EDGE-WIENER TYPE INVARIANTS OF SPLICES AND LINKS OF GRAPHS

Mahdieh AZARI[1], Ali IRANMANESH[2*]

In this paper, we present explicit formulae for the first and second edge-Wiener type invariants of splices and links of graphs. As a consequence, the first and second edge-Wiener and edge hyper-Wiener indices of these classes of composite graphs will be determined.

Keywords: Distance, Edge-Wiener type invariants, Splice, Link.
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1. Introduction

In this paper, we are concerned with connected finite graphs without any loops or multiple edges. Let G be such a graph with the vertex set $V(G)$ and the edge set $E(G)$. For $u \in V(G)$ and $e \in E(G)$, we denote by $\deg_G(u)$, the degree of u in G and by $V(e)$, the set of two end vertices of e. A topological index $Top(G)$ of G is a real number with the property that for every graph H isomorphic to G, $Top(H) = Top(G)$. Vertex version of the Wiener index is the first reported distance-based topological index which was introduced in 1947 by Wiener [1], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index $W(G)$ of G is defined as:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v|G),$$

where $d(u,v|G)$ denotes the distance between the vertices u and v of G which is defined as the length of any shortest path in G connecting u and v. Details on the Wiener index can be found in [2-4].

The hyper-Wiener index of acyclic graphs was introduced by Milan Randić in 1993. Then Klein et al. generalized Randić's definition for all connected graphs in 1995 [5]. The vertex version of hyper-Wiener index of G is defined as:

[1]Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P. O. Box: 14115-137, Tehran, Iran, e-mail: mahdie.azari@gmail.com
[2]Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P. O. Box: 14115-137, Tehran, Iran, e-mail: iranmanesh@modares.ac.ir
*Corresponding author

$$WW(G) = \frac{1}{2}[W(G) + \sum_{\{u,v\}\subseteq V(G)} d(u,v|G)^2].$$

We encourage the reader to consult [6-7], for the mathematical properties of hyper-Wiener index and its applications in chemistry.

Edge versions of the Wiener index based on distance between all pairs of edges in a graph G were introduced in 2009 [8]. Two possible distances between the edges $e = uv$ and $f = zt$ of the graph G can be considered. The first distance is denoted by $d_0(e, f|G)$ and defined as:

$$d_0(e, f|G) = \begin{cases} d_1(e, f|G) + 1 & e \neq f \\ 0 & e = f \end{cases},$$

where $d_1(e, f|G) = \min\{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\}$. It is easy to see that $d_0(e, f|G) = d(e, f|L(G))$, where $L(G)$ is the line graph of G.

The second distance is denoted by $d_4(e, f|G)$ and defined as:

$$d_4(e, f|G) = \begin{cases} d_2(e, f|G) & e \neq f \\ 0 & e = f \end{cases},$$

where $d_2(e, f|G) = \max\{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\}$.

Related to the above distances, two edge versions of the Wiener index can be defined. The first and second edge-Wiener indices of G are denoted by $W_{e_0}(G)$ and $W_{e_4}(G)$, respectively and defined as [8]:

$$W_{e_i}(G) = \sum_{\{e,f\}\subseteq E(G)} d_i(e, f|G), \quad i \in \{0,4\}.$$

Obviously, $W_{e_0}(G) = W(L(G))$. For more information on the edge-Wiener indices see [9-14].

Edge version of hyper-Wiener index are defined based on the distances d_0 and d_4, as follows [15]:

$$WW_{e_i}(G) = \frac{1}{2}[W_{e_i}(G) + \sum_{\{e,f\}\subseteq E(G)} d_i(e, f|G)^2], \quad i \in \{0,4\}.$$

The definitions of the edge-Wiener and edge hyper-Wiener indices can be generalized by the following definition:

$$W_{e_i}^{(\lambda)}(G) = \sum_{\{e,f\}\subseteq E(G)} d_i(e, f|G)^{\lambda}, \quad i \in \{0,4\},$$

where λ is an arbitrary real number. The indices $W_{e_0}{}^{(\lambda)}(G)$ and $W_{e_4}{}^{(\lambda)}(G)$ are called the first and second edge-Wiener type invariants of *G*, respectively.

