ANALYSIS OF AN ITERATIVE ALGORITHM FOR SOLVING GENERALIZED VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

Zhangsong Yao[1], Zhichuan Zhu[2]

In this paper, we investigate iterative algorithms for solving the generalized variational inequalities and fixed point problems in Hilbert spaces. We construct an iterative algorithm for finding a common solution of the generalized variational inequalities involved in inverse strongly monotone operator and relaxed cocoercive operator and fixed point problem of asymptotically pseudocontractive operators. Strong convergence analysis of the constructed algorithm is given.

Keywords: Variational inequality, inverse strongly monotone, relaxed cocoercive operator, fixed point, asymptotically pseudocontractive operator.

MSC2020: 49J53, 49M37, 65K10, 90C25.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $f : C \to H$ and $\varphi : C \to C$ be two operators. Recall that the generalized variational inequality is to find a point $x^\dagger \in C$ such that

$$\langle f(x^\dagger), \varphi(x) - \varphi(x^\dagger) \rangle \geq 0, \ \forall x \in C. \tag{1}$$

The solution set of (1) is denoted by $Sol(C, f, \varphi)$.

If $\varphi = I$, then the generalized variational inequality (1) reduces to find a point $x^\dagger \in C$ such that

$$\langle f(x^\dagger), x - x^\dagger \rangle \geq 0, \ \forall x \in C. \tag{2}$$

The solution set of (2) is denoted by $Sol(C, g)$.

Variational inequality acts as a key role and offers helpful techniques and means for solving many important problems arising in industry, finance, economics, social, ecology, regional, pure and applied sciences and so on ([8, 12, 15]). It has been shown that variational inequality theory provides a simple, natural and unified framework for a general treatment of unrelated problems. Variational inequality (2) was introduced by Stampacchia [28] in 1964. A lot of work and a great deal of algorithms for solving (2) have been proposed and investigated, see, e.g., [1, 18, 44, 45, 46, 49]. One of basic techniques for solving (2) is the projection method which generates a sequence $\{x_n\}$ by the following iterate

$$x_{n+1} = P_C(I - \tau f)x_n, \ n \geq 0, \tag{3}$$

where $\tau > 0$ is step-size and P_C is the orthogonal projection from H onto C.

Projection method (3) delegates a critical tool for finding the approximate solution of assorted types of variational inequalities. The general variational inequality (1) was

[1]School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China, e-mail: `yaozhsong@163.com`

[3]`Corresponding author`, School of Economics, Liaoning University, Shenyang, Liaoning 110136, China, e-mail: `zhuzhichuan@lnu.edu.cn`

introduced and studied by Noor in [20]. It has been shown that the minimum of a differentiable nonconvex function on the nonconvex set can be characterized by the general variational inequality ([21]). On the other hand, we note that the variational inequalities (1) and (2) can be transformed into fixed points problems. These equivalent relations have been applied to construct relevant iterative algorithms including proximal point methods ([2, 10, 53]), projection methods [14, 35, 51], Wiener-Hopf equations ([22, 25]), auxiliary principle techniques ([11, 16], extragradient methods ([4, 7, 33, 54]), subgradient methods ([6, 34]), Tseng's methods ([48]) and splitting methods ([3]) for solving variational inequalities (1) and (2). Especially, iterative algorithms for solving variational inequalities and/or fixed point problems have been investigated extensively by many authors ([5, 19],[24]-[31],[38]-[43],[13, 23, 37, 50, 52, 56]).

The main purpose of this paper is to investigate the following variational inequalities and fixed point problems of finding a point $\tilde{p}$ such that

$$\tilde{p} \in Sol(C, f, \varphi) \text{ and } \varphi(\tilde{p}) \in Sol(C, g) \cap Fix(S), \tag{4}$$

where $g : C \to H$, $S : C \to C$ are two operators and $Fix(S)$ denotes the fixed point set of S.

We construct an iterative algorithm for solving (4) in which the involved operators f, g and S are inverse strongly φ-monotone, relaxed (γ, ϱ)-cocoercive, and asymptotically pseudocontractive, respectively. Under some additional assumptions, we show that the constructed algorithm converges strongly to a special solution of problem (4).

2. Preliminaries

In this section, we collect several relevant notations and lemmas. Let C be a nonempty closed convex subset of a real Hilbert space H. For $\forall x^\dagger \in H$, there exists a unique point in C, denoted by $P_C[x^\dagger]$, such that $\|x^\dagger - P_C[x^\dagger]\| \leq \|x - x^\dagger\|, \forall x \in C$. Furthermore, P_C is firmly nonexpansive, namely,

$$\|P_C[\tilde{u}] - P_C[v^\dagger]\|^2 \leq \langle P_C[\tilde{u}] - P_C[v^\dagger], \tilde{u} - v^\dagger\rangle, \ \forall \tilde{u}, v^\dagger \in H. \tag{5}$$

P_C has the characteristic ([47]), $\forall \tilde{u} \in H$,

$$\langle \tilde{u} - P_C[\tilde{u}], x^\dagger - P_C[\tilde{u}]\rangle \leq 0, \ \forall x^\dagger \in C. \tag{6}$$

In Hilbert space H, we have the following equality

$$\|cp + (1-c)p^\dagger\|^2 = c\|p\|^2 + (1-c)\|p^\dagger\|^2 - c(1-c)\|p - p^\dagger\|^2, \tag{7}$$

for all $p, p^\dagger \in H$ and any constant $c \in \mathbb{R}$.

Recall that an operator $f : C \to H$ is said to be

- σ-strongly monotone, if $\forall u, v \in C$,

$$\langle f(u) - f(v), u - v\rangle \geq \sigma\|u - v\|^2,$$

  where $\sigma > 0$ is a constant.
- α-inverse strongly φ-monotone, if $\forall u, v \in C$,

$$\langle f(u) - f(v), \varphi(u) - \varphi(v)\rangle \geq \alpha\|f(u) - f(v)\|^2,$$

  where $\alpha > 0$ is a constant and $\varphi : C \to C$ is an operator.
- relaxed (γ, ϱ)-cocoercive ([9, 32]), if $\forall u, v \in C$,

$$\langle f(u) - f(v), u - v\rangle \geq (-\gamma)\|f(u) - f(v)\|^2 + \varrho\|u - v\|^2,$$

  where $\gamma > 0$ and $\varrho > 0$ are two constants.

Recall that an operator $S : C \to C$ is said to be

- k_n-asymptotically pseudocontractive if for all $n \geq 1$ and for all $\tilde{p}, v^\dagger \in C$,
$$\langle S^n(\tilde{p}) - S^n(v^\dagger), \tilde{p} - v^\dagger \rangle \leq k_n \|\tilde{p} - v^\dagger\|^2,$$
equivalently,
$$\|S^n(\tilde{p}) - S^n(v^\dagger)\|^2 \leq (2k_n - 1)\|\tilde{p} - v^\dagger\|^2 + \|(I - S^n)\tilde{p} - (I - S^n)v^\dagger\|^2, \tag{8}$$
where $\{k_n\}$ is a real number sequence in $[1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$.
- uniformly L_2-Lipschitz if for all $n \geq 1$ and for all $\tilde{p}, v^\dagger \in C$,
$$\|S^n(\tilde{p}) - S^n(v^\dagger)\| \leq L_2 \|\tilde{p} - v^\dagger\|,$$
where $L_2 > 0$ is a constant.

