\rho}\langle\beta\psi(\hat{p}) - \Phi(\hat{p}), x_{n_i+1}-\hat{p}\rangle$ exists. By (22), the limit

$$\begin{aligned} \lim_{i\to\infty}\Big[&\frac{\vartheta_{n_i}(\vartheta_{n_i}-\gamma_{n_i})}{(\hat{\kappa}-\beta\rho)}\frac{\|\phi[(1-\gamma_{n_i})x_{n_i}+\gamma_{n_i}\phi(x_{n_i})]-x_{n_i}\|^2}{\alpha_{n_i}} \\ &- \frac{(2-\zeta)\zeta(1-\varpi)^2}{(\hat{\kappa}-\beta\rho)}\frac{\|u_{n_i}-v_{n_i}\|^4}{\|u_{n_i}-v_{n_i}-\kappa\varsigma^{int(v_{n_i})}\varphi(u_{n_i})\|^2\alpha_{n_i}}\Big] \end{aligned} \tag{23}$$

exists.

It follows from conditions (C1)-(C3) and (23) that

$$\lim_{i\to\infty}\frac{\|u_{n_i}-v_{n_i}\|^4}{\|u_{n_i}-v_{n_i}-\kappa\varsigma^{int(v_{n_i})}\varphi(u_{n_i})\|^2} = 0 \tag{24}$$

and

$$\lim_{i\to\infty}\|\phi[(1-\gamma_{n_i})x_{n_i}+\gamma_{n_i}\phi(x_{n_i})]-x_{n_i}\|^2 = 0. \tag{25}$$

Now, we show $\omega_w(x_n) \subset Fix(\phi)$. Firstly, by the β_2-Lipschit continuity of ϕ, we derive

$$\begin{aligned} \|\phi(x_{n_i})-x_{n_i}\| &\leq \|\phi(x_{n_i}) - \phi[(1-\gamma_{n_i})x_{n_i}+\gamma_{n_i}\phi(x_{n_i})]\| \\ &\quad + \|\phi[(1-\gamma_{n_i})x_{n_i}+\gamma_{n_i}\phi(x_{n_i})]-x_{n_i}\| \\ &\leq \beta_2\gamma_{n_i}\|\phi(x_{n_i})-x_{n_i}\| + \|\phi[(1-\gamma_{n_i})x_{n_i}+\gamma_{n_i}\phi(x_{n_i})]-x_{n_i}\|, \end{aligned}$$

which leads to

$$\|\phi(x_{n_i}) - x_{n_i}\| \leq \frac{1}{1-\beta_2\gamma_{n_i}}\|\phi[(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\phi(x_{n_i})] - x_{n_i}\|.$$

Which together with (25) implies that

$$\lim_{i\to\infty} \|\phi(x_{n_i}) - x_{n_i}\| = 0.$$

At the same time, noting that $x_{n_i} \rightharpoonup x^\dagger$, by Lemma 2.2 and the last equality, we deduce that $x^\dagger \in Fix(\phi)$. That is, $\omega_w(x_n) \subset Fix(\phi)$. Next, we show that $\omega_w(x_n) \subset VI(C, \varphi)$.

In the light of (5), we obtain

$$\|u_n - \hat{p}\| \leq \|v_n - \hat{p}\| + \kappa\varsigma^{int(v_n)}\|\varphi(v_n)\|. \tag{26}$$

Since $\{v_n\}$ is bounded, $\{\varphi(v_n)\}$ is bounded due to the Lipschitz continuity of φ. Taking account of (26), $\{u_n\}$ is bounded. Thus, $\{u_n - v_n - \kappa\varsigma^{int(v_n)}\varphi(u_n)\}$ is bounded. Thanks to (24), we derive