Obviously for $i \in \{0,4\}$,

$$W_{e_i}{}^{(0)}(G) = \binom{|E(G)|}{2},\ W_{e_i}{}^{(1)}(G) = W_{e_i}(G) \text{ and } \frac{1}{2}[W_{e_i}{}^{(1)}(G) + W_{e_i}{}^{(2)}(G)] = WW_{e_i}(G).$$

In this paper, we present explicit formulae for the first and second edge-Wiener type invariants of splices and links of graphs. Then, we apply our results to compute the first and second edge-Wiener and edge hyper-Wiener indices of these classes of composite graphs. Readers interested in more information on computing topological indices of composite graphs can be referred to [4, 9-12, 16-19].

2. Discussion and results

In this section, we compute the first and second edge-Wiener type invariants of splices and links of graphs. We start by introducing some useful notations.

Let G be a simple connected graph. Two possible distances between a vertex u and an edge $e=ab$ of the graph G can be considered [20]. The first distance is denoted by $D_1(u,e|G)$ and defined as:

$$D_1(u,e|G) = \min\{d(u,a|G), d(u,b|G)\},$$

and the second one is denoted by $D_2(u,e|G)$ and defined as:

$$D_2(u,e|G) = \max\{d(u,a|G), d(u,b|G)\}.$$

Note that, $D_1(u,e|G)$ is a nonnegative integer and $D_1(u,e|G) = 0$ if and only if $u \in V(e)$. Also, $D_2(u,e|G)$ is a positive integer and $D_2(u,e|G) = 1$ if and only if $u \in V(e)$ or *u*, *a* and *b* form a triangle in G.

Let λ be a real number and let $u \in V(G)$. We define:

$$D_1{}^{(\lambda)}(u|G) = \sum_{e \in E(G); u \notin V(e)} D_1(u,e|G)^{\lambda},\ D_2{}^{(\lambda)}(u|G) = \sum_{e \in E(G)} D_2(u,e|G)^{\lambda}.$$

Note that, if λ is a positive number then $D_1{}^{(\lambda)}(u|G) = \sum_{e \in E(G)} D_1(u,e|G)^{\lambda}$. In particular for $\lambda = 0$, $D_1{}^{(0)}(u|G) = |E(G)| - \deg_G(u)$, $D_2{}^{(0)}(u|G) = |E(G)|$.

2.1 Splice

Let G_1 and G_2 be two simple connected graphs with the vertex sets $V(G_1)$ and $V(G_2)$ and the edge sets $E(G_1)$ and $E(G_2)$, respectively. For given vertices $y \in V(G_1)$ and $z \in V(G_2)$, a splice of G_1 and G_2 by vertices y and z is denoted by $(G_1.G_2)(y,z)$ and defined by identifying the vertices y and z in the union of G_1 and G_2 as shown in Fig. 1 [21]. We denote by n_i and e_i the order and size of the graph G_i, respectively. It is easy to see that $|V((G_1.G_2)(y,z))| = n_1 + n_2 - 1$ and $|E((G_1.G_2)(y,z))| = e_1 + e_2$.

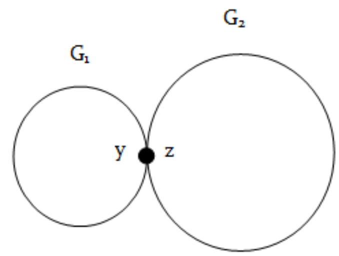

Fig. 1. A splice of G_1 and G_2 by vertices y and z.

In the following Lemma, the distance between vertices of $(G_1.G_2)(y,z)$ is computed. The proof can be easily obtained from the definition of splice of graphs, so is omitted.

Lemma 2.1 Let $u,v \in V((G_1.G_2)(y,z))$. Then

$$d(u,v|(G_1.G_2)(y,z)) = \begin{cases} d(u,v|G_1) & u,v \in V(G_1) \\ d(u,v|G_2) & u,v \in V(G_2) \\ d(u,y|G_1) + d(z,v|G_2) & u \in V(G_1), v \in V(G_2) \end{cases}.$$

Theorem 2.2 Let λ be a positive integer. The first and second edge-Wiener type invariants of $G = (G_1.G_2)(y,z)$ are given by:

(i) $$W_{e_0}{}^{(\lambda)}(G) = W_{e_0}{}^{(\lambda)}(G_1) + W_{e_0}{}^{(\lambda)}(G_2) + \deg_{G_1}(y)\deg_{G_2}(z)$$
$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i}[\deg_{G_2}(z)D_1{}^{(i)}(y|G_1) + \deg_{G_1}(y)D_1{}^{(i)}(z|G_2)]$$
$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i}D_1{}^{(i)}(y|G_1)\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j}D_1{}^{(j)}(z|G_2),$$