An operator $h : C \to C$ is said to be κ-contractive if for all $\tilde{p}, v^\dagger \in C$,
$$\|h(\tilde{p}) - h(v^\dagger)\| \leq \kappa \|\tilde{p} - v^\dagger\|,$$
where κ is a constant in $[0, 1)$.

Let T be a multi-valued operator of H into 2^H. The effective domain of T is denoted by $dom(T)$, that is, $dom(T) = \{x \in H : T(x) \neq \emptyset\}$. A multi-valued operator T is said to be monotone iff $\langle x - y, x^* - y^* \rangle \geq 0$ for all $x, y \in dom(T)$, $x^* \in T(x)$, and $y^* \in T(y)$. A multi-valued operator T is said to be a maximal monotone operator iff T is monotone and its graph is not properly contained in the graph of any other monotone operator on H.

Lemma 2.1 ([42]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be an α-inverse strongly φ-monotone operator. Then, $\forall x, y \in C$, we have*
$$\|(\varphi(x) - \beta f(x)) - (\varphi(y) - \beta f(y))\|^2 \leq \|\varphi(x) - \varphi(y)\|^2 + \beta(\beta - 2\alpha)\|f(x) - f(y)\|^2.$$

Lemma 2.2 ([55]). *Let $S\colon C \to C$ be a uniformly L-Lipschtzian and asymptotically pseudocontractive operator. Then, $I - S$ is demiclosed at zero.*

Lemma 2.3 ([36]). *Let $\{\varrho_n\} \subset [0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\zeta_n\}$ be real number sequences. Suppose that the following conditions are satisfied*

(i) $\varrho_{n+1} \leq (1 - \alpha_n)\varrho_n + \zeta_n, \forall n \geq 1$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \frac{\zeta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^\infty |\zeta_n| < \infty$.

Then $\lim_{n\to\infty} \varrho_n = 0$.

Lemma 2.4 ([17]). *Let $\{\phi_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\phi_{n_i}\}$ of $\{\phi_n\}$ such that $\phi_{n_i} \leq \phi_{n_i+1}$ for all $i \geq 0$. For every $n \geq n_0$, define an integer sequence $\{\gamma(n)\}$ as*
$$\gamma(n) = \max\{k \leq n : \phi_{n_i} < \phi_{n_i+1}\}.$$
Then $\gamma(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,
$$\max\{\phi_{\gamma(n)}, \phi_n\} \leq \phi_{\gamma(n)+1}.$$

3. Main results

In this section, we introduce our main results. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the operators h, φ, f, g and S satisfy the following conditions

(C1): $h : C \to C$ is κ-contractive;
(C2): $\varphi : C \to C$ is σ-strongly monotone and weakly continuous with $R(\varphi) = C$;
(C3): $f : C \to H$ is α-inverse strongly φ-monotone;
(C4): $g : C \to H$ is relaxed (γ, ϱ)-cocoercive and L_1-Lipschitz continuous;

(C5): $S : C \to C$ is k_n-asymptotically pseudocontractive and uniformly L_2-Lipschitz continuous.

Let $\{\alpha_n\}$, $\{\vartheta_n\}$ and $\{\zeta_n\}$ be three real number sequences in $[0,1]$ and $\{\beta_n\}$ and $\{\tau_n\}$ be two real number sequences in $(0,\infty)$. Let η be a positive constant in $(0,1)$. Use Δ to denote the solution set of problem (4), that is, $\Delta = Sol(C,f,\varphi)\bigcap\varphi^{-1}(Sol(C,g)\cap Fix(S))$. Now, we construct an iterative algorithm for solving problem (4).

Algorithm 3.1. *Let $x_0 \in C$ be a fixed point. Let $\{x_n\}$ be a sequence generated by the following iterative format*

$$\begin{cases} s_n = \alpha_n h(x_n) + (1-\alpha_n)P_C[\varphi(x_n) - \beta_n f(x_n)], \\ t_n = P_C[s_n - \tau_n g(s_n)], \\ w_n = (1-\vartheta_n)t_n + \vartheta_n S^n[(1-\zeta_n)t_n + \zeta_n S^n(t_n)], \\ \varphi(x_{n+1}) = (1-\eta)\varphi(x_n) + \eta w_n,\ n \geq 0. \end{cases} \tag{9}$$

Theorem 3.1. *Suppose that $\Delta \neq \emptyset$. Suppose that the following restrictions hold:*

(r1): $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(r2): $0 \leq \kappa < \sigma < 2\alpha$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 2\alpha$;

(r3): $\varrho > \gamma L_1^2 + \frac{1}{2}$ *and* $0 < a_1 \leq \tau_n \leq a_2 < \frac{2(\varrho-\gamma L_1^2)}{L_1^2}$ *for all* $n \geq 0$;

(r4): $L_2 > 1$ *and* $0 < b_1 < \vartheta_n < b_2 < \zeta_n < \frac{1}{2+\sqrt{L_2^2+4}}$ *for all* $n \geq 0$;

(r5): $1 \leq k_n \leq 2$, $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n} = 0$ *and* $\sum_{n=1}^{\infty}(k_n-1) < +\infty$.

Then the sequence $\{x_n\}$ generated by (9) converges strongly to $\tilde{p} \in \Delta$ which solves the following VI

$$\langle h(\tilde{p}) - \varphi(\tilde{p}), \varphi(x^\dagger) - \varphi(\tilde{p})\rangle \leq 0, \quad \forall x^\dagger \in \Delta. \tag{10}$$

Proof. Since φ is σ-strongly monotone, we deduce $\|\varphi(\tilde{u})-\varphi(\tilde{v})\| \geq \sigma\|\tilde{u}-\tilde{v}\|$ for all $\tilde{u},\tilde{v} \in C$. This indicates that VI (10) has a unique solution $\tilde{p}$. Then, $\tilde{p} \in Sol(C,f,\varphi)$ and $\varphi(\tilde{p}) \in Sol(C,g)\cap Fix(S)$. By inequality (6), we receive $\varphi(\tilde{p}) = P_C[\varphi(\tilde{p}) - \beta_n f(\tilde{p})]$ for all $n \geq 0$. Set $y_n = P_C[\varphi(x_n) - \beta_n f(x_n)]$ and $v_n = \varphi(x_n) - \beta_n f(x_n) - (\varphi(\tilde{p}) - \beta_n f(\tilde{p}))$ for all $n \geq 0$. According to Lemma 2.1, we deduce

$$\|y_n - \varphi(\tilde{p})\|^2 \leq \|v_n\|^2 \leq \|\varphi(x_n) - \varphi(\tilde{p})\|^2 - \beta_n(2\alpha-\beta_n)\|f(x_n) - f(\tilde{p})\|^2. \tag{11}$$