$$\lim_{i\to\infty} \|u_{n_i} - v_{n_i}\| = 0. \tag{27}$$

Combining (6) and (27), we deduce

$$\lim_{i\to\infty} \|\varphi(u_{n_i}) - \varphi(v_{n_i})\| = 0. \tag{28}$$

As a result of (7), we have the following estimate

$$\begin{aligned}\|y_{n_i} - v_{n_i}\| &= \left\|P_C\left[v_{n_i} + \zeta(1-\varpi)\|u_{n_i} - v_{n_i}\|^2 \frac{u_{n_i} - v_{n_i} - \kappa\varsigma^{int(v_{n_i})}\varphi(u_{n_i})}{\|u_{n_i} - v_{n_i} - \kappa\varsigma^{int(v_{n_i})}\varphi(u_{n_i})\|^2}\right] - P_C[v_{n_i}]\right\| \\ &\leq \frac{\zeta(1-\varpi)\|u_{n_i} - v_{n_i}\|^2}{\|u_{n_i} - v_{n_i} - \kappa\varsigma^{int(v_{n_i})}\varphi(u_{n_i})\|}.\end{aligned}$$

This together with (24) implies that

$$\lim_{i\to\infty} \|y_{n_i} - v_{n_i}\| = 0. \tag{29}$$

By (10), we have

$$\langle v_{n_i} - \kappa\varsigma^{int(v_{n_i})}\varphi(v_{n_i}) - u_{n_i}, u_{n_i} - p^\dagger\rangle \geq 0, \ \forall p^\dagger \in C.$$

It results in that

$$\langle\varphi(v_{n_i}), p^\dagger - v_{n_i}\rangle \geq \langle\varphi(v_{n_i}), u_{n_i} - v_{n_i}\rangle + \frac{1}{\kappa\varsigma^{int(v_{n_i})}}\langle u_{n_i} - v_{n_i}, u_{n_i} - p^\dagger\rangle, \ \forall p^\dagger \in C. \tag{30}$$

In view of (27) and (30), we obtain

$$\liminf_{i\to\infty}\langle\varphi(v_{n_i}), p^\dagger - v_{n_i}\rangle \geq 0, \ \forall p^\dagger \in C. \tag{31}$$

On account of (4), we have

$$\|v_{n_i} - x_{n_i}\| = \vartheta_{n_i}\|\phi[(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\phi(x_{n_i})] - x_{n_i}\|,$$

which together with (25) implies that

$$\lim_{n\to\infty} \|v_{n_i} - x_{n_i}\| = 0.$$

So, $v_{n_i} \rightharpoonup x^\dagger$.

Now, we consider two cases: (i) $\liminf_{i\to\infty}\|\varphi(v_{n_i})\| = 0$; (ii) $\liminf_{i\to\infty}\|f(v_{n_i})\| > 0$.

If $\liminf_{i\to\infty}\|\varphi(v_{n_i})\| = 0$, by the property (WSC) of φ, we deduce that $\varphi(x^\dagger) = 0$. Therefore, $x^\dagger \in VI(C, \varphi)$ and $\omega(x_n) \subset VI(C, \varphi)$. If $\liminf_{i\to\infty}\|\varphi(v_{n_i})\| > 0$, without loss of generality, we assume that $\|\varphi(v_{n_i})\| \geq \hat{v}(\forall i \geq 0)$ for some $\hat{v} > 0$. Set $\hat{v}_{n_i} = \frac{f(v_{n_i})}{\|f(v_{n_i})\|^2}(\forall i \geq 0)$. Then, $\langle \varphi(v_{n_i}), \hat{v}_{n_i} \rangle = 1(\forall i \geq 0)$. From (31), we have

$$\liminf_{i\to\infty} \langle \frac{\varphi(v_{n_i})}{\|\varphi(v_{n_i})\|}, p^\dagger - v_{n_i} \rangle \geq 0. \tag{32}$$