(ii) $$W_{e_4}{}^{(\lambda)}(G) = W_{e_4}{}^{(\lambda)}(G_1) + W_{e_4}{}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda}\binom{\lambda}{i}D_2{}^{(i)}(y|G_1)D_2{}^{(\lambda-i)}(z|G_2).$$

Proof. (i) By definition of the first edge-Wiener type invariant,

$$W_{e_0}{}^{(\lambda)}(G) = \sum_{\{e,f\}\subseteq E(G)} d_0(e,f|G)^{\lambda} .$$

Now, we partition the above sum into three sums as follows:

The first sum S_1 consists of contributions to $W_{e_0}{}^{(\lambda)}(G)$ of pairs of edges from G_1. For edges $e,f \in E(G_1)$, $d_0(e,f|G) = d_0(e,f|G_1)$. So,

$$S_1 = \sum_{\{e,f\}\subseteq E(G_1)} d_0(e,f|G_1)^{\lambda} = W_{e_0}{}^{(\lambda)}(G_1) .$$

The second sum S_2 consists of contributions to $W_{e_0}{}^{(\lambda)}(G)$ of pairs of edges from G_2. For edges $e,f \in E(G_2)$, $d_0(e,f|G) = d_0(e,f|G_2)$. So,

$$S_2 = \sum_{\{e,f\}\subseteq E(G_2)} d_0(e,f|G_2)^{\lambda} = W_{e_0}{}^{(\lambda)}(G_2) .$$

The third sum S_3 is taken over all pairs of edges $e,f \in E(G)$ such that $e \in E(G_1)$ and $f \in E(G_2)$. It is easy to see that, $d_0(e,f|G) = 1 + D_1(y,e|G_1) + D_1(z,f|G_2)$. So,

$$S_3 = \sum_{e\in E(G_1), f\in E(G_2)} [1 + D_1(y,e|G_1) + D_1(z,f|G_2)]^{\lambda} .$$

In order to compute the sum S_3, we partition it into four sums S_{31}, S_{32}, S_{33} and S_{34} as follows:

The sum S_{31} is equal to:

$$\begin{aligned} S_{31} &= \sum_{e\in E(G_1); y\notin V(e)} \sum_{f\in E(G_2); z\notin V(f)} [1 + D_1(y,e|G_1) + D_1(z,f|G_2)]^{\lambda} \\ &= \sum_{e\in E(G_1); y\notin V(e)} \sum_{f\in E(G_2); z\notin V(f)} \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1(y,e|G_1)^{i} \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_1(z,f|G_2)^{j} \\ &= \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{e\in E(G_1); y\notin V(e)} D_1(y,e|G_1)^{i} \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} \sum_{f\in E(G_2); z\notin V(f)} D_1(z,f|G_2)^{j} \\ &= \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1{}^{(i)}(y|G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_1{}^{(j)}(z|G_2) . \end{aligned}$$

The sum S_{32} is equal to:

$$S_{32} = \sum_{e\in E(G_1); y\notin V(e)} \sum_{f\in E(G_2); z\in V(f)} [1 + D_1(y,e|G_1)]^{\lambda}$$

$$= \sum_{e\in E(G_1);y\notin V(e)} \sum_{f\in E(G_2);z\in V(f)} \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1(y,e|G_1)^i$$

$$= \deg_{G_2}(z)\sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1{}^{(i)}(y|G_1).$$

The sum S_{33} is equal to:

$$S_{33} = \sum_{e\in E(G_1);y\in V(e)} \sum_{f\in E(G_2);z\notin V(f)} [1+D_1(z,f|G_2)]^{\lambda}$$

$$= \sum_{e\in E(G_1);y\in V(e)} \sum_{f\in E(G_2);z\notin V(f)} \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1(z,f|G_2)^i$$

$$= \deg_{G_1}(y)\sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1{}^{(i)}(z|G_2).$$

The sum S_{34} is equal to:

$$S_{34} = \sum_{e\in E(G_1);y\in V(e)} \sum_{f\in E(G_2);z\in V(f)} 1^{\lambda} = \deg_{G_1}(y)\deg_{G_2}(z).$$