Note that $\|\varphi(x_n) - \varphi(\tilde{p})\| \geq \sigma\|x_n - \tilde{p}\|$. From (9), (11) and (r2), we achieve

$$\begin{aligned} \|s_n - \varphi(\tilde{p})\| &= \|\alpha_n h(x_n) + (1-\alpha_n)y_n - P_C[\varphi(\tilde{p}) - \beta_n f(\tilde{p})]\| \\ &\leq \|\alpha_n(h(x_n) - \varphi(\tilde{p}) + \beta_n f(\tilde{p})) + (1-\alpha_n)v_n\| \\ &\leq \alpha_n\|h(x_n) - h(\tilde{p})\| + \alpha_n\|h(\tilde{p}) - \varphi(\tilde{p}) + \beta_n f(\tilde{p})\| + (1-\alpha_n)\|v_n\| \\ &\leq [1-(1-\kappa/\sigma)\alpha_n]\|\varphi(x_n) - \varphi(\tilde{p})\| + \alpha_n(\|h(\tilde{p}) - \varphi(\tilde{p})\| + 2\alpha\|f(\tilde{p})\|). \end{aligned} \tag{12}$$

Taking into account (11) and (12), we obtain

$$\begin{aligned} \|s_n - \varphi(\tilde{p})\|^2 &\leq \|\alpha_n(h(x_n) - \varphi(\tilde{p}) + \beta_n f(\tilde{p})) + (1-\alpha_n)v_n\|^2 \\ &\leq \alpha_n\|h(x_n) - \varphi(\tilde{p}) + \beta_n f(\tilde{p})\|^2 + (1-\alpha_n)\|v_n\|^2 \\ &\leq \alpha_n\|h(x_n) - \varphi(\tilde{p}) + \beta_n f(\tilde{p})\|^2 + (1-\alpha_n)[\|\varphi(x_n) - \varphi(\tilde{p})\|^2 \\ &\quad + \beta_n(2\alpha-\beta_n)\|f(x_n) - f(\tilde{p})\|^2]. \end{aligned} \tag{13}$$

Since g is relaxed (γ,ϱ)-cocoercive and L_1-Lipschitz, for all $x, y \in C$, we have

$$\langle g(x) - g(y), x-y\rangle \geq (-\gamma)\|g(x) - g(y)\|^2 + \varrho\|x-y\|^2 \geq (\varrho - \gamma L_1^2)\|x-y\|^2 \geq 0, \tag{14}$$

which implies that g is monotone and it follows from (14) that

$$\langle g(s_n) - g(\varphi(\tilde{p})), s_n - \varphi(\tilde{p})\rangle \geq (\varrho - \gamma L_1^2)\|s_n - \varphi(\tilde{p})\|^2.$$

Thus,

$$\begin{aligned}
&\|s_n - \varphi(\tilde{p}) - \tau_n(g(s_n) - g(\varphi(\tilde{p})))\|^2 \\
&\qquad = \|s_n - \varphi(\tilde{p})\|^2 - 2\tau_n\langle g(s_n) - g(\varphi(\tilde{p})), s_n - \varphi(\tilde{p})\rangle + \tau_n^2\|g(s_n) - g(\varphi(\tilde{p}))\|^2 \\
&\qquad \le \|s_n - \varphi(\tilde{p})\|^2 + 2\tau_n\gamma L_1^2\|s_n - \varphi(\tilde{p})\|^2 - 2\tau_n\varrho\|s_n - \varphi(\tilde{p})\|^2 \\
&\qquad\quad + \tau_n^2 L_1^2\|s_n - \varphi(\tilde{p})\|^2 \\
&\qquad = (1 + 2\tau_n\gamma L_1^2 - 2\tau_n\varrho + \tau_n^2 L_1^2)\|s_n - \varphi(\tilde{p})\|^2.
\end{aligned} \tag{15}$$

Since $0 < \tau_n < \frac{2(\varrho - \gamma L_1^2)}{L_1^2}$, $0 < 1 + 2\tau_n\gamma L_1^2 - 2\tau_n\varrho + \tau_n^2 L_1^2 < 1$. Hence, from (15), we obtain

$$\|s_n - \varphi(\tilde{p}) - \tau_n(g(s_n) - g(\varphi(\tilde{p})))\| \le \|s_n - \varphi(\tilde{p})\|.$$

Therefore,

$$\begin{aligned}
\|t_n - \varphi(\tilde{p})\| &= \|P_C(I - \tau_n g)s_n - P_C(I - \tau_n g)\varphi(\tilde{p})\| \\
&\le \|(I - \tau_n g)s_n - (I - \tau_n g)\varphi(\tilde{p})\| \\
&\le \|s_n - \varphi(\tilde{p})\|.
\end{aligned} \tag{16}$$

Set $u_n = (1 - \zeta_n)t_n + \zeta_n S^n(t_n)$ for all $n \ge 0$. By (8), we have

$$\|S^n(t_n) - \varphi(\tilde{p})\|^2 = \|S^n(t_n) - S^n(\varphi(\tilde{p}))\|^2 \le (2k_n - 1)\|t_n - \varphi(\tilde{p})\|^2 + \|t_n - S^n(t_n)\|^2, \tag{17}$$

and

$$\|S^n(u_n) - \varphi(\tilde{p})\|^2 \le (2k_n - 1)\|u_n - \varphi(\tilde{p})\|^2 + \|u_n - S^n(u_n)\|^2. \tag{18}$$

Using (7) and (17), we have

$$\begin{aligned}
\|u_n - \varphi(\tilde{p})\|^2 &= \|(1 - \zeta_n)(t_n - \varphi(\tilde{p})) + \zeta_n(S^n(t_n) - \varphi(\tilde{p}))\|^2 \\
&= (1 - \zeta_n)\|t_n - \varphi(\tilde{p})\|^2 + \zeta_n\|S^n(t_n) - \varphi(\tilde{p})\|^2 - \zeta_n(1 - \zeta_n)\|t_n - S^n(t_n)\|^2 \\
&\le (1 - \zeta_n)\|t_n - \varphi(\tilde{p})\|^2 + \zeta_n\big((2k_n - 1)\|t_n - \varphi(\tilde{p})\|^2 + \|t_n - S^n(t_n)\|^2\big) \\
&\quad - \zeta_n(1 - \zeta_n)\|t_n - S^n(t_n)\|^2 \\
&= [1 + 2(k_n - 1)\zeta_n]\|t_n - \varphi(\tilde{p})\|^2 + \zeta_n^2\|t_n - S^n(t_n)\|^2.
\end{aligned} \tag{19}$$