Let $\{\epsilon_i\}$ be a positive real number sequence satisfying $\epsilon_i \to 0$ as $i \to \infty$. According to (32), for each ϵ_i, there exists the smallest positive integer N_i such that

$$\langle \frac{\varphi(v_{n_i})}{\|\varphi(v_{n_i})\|}, p^\dagger - v_{n_i} \rangle + \epsilon_i \geq 0, \ \forall i \geq N_i,$$

which implies that

$$\langle \varphi(v_{n_i}), p^\dagger - v_{n_i} \rangle + \epsilon_i \|\varphi(v_{n_i})\| \geq 0, \ \forall i \geq N_i.$$

Namely,

$$\langle \varphi(v_{n_i}), p^\dagger + \epsilon_i \|\varphi(v_{n_i})\| \hat{v}_{n_i} - v_{n_i} \rangle \geq 0, \ \forall i \geq N_i.$$

This together with the pseudomonotonicity of φ implies that

$$\langle \varphi(p^\dagger + \epsilon_i \|\varphi(v_{n_i})\| \hat{v}_{n_i}), p^\dagger + \epsilon_i \|\varphi(v_{n_i})\| \hat{v}_{n_i} - v_{n_i} \rangle \geq 0, \ \forall i \geq N_i. \tag{33}$$

Since $\lim_{i\to\infty} \epsilon_i \|\varphi(v_{n_i})\| \|\hat{v}_{n_i}\| = \lim_{i\to\infty} \epsilon_i = 0$, letting $i \to \infty$ in (33), we deduce

$$\langle \varphi(p^\dagger), p^\dagger - x^\dagger \rangle \geq 0, \forall p^\dagger \in C. \tag{34}$$

B Lemma 2.3 and (34), we conclude that $x^\dagger \in VI(C, \varphi)$. Thus, $\omega_w(x_n) \subset VI(C, \varphi)$. Therefore, $x^\dagger \in \Omega$.

Finally, we show $x_n \to P_\Omega(I - \Phi + \beta\psi)q^\dagger = q^\dagger$. With the help of (19), we deduce

$$\begin{aligned} \|x_{n+1} - q^\dagger\|^2 &\leq [1 - (\hat{\kappa} - \beta\rho)\alpha_n]\|x_n - q^\dagger\|^2 \\ &\quad + (\hat{\kappa} - \beta\rho)\alpha_n \times \frac{2\alpha_n}{(\hat{\kappa} - \beta\rho)\alpha_n} \langle \beta\psi(q^\dagger) - \Phi(q^\dagger), x_{n+1} - q^\dagger \rangle. \end{aligned} \tag{35}$$

It is obviously that

$$\limsup_{n\to\infty} \langle \beta\psi(q^\dagger) - \Phi(q^\dagger), x_{n+1} - q^\dagger \rangle \leq 0.$$

Therefore, applying Lemma 2.4 to (35), we conclude that $x_n \to q^\dagger$. This completes the proof. □

Algorithm 3.2. *Take any initial value $x_0 \in C$ and set $n = 0$.*

Step 1. Assume that x_n is known. Finding the smallest nonnegative integer $int(x_n)$ such that

$$u_n = P_C[x_n - \kappa\varsigma^{int(x_n)}\varphi(x_n)],$$

and

$$\kappa\varsigma^{int(x_n)}\|\varphi(u_n) - \varphi(x_n)\| \leq \varpi\|u_n - x_n\|.$$

If $u_n = x_n$, then set $y_n = x_n$ and go to Step 2. Otherwise, calculate

$$y_n = P_C\Big[x_n + \zeta(1 - \varpi)\|u_n - x_n\|^2 \frac{u_n - x_n - \kappa\varsigma^{int(x_n)}\varphi(u_n)}{\|u_n - x_n - \kappa\varsigma^{int(x_n)}\varphi(u_n)\|^2}\Big].$$

Step 2. Calculate

$$x_{n+1} = P_C[\alpha_n\beta\psi(x_n) + (I - \alpha_n\Phi)y_n].$$

Step 3. Set $n := n + 1$ and return to Step 1.

Corollary 3.1. *The sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to* $q_1^\dagger = P_{VI(C,\varphi}(I - \Phi + \beta\psi)q_1^\dagger$.

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