By adding the quantities S_{31}, S_{32}, S_{33} and S_{34}, we obtain:

$$S_3 = \deg_{G_1}(y)\deg_{G_2}(z) + \sum_{i=0}^{\lambda}\binom{\lambda}{i}[\deg_{G_2}(z)D_1{}^{(i)}(y|G_1) + \deg_{G_1}(y)D_1{}^{(i)}(z|G_2)]$$

$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1{}^{(i)}(y|G_1)\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j} D_1{}^{(j)}(z|G_2).$$

The formula of $W_{e_0}{}^{(\lambda)}(G)$ is obtained by adding the quantities S_1, S_2 and S_3.

(ii) Using a similar method as in the proof of part (i), we have:

$$W_{e_4}{}^{(\lambda)}(G) = \sum_{\{e,f\}\subseteq E(G_1)} d_4(e,f|G_1)^{\lambda} + \sum_{\{e,f\}\subseteq E(G_2)} d_4(e,f|G_2)^{\lambda}$$

$$+ \sum_{e\in E(G_1), f\in E(G_2)} [D_2(y,e|G_1) + D_2(z,f|G_2)]^{\lambda} = W_{e_4}{}^{(\lambda)}(G_1) + W_{e_4}{}^{(\lambda)}(G_2)$$

$$+ \sum_{e\in E(G_1), f\in E(G_2)} \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_2(y,e|G_1)^i D_2(z,f|G_2)^{\lambda-i}$$

$$= W_{e_4}{}^{(\lambda)}(G_1) + W_{e_4}{}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_2{}^{(i)}(y|G_1) D_2{}^{(\lambda-i)}(z|G_2).$$

This completes the proof. □

Using Theorem 2.2, we can get the formulae for the edge-Wiener and edge hyper-Wiener indices of $(G_1.G_2)(y,z)$.

Corollary 2.3 The first and second edge-Wiener indices of $G=(G_1.G_2)(y,z)$ are given by:

(i) $W_{e_0}(G)=W_{e_0}(G_1)+W_{e_0}(G_2)+e_2D_1^{(1)}(y|G_1)+e_1D_1^{(1)}(z|G_2)+e_1e_2$,

(ii) $W_{e_4}(G)=W_{e_4}(G_1)+W_{e_4}(G_2)+e_2D_2^{(1)}(y|G_1)+e_1D_2^{(1)}(z|G_2)$.

Corollary 2.4 The first and second edge hyper-Wiener indices of $G=(G_1.G_2)(y,z)$ are given by:

(i) $$WW_{e_0}(G)=WW_{e_0}(G_1)+WW_{e_0}(G_2)+\frac{3}{2}e_2D_1^{(1)}(y|G_1)+\frac{3}{2}e_1D_1^{(1)}(z|G_2)$$
$$+\frac{1}{2}e_2D_1^{(2)}(y|G_1)+\frac{1}{2}e_1D_1^{(2)}(z|G_2)+D_1^{(1)}(y|G_1)D_1^{(1)}(z|G_2)+e_1e_2,$$

(ii) $$WW_{e_4}(G)=WW_{e_4}(G_1)+WW_{e_4}(G_2)+\frac{1}{2}e_2[D_2^{(1)}(y|G_1)+D_2^{(2)}(y|G_1)]$$
$$+\frac{1}{2}e_1[D_2^{(1)}(z|G_2)+D_2^{(2)}(z|G_2)]+D_2^{(1)}(y|G_1)D_2^{(1)}(z|G_2).$$

2.2 Link

Let G_1 and G_2 be two simple connected graphs with the vertex sets $V(G_1)$ and $V(G_2)$ and the edge sets $E(G_1)$ and $E(G_2)$, respectively. For vertices $y\in V(G_1)$ and $z\in V(G_2)$, a link of G_1 and G_2 by vertices y and z is denoted by $(G_1\sim G_2)(y,z)$ and obtained by joining y and z by an edge in the union of these graph, as shown in Fig. 2 [21].

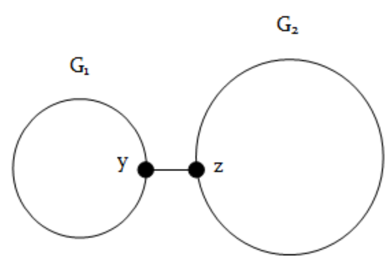

Fig. 2. A link of G_1 and G_2 by vertices y and z.