As a result of uniform L_2-Lipschitz continuity of S, $\|S^n(u_n) - S^n(t_n)\| \le L_2\|u_n - t_n\| = L_2\zeta_n\|t_n - S^n(t_n)\|$. This together with (7) implies that

$$\begin{aligned}
\|u_n - S^n(u_n)\|^2 &= \|(1 - \zeta_n)(t_n - S^n(u_n)) + \zeta_n(S^n(t_n) - S^n(u_n))\|^2 \\
&= (1 - \zeta_n)\|t_n - S^n(u_n)\|^2 + \zeta_n\|S^n(t_n) - S^n(u_n)\|^2 \\
&\quad - \zeta_n(1 - \zeta_n)\|t_n - S^n(t_n)\|^2 \\
&\le (1 - \zeta_n)\|t_n - S^n(u_n)\|^2 - \zeta_n(1 - \zeta_n - L_2^2\zeta_n^2)\|t_n - S^n(t_n)\|^2.
\end{aligned} \tag{20}$$

By virtue of (18)-(20), we obtain

$$\begin{aligned}
\|S^n(u_n) - \varphi(\tilde{p})\|^2 &\le (2k_n - 1)[1 + 2(k_n - 1)\zeta_n]\|t_n - \varphi(\tilde{p})\|^2 + (2k_n - 1)\zeta_n^2\|t_n - S^n(t_n)\|^2 \\
&\quad + (1 - \zeta_n)\|t_n - S^n(u_n)\|^2 - \zeta_n(1 - \zeta_n - L_2^2\zeta_n^2)\|t_n - S^n(t_n)\|^2 \\
&= (2k_n - 1)[1 + 2(k_n - 1)\zeta_n]\|t_n - \varphi(\tilde{p})\|^2 + (1 - \zeta_n)\|t_n - S^n(u_n)\|^2 \\
&\quad - \zeta_n(1 - 2k_n\zeta_n - L_2^2\zeta_n^2)\|t_n - S^n(t_n)\|^2.
\end{aligned} \tag{21}$$

Since $\zeta_n < \frac{1}{2 + \sqrt{L_2^2 + 4}} \le \frac{1}{k_n + \sqrt{k_n^2 + L_2^2}}$, $1 - 2k_n\zeta_n - \zeta_n^2 L_2^2 > 0$. On account of (21), we deduce

$$\|S^n(u_n) - \varphi(\tilde{p})\|^2 \le (2k_n - 1)[1 + 2(k_n - 1)\zeta_n]\|t_n - \varphi(\tilde{p})\|^2 + (1 - \zeta_n)\|t_n - S^n(u_n)\|^2. \tag{22}$$

In the light of (7) and (22), we get

$$\begin{aligned}\|w_n-\varphi(\tilde{p})\|^2 &= \|(1-\vartheta_n)(t_n-\varphi(\tilde{p}))+\vartheta_n\big(S^n(u_n)-\varphi(\tilde{p})\big)\|^2\\ &= (1-\vartheta_n)\|t_n-\varphi(\tilde{p})\|^2+\vartheta_n\|S^n(u_n)-\varphi(\tilde{p})\|^2\\ &\quad -\vartheta_n(1-\vartheta_n)\|t_n-S^n(u_n)\|^2\\ &\le \vartheta_n(2k_n-1)[1+2(k_n-1)\zeta_n]\|t_n-\varphi(\tilde{p})\|^2+(1-\vartheta_n)\|t_n-\varphi(\tilde{p})\|^2\\ &\quad +\vartheta_n(1-\zeta_n)\|t_n-S^n(u_n)\|^2-\vartheta_n(1-\vartheta_n)\|t_n-S^n(u_n)\|^2\\ &= [1+2\vartheta_n(k_n-1)+2\zeta_n\vartheta_n(2k_n-1)(k_n-1)]\|t_n-\varphi(\tilde{p})\|^2\\ &\quad +\vartheta_n(\vartheta_n-\zeta_n)\|t_n-S^n(u_n)\|^2\\ &\le [1+8(k_n-1)]\|t_n-\varphi(\tilde{p})\|^2-\vartheta_n(\zeta_n-\vartheta_n)\|t_n-S^n(u_n)\|^2.\end{aligned} \tag{23}$$

Furthermore,

$$\|w_n-\varphi(\tilde{p})\|\le [1+4(k_n-1)]\|t_n-\varphi(\tilde{p})\|. \tag{24}$$

From (9), (12), (16) and (24), we obtain

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{p})\| &\le (1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|+\eta\|w_n-\varphi(\tilde{p})\|\\ &\le (1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|+\eta[1+4(k_n-1)]\|s_n-\varphi(\tilde{p})\|\\ &\le \eta[1+4(k_n-1)][1-(1-\kappa/\sigma)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{p})\|\\ &\quad +\eta[1+4(k_n-1)]\alpha_n(\|h(\tilde{p})-\varphi(\tilde{p})\|+2\alpha\|f(\tilde{p})\|)\\ &\quad +(1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|\\ &\le [1+4(k_n-1)][1-(1-\kappa/\sigma)\eta\alpha_n]\|\varphi(x_n)-\varphi(\tilde{p})\|\\ &\quad +[1+4(k_n-1)](1-\kappa/\sigma)\eta\alpha_n\frac{\|h(\tilde{p})-\varphi(\tilde{p})\|+2\alpha\|f(\tilde{p})\|}{1-\kappa/\sigma}.\end{aligned} \tag{25}$$

It follows that

$$\|\varphi(x_n)-\varphi(\tilde{p})\|\le\prod_{i=1}^{n}[1+4(k_i-1)]\max\Big\{\|\varphi(x_0)-\varphi(\tilde{p})\|,\frac{\|h(\tilde{p})-\varphi(\tilde{p})\|+2\alpha\|f(\tilde{p})\|}{1-\kappa/\sigma}\Big\}.$$

Then, $\{\varphi(x_n)\}$ is bounded. Note that $\|x_n-\tilde{p}\|\le\frac{1}{\sigma}\|\varphi(x_n)-\varphi(\tilde{p})\|$. So, $\{x_n\}$, $\{s_n\}$ $\{t_n\}$ and $\{w_n\}$ are bounded. By (9), we receive

$$\begin{aligned}&\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2-\|\varphi(x_n)-\varphi(\tilde{p})\|^2\\ &\quad=\eta[\|w_n-\varphi(\tilde{p})\|^2-\|\varphi(x_n)-\varphi(\tilde{p})\|^2-\|w_n-\varphi(x_n)\|^2]+\eta^2\|w_n-\varphi(x_n)\|^2\\ &\quad=\eta[\|w_n-\varphi(\tilde{p})\|^2-\|\varphi(x_n)-\varphi(\tilde{p})\|^2]-\eta(1-\eta)\|w_n-\varphi(x_n)\|^2.\end{aligned} \tag{26}$$

In terms of (12), (16) and (23), we get

$$\begin{aligned}\|w_n-\varphi(\tilde{p})\|^2 &\le [1+8(k_n-1)]\|s_n-\varphi(\tilde{p})\|^2\\ &\le [1+8(k_n-1)][1-(1-\kappa/\sigma)\alpha_n]\|\varphi(x_n)-\varphi(\tilde{p})\|^2\\ &\quad +[1+8(k_n-1)](1-\kappa/\sigma)\alpha_n\Big(\frac{\|h(\tilde{p})-\varphi(\tilde{p})\|+2\alpha\|f(\tilde{p})\|}{1-\kappa/\sigma}\Big)^2.\end{aligned} \tag{27}$$

Next, we consider two possibilities: the sequence $\{\|\varphi(x_n)-\varphi(\tilde{p})\|\}$ is either monotone decreasing (Case 1) or not (Case 2), i.e.,

Case 1. There exists positive integer n_0 such that $\{\|\varphi(x_n)-\varphi(\tilde{p})\|\}$ is decreasing for all $n\ge n_0$.

Case 2. For any positive integer N, there exists at least a positive integer $n_0>N$ such that $\|\varphi(x_{n_0})-\varphi(\tilde{p})\|\le\|\varphi(x_{n_0+1})-\varphi(\tilde{p})\|$.

For Case 1, it is obviously that $\lim_{n\to\infty}\|\varphi(x_n)-\varphi(\tilde{p})\|$ exists. Owing to (26) and (27), we obtain

$$\begin{aligned}\eta(1-\eta)\|w_n-\varphi(x_n)\|^2 &\le \|\varphi(x_n)-\varphi(\tilde{p})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2\\ &\quad+\eta[\|w_n-\varphi(\tilde{p})\|^2-\|\varphi(x_n)-\varphi(\tilde{p})\|^2]\\ &\le \|\varphi(x_n)-\varphi(\tilde{p})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2+8(k_n-1)\|\varphi(x_n)-\varphi(\tilde{p})\|^2\\ &\quad+[1+8(k_n-1)](1-\kappa/\sigma)\alpha_n\left(\frac{\|h(\tilde{p})-\varphi(\tilde{p})\|+2\alpha\|f(\tilde{p})\|}{1-\kappa/\sigma}\right)^2\\ &\to 0,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|w_n-\varphi(x_n)\|=0. \tag{28}$$

Therefore,

$$\lim_{n\to\infty}\|\varphi(x_{n+1})-\varphi(x_n)\|=\lim_{n\to\infty}\eta\|w_n-\varphi(x_n)\|=0. \tag{29}$$

From (9), (13) and (23), we achieve

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2 &\le (1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|^2+\eta\|w_n-\varphi(\tilde{p})\|^2\\ &\le (1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|^2+\eta[1+8(k_n-1)]\|s_n-\varphi(\tilde{p})\|^2\\ &\le [1+8(k_n-1)]\eta\alpha_n\|h(x_n)-\varphi(\tilde{p})+\beta_n f(\tilde{p})\|^2\\ &\quad+[1+8(k_n-1)]\eta(1-\alpha_n)\beta_n(\beta_n-2\alpha)\|f(x_n)-f(\tilde{p})\|^2\\ &\quad+[1+8(k_n-1)]\eta(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{p})\|^2\\ &\quad+(1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|^2\\ &\le [1+8(k_n-1)]\eta\alpha_n\|h(x_n)-\varphi(\tilde{p})+\beta_n f(\tilde{p})\|^2\\ &\quad+[1+8(k_n-1)]\eta(1-\alpha_n)\beta_n(\beta_n-2\alpha)\|f(x_n)-f(\tilde{p})\|^2\\ &\quad+[1+8(k_n-1)]\|\varphi(x_n)-\varphi(\tilde{p})\|^2.\end{aligned} \tag{30}$$

It results in that

$$\begin{aligned}&\eta[1+8(k_n-1)](1-\alpha_n)\beta_n(2\alpha-\beta_n)\|f(x_n)-f(\tilde{p})\|^2\\ &\qquad\le [1+8(k_n-1)]\|\varphi(x_n)-\varphi(\tilde{p})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2\\ &\qquad\quad+[1+8(k_n-1)]\eta\alpha_n\|h(x_n)-\varphi(\tilde{p})+\beta_n f(\tilde{p})\|^2\\ &\qquad\to 0.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|f(x_n)-f(\tilde{p})\|=0. \tag{31}$$

Using (6) and (11), we have

$$\begin{aligned}\|y_n-\varphi(\tilde{p})\|^2 &= \|P_C[\varphi(x_n)-\beta_n f(x_n)]-P_C[\varphi(\tilde{p})-\beta_n f(\tilde{p})]\|^2\\ &\le \langle v_n, y_n-\varphi(\tilde{p})\rangle\\ &= \frac{1}{2}\Big\{\|v_n\|^2+\|y_n-\varphi(\tilde{p})\|^2-\|\varphi(x_n)-y_n-\beta_n(f(x_n)-f(\tilde{p}))\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|\varphi(x_n)-\varphi(\tilde{p})\|^2+\|y_n-\varphi(\tilde{p})\|^2-\|\varphi(x_n)-y_n\|^2-\beta_n^2\|f(x_n)-f(\tilde{p})\|\\ &\quad+2\beta_n\langle\varphi(x_n)-y_n, f(x_n)-f(\tilde{p})\rangle\Big\}.\end{aligned}$$

It leads to

$$\begin{aligned}\|y_n-\varphi(\tilde{p})\|^2 \le \|\varphi(x_n)-\varphi(\tilde{p})\|^2-\beta_n^2\|f(x_n)-f(\tilde{p})\|-\|\varphi(x_n)-y_n\|^2 \\ +2\beta_n\langle\varphi(x_n)-y_n,f(x_n)-f(\tilde{p})\rangle.\end{aligned} \tag{32}$$

On the basis of (9) and (32), we have

$$\begin{aligned}\|s_n-\varphi(\tilde{p})\|^2 &\le \alpha_n\|h(x_n)-\varphi(\tilde{p})\|^2+(1-\alpha_n)\|y_n-\varphi(\tilde{p})\|^2 \\ &\le \alpha_n\|h(x_n)-\varphi(\tilde{p})\|^2+(1-\alpha_n)\|\varphi(x_n)-\varphi(\tilde{p})\|^2 \\ &\quad +2\beta_n\|\varphi(x_n)-y_n\|\|f(x_n)-f(\tilde{p})\|-(1-\alpha_n)\|\varphi(x_n)-y_n\|^2.\end{aligned} \tag{33}$$

In view of (30) and (33), we obtain

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2 \le{}& [1+8(k_n-1)]\|\varphi(x_n)-\varphi(\tilde{p})\|^2+[1+8(k_n-1)]\alpha_n\|h(x_n)-\varphi(\tilde{p})\|^2 \\ &-[1+8(k_n-1)]\eta(1-\alpha_n)\|\varphi(x_n)-y_n\|^2 \\ &+2[1+8(k_n-1)]\beta_n\|\varphi(x_n)-y_n\|\|f(x_n)-f(\tilde{p})\|.\end{aligned}$$

Hence,

$$\begin{aligned}&[1+8(k_n-1)]\eta(1-\alpha_n)\|\varphi(x_n)-y_n\|^2 \\ &\le [1+8(k_n-1)]\|\varphi(x_n)-\varphi(\tilde{p})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2 \\ &\quad +2[1+8(k_n-1)]\eta\beta_n\|\varphi(x_n)-y_n\|\|f(x_n)-f(\tilde{p})\| \\ &\quad +[1+8(k_n-1)]\eta\alpha_n\|h(x_n)-\varphi(\tilde{p})\|^2.\end{aligned} \tag{34}$$