We denote by n_i and e_i the order and size of the graph G_i, respectively. It is easy to see that $|V((G_1\sim G_2)(y,z))|=n_1+n_2$ and $|E((G_1\sim G_2)(y,z))|=e_1+e_2+1$.

In the following Lemma, the distance between vertices of $(G_1\sim G_2)(y,z)$ is computed. The proof is easy, so we omit it.

Lemma 2.5 Let $u,v\in V((G_1\sim G_2)(y,z))$. Then

$$d(u,v|(G_1 \sim G_2)(y,z)) = \begin{cases} d(u,v|G_1) & u,v \in V(G_1) \\ d(u,v|G_2) & u,v \in V(G_2) \\ d(u,y|G_1) + d(z,v|G_2) + 1 & u \in V(G_1), v \in V(G_2) \end{cases}.$$

Theorem 2.6 Let λ be a positive integer. The first and second edge-Wiener type invariants of $G = (G_1 \sim G_2)(y,z)$ are given by:

(i) $$W_{e_0}{}^{(\lambda)}(G) = W_{e_0}{}^{(\lambda)}(G_1) + W_{e_0}{}^{(\lambda)}(G_2) + 2^{\lambda} \deg_{G_1}(y)\deg_{G_2}(z) + \deg_{G_1}(y) + \deg_{G_2}(z) + \sum_{i=0}^{\lambda}\binom{\lambda}{i}[D_1{}^{(i)}(y|G_1) + D_1{}^{(i)}(z|G_2)]$$
$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i}2^{\lambda-i}[\deg_{G_2}(z)D_1{}^{(i)}(y|G_1) + \deg_{G_1}(y)D_1{}^{(i)}(z|G_2)]$$
$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i}D_1{}^{(i)}(y|G_1)\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j}2^{\lambda-i-j}D_1{}^{(j)}(z|G_2),$$

(ii) $$W_{e_4}{}^{(\lambda)}(G) = W_{e_4}{}^{(\lambda)}(G_1) + W_{e_4}{}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda}\binom{\lambda}{i}[D_2{}^{(i)}(y|G_1) + D_2{}^{(i)}(z|G_2)]$$
$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i}D_2{}^{(i)}(y|G_1)\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j}D_2{}^{(j)}(z|G_2).$$

Proof. (i) By definition of the first edge-Wiener type invariant,

$$W_{e_0}{}^{(\lambda)}(G) = \sum_{\{e,f\}\subseteq E(G)} d_0(e,f|G)^{\lambda}.$$

Now, we partition the above sum into fifth sums as follows:

The first sum S_1 consists of contributions to $W_{e_0}{}^{(\lambda)}(G)$ of pairs of edges from G_1. For edges $e,f \in E(G_1)$, $d_0(e,f|G) = d_0(e,f|G_1)$. So,

$$S_1 = \sum_{\{e,f\}\subseteq E(G_1)} d_0(e,f|G_1)^{\lambda} = W_{e_0}{}^{(\lambda)}(G_1).$$

The second sum S_2 consists of contributions to $W_{e_0}{}^{(\lambda)}(G)$ of pairs of edges from G_2. For edges $e,f \in E(G_2)$, $d_0(e,f|G) = d_0(e,f|G_2)$. So,

$$S_2 = \sum_{\{e,f\}\subseteq E(G_2)} d_0(e,f|G_2)^{\lambda} = W_{e_0}{}^{(\lambda)}(G_2).$$

The third sum S_3 is taken over all pairs of edges $e,f \in E(G)$ such that $e \in E(G_1)$ and $f = yz$. It is easy to see that, $d_0(e,f|G) = 1 + D_1(y,e|G_1)$. So,

$$S_3 = \sum_{e\in E(G_1)}[1+D_1(y,e|G_1)]^\lambda = \sum_{e\in E(G_1);\, y\notin V(e)}[1+D_1(y,e|G_1)]^\lambda + \sum_{e\in E(G_1);\, y\in V(e)} 1^\lambda$$
$$= \sum_{e\in E(G_1);\, y\notin V(e)} \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1(y,e|G_1)^i + \deg_{G_1}(y)$$
$$= \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1{}^{(i)}(y|G_2) + \deg_{G_2}(y).$$