By virtue of (29), (31) and (34), we deduce

$$\lim_{n\to\infty}\|\varphi(x_n)-y_n\|=0. \tag{35}$$

Since $s_n-y_n=\alpha_n(h(x_n)-y_n)\to 0$, from (28), (29) and (35), we have

$$\lim_{n\to\infty}\|\varphi(x_n)-s_n\|=\lim_{n\to\infty}\|\varphi(x_{n+1})-s_n\|=\lim_{n\to\infty}\|w_n-s_n\|=0. \tag{36}$$

From (15) and (16), we get

$$\begin{aligned}\|t_n-\varphi(\tilde{p})\|^2 &\le \|s_n-\varphi(\tilde{p})\|^2-2\tau_n[-\gamma\|g(s_n)-g(\varphi(\tilde{p}))\|^2+\varrho\|s_n-\varphi(\tilde{p})\|^2] \\ &\quad +\tau_n^2\|g(s_n)-g(\varphi(\tilde{p}))\|^2 \\ &\le \|s_n-\varphi(\tilde{p})\|^2+(2\tau_n\gamma+\tau_n^2-\frac{2\tau_n\varrho}{L_1^2})\|g(s_n)-g(\varphi(\tilde{p}))\|^2.\end{aligned}$$

This together with (23) and (30) implies that

$$\begin{aligned}\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2 \le{}& (1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|^2+[1+8(k_n-1)]\eta\|s_n-\varphi(\tilde{p})\|^2 \\ &+[1+8(k_n-1)]\eta(2\tau_n\gamma+\tau_n^2-\frac{2\tau_n\varrho}{L_1^2})\|g(s_n)-g(\varphi(\tilde{p}))\|^2,\end{aligned}$$

which together with (34) implies that

$$\begin{aligned}&-[1+8(k_n-1)]\eta(2\tau_n\gamma+\tau_n^2-\frac{2\tau_n\varrho}{L_1^2})\|g(s_n)-g(\varphi(\tilde{p}))\|^2 \\ &\quad\le (1-\eta)\|\varphi(x_n)-\varphi(\tilde{p})\|^2-\|\varphi(x_{n+1})-\varphi(\tilde{p})\|^2+[1+8(k_n-1)]\eta\|s_n-\varphi(\tilde{p})\|^2 \\ &\quad\to 0.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|g(s_n)-g(\varphi(\tilde{p}))\|=0. \tag{37}$$

In accordance with (6) and (16), we have

$$\begin{aligned}\|t_n - \varphi(\tilde{p})\|^2 &= \|P_C(I-\tau_n g)s_n - P_C(I-\tau_n g)\varphi(\tilde{p})\|^2 \\ &\le \langle (I-\tau_n g)s_n - (I-\tau_n g)\varphi(\tilde{p}), t_n - \varphi(\tilde{p})\rangle \\ &= \frac{1}{2}\Big\{\|(I-\tau_n g)s_n - (I-\tau_n g)\varphi(\tilde{p})\|^2 + \|t_n - \varphi(\tilde{p})\|^2 \\ &\quad - \|(I-\tau_n g)s_n - (I-\tau_n g)\varphi(\tilde{p}) - (t_n - \varphi(\tilde{p}))\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|s_n - \varphi(\tilde{p})\|^2 + \|t_n - \varphi(\tilde{p})\|^2 - \|s_n - t_n - \tau_n(g(s_n) - g(\varphi(\tilde{p})))\|^2\Big\} \\ &= \frac{1}{2}\Big\{\|s_n - \varphi(\tilde{p})\|^2 + \|t_n - \varphi(\tilde{p})\|^2 - \|s_n - t_n\|^2 - \tau_n^2\|g(s_n) - g(\varphi(\tilde{p}))\|^2 \\ &\quad + 2\tau_n\langle g(s_n) - g(\varphi(\tilde{p})), s_n - t_n\rangle\Big\},\end{aligned}$$

which yields

$$\|t_n - \varphi(\tilde{p})\|^2 \le \|s_n - \varphi(\tilde{p})\|^2 - \|s_n - t_n\|^2 + 2\tau_n\|g(s_n) - g(\varphi(\tilde{p}))\|\|s_n - t_n\|.$$

This together with (28) implies that

$$\begin{aligned}\|\varphi(x_{n+1}) - \varphi(\tilde{p})\|^2 \le (1-\eta)\|\varphi(x_n) - \varphi(\tilde{p})\|^2 &+ \eta[1+8(k_n-1)]\|s_n - \varphi(\tilde{p})\|^2 \\ &+ 2[1+8(k_n-1)]\eta\tau_n\|g(s_n) - g(\varphi(\tilde{p}))\|\|s_n - t_n\| \\ &- \eta[1+8(k_n-1)]\|s_n - t_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\eta[1+8(k_n-1)]\|s_n - t_n\|^2 \le (1-\eta)\|\varphi(x_n) - \varphi(\tilde{p})\|^2 &+ \eta[1+8(k_n-1)]\|s_n - \varphi(\tilde{p})\|^2 \\ &+ 2[1+8(k_n-1)]\eta\tau_n\|g(s_n) - g(\varphi(\tilde{p}))\|\|s_n - t_n\| \\ &- \|\varphi(x_{n+1}) - \varphi(\tilde{p})\|^2.\end{aligned} \tag{38}$$

By (29), (37) and (38), we gain

$$\lim_{n\to\infty}\|s_n - t_n\| = 0. \tag{39}$$

Combining (36) and (39), we have

$$\lim_{n\to\infty}\|w_n - t_n\| = 0. \tag{40}$$

In view of (23), we get

$$\vartheta_n(\zeta_n - \vartheta_n)\|t_n - S^n(u_n)\|^2 \le [1+8(k_n-1)]\|t_n - \varphi(\tilde{p})\|^2 - \|w_n - \varphi(\tilde{p})\|^2. \tag{41}$$

It follows from (40) and (41) that

$$\lim_{n\to\infty}\|t_n - S^n(u_n)\| = 0. \tag{42}$$

Since S is uniformly L_2-Lipschitz, we have

$$\begin{aligned}\|t_n - S^n(t_n)\| &\le \|t_n - S^n(u_n)\| + \|S^n(u_n) - S^n(t_n)\| \\ &\le \|t_n - S^n(u_n)\| + L_2\zeta_n\|t_n - S^n(t_n)\|.\end{aligned}$$