The forth sum S_4 is taken over all pairs of edges $e, f \in E(G)$ such that $e \in E(G_2)$ and $f = yz$. It is easy to see that, $d_0(e,f|G) = 1 + D_1(z,e|G_2)$. So by a similar argument as used in the computation of S_3, we have:

$$S_4 = \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1{}^{(i)}(z|G_2) + \deg_{G_2}(z).$$

The last sum S_5 is taken over all pairs of edges $e, f \in E(G)$ such that $e \in E(G_1)$ and $f \in E(G_2)$. In this case,

$$d_0(e,f|G) = 2 + D_1(y,e|G_1) + D_1(z,f|G_2).$$

So,

$$S_5 = \sum_{e\in E(G_1),\, f\in E(G_2)}[2 + D_1(y,e|G_1) + D_1(z,f|G_2)]^\lambda .$$

Now, we partition the sum S_5 into four sums S_{51}, S_{52}, S_{53} and S_{54} as follows:
The sum S_{51} is equal to:

$$S_{51} = \sum_{e\in E(G_1);\, y\notin V(e)} \sum_{f\in E(G_2);\, z\notin V(f)}[2 + D_1(y,e|G_1) + D_1(z,f|G_2)]^\lambda$$
$$= \sum_{e\in E(G_1);\, y\notin V(e)} \sum_{f\in E(G_2);\, z\notin V(f)} \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1(y,e|G_1)^i \sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j} 2^{\lambda-i-j} D_1(z,f|G_2)^j$$
$$= \sum_{i=0}^{\lambda}\binom{\lambda}{i} D_1{}^{(i)}(y|G_1) \sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j} 2^{\lambda-i-j} D_1{}^{(j)}(z|G_2).$$

The sum S_{52} is equal to:

$$S_{52} = \sum_{e\in E(G_1);\, y\notin V(e)} \sum_{f\in E(G_2);\, z\in V(f)}[2 + D_1(y,e|G_1)]^\lambda$$
$$= \sum_{e\in E(G_1);\, y\notin V(e)} \sum_{f\in E(G_2);\, z\in V(f)} \sum_{i=0}^{\lambda}\binom{\lambda}{i} 2^{\lambda-i} D_1(y,e|G_1)^i$$

$$= \deg_{G_2}(z)\sum_{i=0}^{\lambda}\binom{\lambda}{i}2^{\lambda-i}D_1^{(i)}(y|G_1).$$

The sum S_{53} is equal to:

$$S_{53} = \sum_{e\in E(G_1);y\in V(e)}\ \sum_{f\in E(G_2);z\notin V(f)}[2+D_1(z,f|G_2)]^{\lambda}$$

$$= \sum_{e\in E(G_1);y\in V(e)}\ \sum_{f\in E(G_2);z\notin V(f)}\sum_{i=0}^{\lambda}\binom{\lambda}{i}2^{\lambda-i}D_1(z,f|G_2)^{i}$$

$$= \deg_{G_1}(y)\sum_{i=0}^{\lambda}\binom{\lambda}{i}2^{\lambda-i}D_1^{(i)}(z|G_2).$$

The sum S_{54} is equal to:

$$S_{54} = \sum_{e\in E(G_1);y\in V(e)}\ \sum_{f\in E(G_2);z\in V(f)}2^{\lambda} = 2^{\lambda}\deg_{G_1}(y)\deg_{G_2}(z).$$

By adding the quantities S_{51}, S_{52}, S_{53} and S_{54}, we obtain:

$$S_5 = \sum_{i=0}^{\lambda}\binom{\lambda}{i}2^{\lambda-i}[\deg_{G_2}(z)D_1^{(i)}(y|G_1)+\deg_{G_1}(y)D_1^{(i)}(z|G_2)]$$
$$+2^{\lambda}\deg_{G_1}(y)\deg_{G_2}(z)+\sum_{i=0}^{\lambda}\binom{\lambda}{i}D_1^{(i)}(y|G_1)\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j}2^{\lambda-i-j}D_1^{(j)}(z|G_2).$$

Now, the formula of $W_{e_0}^{(\lambda)}(G)$ is obtained by adding the quantities S_1, S_2, S_3, S_4 and S_5.

(ii) Using a similar method as in the proof of part (i), we have:

$$W_{e_4}^{(\lambda)}(G) = \sum_{\{e,f\}\subseteq E(G_1)}d_4(e,f|G_1)^{\lambda}+\sum_{\{e,f\}\subseteq E(G_2)}d_4(e,f|G_2)^{\lambda}+\sum_{e\in E(G_1)}[1+D_2(y,e|G_1)]^{\lambda}$$
$$+\sum_{e\in E(G_2)}[1+D_2(z,e|G_2)]^{\lambda}+\sum_{e\in E(G_1),f\in E(G_2)}[1+D_2(y,e|G_1)+D_2(z,f|G_2)]^{\lambda}$$
$$= W_{e_4}^{(\lambda)}(G_1)+W_{e_4}^{(\lambda)}(G_2)$$
$$+\sum_{e\in E(G_1)}\sum_{i=0}^{\lambda}\binom{\lambda}{i}D_2(y,e|G_1)^{i}+\sum_{e\in E(G_2)}\sum_{i=0}^{\lambda}\binom{\lambda}{i}D_2(z,f|G_2)^{i}$$
$$+\sum_{e\in E(G_1)}\sum_{f\in E(G_2)}\sum_{i=0}^{\lambda}\binom{\lambda}{i}D_2(y,e|G_1)^{i}\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j}D_2(z,f|G_2)^{j}$$

$$= W_{e_4}{}^{(\lambda)}(G_1) + W_{e_4}{}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda}\binom{\lambda}{i}[D_2{}^{(i)}(y|G_1) + D_2{}^{(i)}(z|G_2)]$$

$$+ \sum_{i=0}^{\lambda}\binom{\lambda}{i}D_2{}^{(i)}(y|G_1)\sum_{j=0}^{\lambda-i}\binom{\lambda-i}{j}D_2{}^{(j)}(z|G_2) . \qquad \square$$

Using Theorem 2.6, we can get the formulae for the edge-Wiener and edge hyper-Wiener indices of $(G_1 \sim G_2)(y,z)$.

Corollary 2.7 The first and second edge-Wiener indices of $G = (G_1 \sim G_2)(y,z)$ are given by:

(i) $W_{e_0}(G) = W_{e_0}(G_1) + W_{e_0}(G_2) + (e_2+1)D_1{}^{(1)}(y|G_1) + (e_1+1)D_1{}^{(1)}(z|G_2)$
$+ e_1 + e_2 + 2e_1e_2$,

(ii) $W_{e_4}(G) = W_{e_4}(G_1) + W_{e_4}(G_2) + (e_2+1)D_2{}^{(1)}(y|G_1) + (e_1+1)D_2{}^{(1)}(z|G_2)$
$+ e_1 + e_2 + e_1e_2$.

Corollary 2.8 The first and second edge hyper-Wiener indices of $G = (G_1 \sim G_2)(y,z)$ are given by:

(i) $WW_{e_0}(G) = WW_{e_0}(G_1) + WW_{e_0}(G_2) + \frac{1}{2}(5e_2+3)D_1{}^{(1)}(y|G_1) + \frac{1}{2}(5e_1+3)D_1{}^{(1)}(z|G_2)$
$+ \frac{1}{2}(e_2+1)D_1{}^{(2)}(y|G_1) + \frac{1}{2}(e_1+1)D_1{}^{(2)}(z|G_2) + D_1{}^{(1)}(y|G_1)D_1{}^{(1)}(z|G_2)$
$+ e_1 + e_2 + 3e_1e_2$,

(ii) $WW_{e_4}(G) = WW_{e_4}(G_1) + WW_{e_4}(G_2) + \frac{3}{2}(e_2+1)D_2{}^{(1)}(y|G_1) + \frac{3}{2}(e_1+1)D_2{}^{(1)}(z|G_2)$
$+ \frac{1}{2}(e_2+1)D_2{}^{(2)}(y|G_1) + \frac{1}{2}(e_1+1)D_2{}^{(2)}(z|G_2)$
$+ D_2{}^{(1)}(y|G_1)D_2{}^{(1)}(z|G_2) + e_1 + e_2 + e_1e_2$.

3. Conclusions

In this paper, we studied the behavior of the edge-Wiener type invariants under the splices and links of graphs. Results were applied to compute the edge-Wiener and edge hyper-Wiener indices of these classes of composite graphs. It is also interesting to find explicit formulae for the edge-Wiener type invariants of other classes of composite graphs such as bridge and chain graphs. In order to achieve that goal, further research into mathematical properties of the edge-Wiener type invariants will be necessary.

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