It follows that

$$\|t_n - S^n(t_n)\| \le \frac{1}{1 - L_2\zeta_n}\|t_n - S^n(u_n)\|. \tag{43}$$

Based on (42) and (43), we deduce

$$\lim_{n\to\infty}\|t_n - S^n(t_n)\| = 0. \tag{44}$$

Observe that

$$\begin{aligned} \|t_{n+1}-S(t_{n+1})\| &\le \|t_{n+1}-S^{n+1}(t_{n+1})\| + \|S^{n+1}(t_{n+1})-S^{n+1}(t_n)\| \\ &\quad + \|S^{n+1}(t_n)-S(t_{n+1})\| \\ &\le \|t_{n+1}-S^{n+1}(t_{n+1})\| + L_2\|t_{n+1}-t_n\| + L_2\|S^n(t_n)-t_{n+1}\| \\ &\le \|t_{n+1}-S^{n+1}(t_{n+1})\| + 2L_2\|t_{n+1}-t_n\| + L_2\|S^n(t_n)-t_n\|. \end{aligned} \tag{45}$$

Meanwhile, from (9), we have

$$\begin{aligned} \|t_{n+1}-t_n\| &\le \|t_{n+1}-w_{n+1}\| + \|w_{n+1}-w_n\| + \|w_n-t_n\| \\ &\le \|t_{n+1}-w_{n+1}\| + \|w_n-t_n\| + \frac{1}{\eta}\|\varphi(x_{n+2})-\varphi(x_{n+1})\| \\ &\quad + \frac{1-\eta}{\eta}\|\varphi(x_{n+1})-\varphi(x_n)\|. \end{aligned} \tag{46}$$

On the basis of (29), (40), (44), (45) and (46), we deduce

$$\lim_{n\to\infty}\|t_n-S(t_n)\|=0.$$

This together with (39) implies that

$$\lim_{n\to\infty}\|s_n-S(s_n)\|=0. \tag{47}$$

Since $\{s_n\}$ is bounded, choose be a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that

$$\limsup_{n\to\infty}\langle h(\tilde{p})-\varphi(\tilde{p}), s_n-\varphi(\tilde{p})\rangle = \lim_{i\to\infty}\langle h(\tilde{p})-\varphi(\tilde{p}), s_{n_i}-\varphi(\tilde{p})\rangle. \tag{48}$$

Furthermore, by the boundedness of $\{x_{n_i}\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ satisfying $x_{n_{i_j}} \rightharpoonup z \in C$. For convenience, we assume that $x_{n_i} \rightharpoonup z$. It follows that $\varphi(x_{n_i}) \rightharpoonup \varphi(z)$ due to the weak continuity of φ. Then, $t_{n_i} \rightharpoonup \varphi(z)$ and $s_{n_i} \rightharpoonup \varphi(z)$. From Lemma 2.2 and (47), we obtain $\varphi(z) \in Fix(S)$. Next, we show that $\varphi(z) \in Sol(C, g)$. Set

$$S_1(x) = \begin{cases} g(x)+N_C(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

It is clearly that S_1 is maximal monotone. Let $(x^\dagger, y^\dagger) \in G(S_1)$. Owing to $y^\dagger - g(x^\dagger) \in N_C(x^\dagger)$ and $t_{n_i} \in C$, we get

$$\langle x^\dagger - t_{n_i}, y^\dagger - g(x^\dagger)\rangle \ge 0. \tag{49}$$

According to (6), we obtain

$$\langle x^\dagger - t_{n_i}, t_{n_i} - (I-\tau_{n_i}g)s_{n_i}\rangle \ge 0.$$

It yields

$$\langle x^\dagger - t_{n_i}, \frac{t_{n_i}-s_{n_i}}{\tau_{n_i}} + g(s_{n_i})\rangle \ge 0. \tag{50}$$

Combining (49) and (50), we achieve

$$\begin{aligned} \langle x^\dagger - t_{n_i}, y^\dagger\rangle &\ge \langle x^\dagger - t_{n_i}, g(x^\dagger)-g(t_{n_i})\rangle + \langle x^\dagger - t_{n_i}, g(t_{n_i})-g(s_{n_i})\rangle \\ &\quad - \langle x^\dagger - t_{n_i}, \frac{t_{n_i}-s_{n_i}}{\tau_{n_i}}\rangle \\ &\ge \langle x^\dagger - t_{n_i}, g(t_{n_i})-g(s_{n_i})\rangle - \langle x^\dagger - t_{n_i}, \frac{t_{n_i}-s_{n_i}}{\tau_{n_i}}\rangle. \end{aligned} \tag{51}$$

Since $t_{n_i} \rightharpoonup \varphi(z)$, $\|t_{n_i} - s_{n_i}\| \to 0$ and $\|g(t_{n_i}) - g(s_{n_i})\| \to 0$, it follows from (51) that that $\langle x^\dagger - \varphi(z), y^\dagger \rangle \geq 0$. Therefore, $\varphi(z) \in S_1^{-1}(0)$ and $\varphi(z) \in Sol(C, g)$. Next, we prove $z \in Sol(C, f, \varphi)$. Set

$$S_2(x) = \begin{cases} f(x) + N_C(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

It is known that S_2 is maximal φ-monotone. Take $(z^\dagger, v^\dagger) \in G(S_2)$. In virtue of $v^\dagger - f(z^\dagger) \in N_C(z^\dagger)$ and $x_{n_i} \in C$, we have

$$\langle \varphi(z^\dagger) - \varphi(x_{n_i}), v^\dagger - f(z^\dagger) \rangle \geq 0. \tag{52}$$

By (6), we receive

$$\langle \varphi(z^\dagger) - y_{n_i}, y_{n_i} - [\varphi(x_{n_i}) - \beta_{n_i} f(x_{n_i})] \rangle \geq 0.$$

It follows that

$$\langle \varphi(z^\dagger) - y_{n_i}, \frac{y_{n_i} - \varphi(x_{n_i})}{\beta_{n_i}} + f(x_{n_i}) \rangle \geq 0. \tag{53}$$

Combining (52) and (53), we deduce

$$\begin{aligned} \langle \varphi(z^\dagger) - \varphi(x_{n_i}), v^\dagger \rangle &\geq \langle \varphi(z^\dagger) - \varphi(x_{n_i}), f(z^\dagger) - f(x_{n_i}) \rangle + \langle \varphi(z^\dagger) - \varphi(x_{n_i}), f(x_{n_i}) \rangle \\ &\quad - \langle \varphi(z^\dagger) - y_{n_i}, \frac{y_{n_i} - \varphi(x_{n_i})}{\beta_{n_i}} \rangle - \langle \varphi(z^\dagger) - y_{n_i}, f(x_{n_i}) \rangle \\ &\geq -\langle \varphi(z^\dagger) - y_{n_i}, \frac{y_{n_i} - \varphi(x_{n_i})}{\beta_{n_i}} \rangle - \langle \varphi(x_{n_i}) - y_{n_i}, f(x_{n_i}) \rangle. \end{aligned} \tag{54}$$

Since $\|\varphi(x_{n_i}) - y_{n_i}\| \to 0$ and $\varphi(x_{n_i}) \rightharpoonup \varphi(z)$, we deduce that $\langle \varphi(z^\dagger) - \varphi(z), v^\dagger \rangle \geq 0$ by taking $i \to \infty$ in (54). Thus, $z \in S_2^{-1}(0)$ by the maximal φ-monotonicity of S_2. Hence, $z \in Sol(C, f, \varphi)$. Therefore, $z \in \varphi^{-1}(Fix(S) \cap Sol(C, g)) \bigcap Sol(C, f, \varphi) = \Delta$.

By (10) and (48), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle &= \lim_{i\to\infty} \langle h(\tilde{p}) - \varphi(\tilde{p}), \varphi(x_{n_i}) - \varphi(\tilde{p}) \rangle \\ &= \langle h(\tilde{p}) - \varphi(\tilde{p}), \varphi(z) - \varphi(\tilde{p}) \rangle \leq 0. \end{aligned} \tag{55}$$

Thanks to (9) and (11), we have

$$\begin{aligned} \|s_n - \varphi(\tilde{p})\|^2 &\leq (1 - \alpha_n)^2 \|y_n - \varphi(\tilde{p})\|^2 + 2\alpha_n \langle h(x_n) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle \\ &\leq (1 - \alpha_n)^2 \|\varphi(x_n) - \varphi(\tilde{p})\|^2 + 2\alpha_n \langle h(x_n) - h(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle \\ &\quad + 2\alpha_n \langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle \\ &\leq (1 - \alpha_n)^2 \|\varphi(x_n) - \varphi(\tilde{p})\|^2 + 2\alpha_n \kappa/\sigma \|\varphi(x_n) - \varphi(\tilde{p})\| \|s_n - \varphi(\tilde{p})\| \\ &\quad + 2\alpha_n \langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle \\ &\leq (1 - \alpha_n)^2 \|\varphi(x_n) - \varphi(\tilde{p})\|^2 + \alpha_n \kappa/\sigma \|\varphi(x_n) - \varphi(\tilde{p})\|^2 \\ &\quad + \alpha_n \kappa/\sigma \|s_n - \varphi(\tilde{p})\|^2 + 2\alpha_n \langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|s_n - \varphi(\tilde{p})\|^2 &\leq \left[1 - \frac{2(1 - \kappa/\sigma)\alpha_n}{1 - \alpha_n \kappa/\sigma}\right] \|\varphi(x_n) - \varphi(\tilde{p})\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \kappa/\sigma} \|\varphi(x_n) - \varphi(\tilde{p})\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \kappa/\sigma} \langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p}) \rangle. \end{aligned}$$

Set $M = \sup_n\{\|\varphi(x_n) - \varphi(\tilde{p})\|^2, \|s_n - \varphi(\tilde{p})\|^2\}$. Therefore,

$$\begin{aligned}\|\varphi(x_{n+1}) - \varphi(\tilde{p})\|^2 &\leq (1-\eta)\|\varphi(x_n) - \varphi(\tilde{p})\|^2 + \eta[1 + 8(k_n - 1)]\|s_n - \varphi(\tilde{p})\|^2 \\ &\leq \left[1 - \frac{2(1-\kappa/\sigma)\alpha_n\eta}{1-\alpha_n\kappa/\sigma}\right]\|\varphi(x_n) - \varphi(\tilde{p})\|^2 + \frac{\alpha_n^2\eta}{1-\alpha_n\kappa/\sigma}\|\varphi(x_n) - \varphi(\tilde{p})\|^2 \\ &\quad + \frac{2\alpha_n\eta}{1-\alpha_n\kappa/\sigma}\langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p})\rangle + 8\eta M(k_n - 1) \\ &= \left[1 - \frac{2(1-\kappa/\sigma)\alpha_n\eta}{1-\alpha_n\kappa/\sigma}\right]\|\varphi(x_n) - \varphi(\tilde{p})\|^2 + \frac{2(1-\kappa/\sigma)\alpha_n\eta}{1-\alpha_n\kappa/\sigma} \\ &\quad \times \left\{\frac{\alpha_n}{2(1-\kappa/\sigma)}M + \frac{4M(k_n-1)}{(1-\kappa/\sigma)\alpha_n} + \frac{1}{1-\kappa/\sigma}\langle h(\tilde{p}) - \varphi(\tilde{p}), s_n - \varphi(\tilde{p})\rangle\right\}.\end{aligned} \tag{56}$$

By Lemma 2.3, (55) and (56), we conclude that $\varphi(x_n) \to \varphi(\tilde{p})$ and $x_n \to \tilde{p}$.

For Case 2, setting $\phi_n = \{\|\varphi(x_n) - \varphi(\tilde{p})\|^2\}$, we have $\phi_{n_0} \leq \phi_{n_0+1}$. Let $\{\gamma_n\}$ be an integer sequence defined by, for all $n \geq n_0$,

$$\gamma(n) = \max\{l \in \mathbb{N} | n_0 \leq l \leq n, \phi_l \leq \phi_{l+1}\}.$$

It is obvious that $\gamma(n)$ is non-decreasing and there hold $\lim_{n\to\infty}\gamma(n) = \infty$ and $\phi_{\gamma(n)} \leq \phi_{\gamma(n)+1}$ for all $n \geq n_0$. Similarly, we have

$$\limsup_{n\to\infty}\langle h(\tilde{p}) - \varphi(\tilde{p}), s_{\gamma(n)} - \varphi(\tilde{p})\rangle \leq 0 \tag{57}$$

and

$$\begin{aligned}\phi_{\gamma(n)+1} &\leq \left[1 - \frac{2(1-\kappa/\sigma)\alpha_\gamma(n)\eta}{1-\alpha_\gamma(n)\kappa/\sigma}\right]\phi_{\gamma(n)} + \frac{2(1-\kappa/\sigma)\alpha_\gamma(n)\eta}{1-\alpha_{\gamma(n)}\kappa/\sigma} \\ &\quad \times \left\{\frac{\alpha_\gamma(n)}{2(1-\kappa/\sigma)}M + \frac{4M(k_{\gamma(n)}-1)}{(1-\kappa/\sigma)\alpha_{\gamma(n)}} + \frac{1}{1-\kappa/\sigma}\langle h(\tilde{p}) - \varphi(\tilde{p}), s_{\gamma(n)} - \varphi(\tilde{p})\rangle\right\}.\end{aligned} \tag{58}$$

Since $\phi_{\gamma(n)} \leq \phi_{\gamma(n)+1}$, it follows from (58) that

$$\phi_{\gamma(n)} \leq \frac{\alpha_\gamma(n)}{2(1-\kappa/\sigma)}M + \frac{4M(k_{\gamma(n)}-1)}{(1-\kappa/\sigma)\alpha_{\gamma(n)}} + \frac{1}{1-\kappa/\sigma}\langle h(\tilde{p}) - \varphi(\tilde{p}), s_{\gamma(n)} - \varphi(\tilde{p})\rangle. \tag{59}$$

According to (r1), (r5), (57) and (59), we derive $\limsup_{n\to\infty}\phi_{\gamma(n)} \leq 0$ which yields

$$\lim_{n\to\infty}\phi_{\gamma(n)} = 0. \tag{60}$$

Combining (57) and (58) to deduce that $\limsup_{n\to\infty}\phi_{\gamma(n)+1} \leq \limsup_{n\to\infty}\phi_{\gamma(n)}$. This together with (60) implies that $\lim_{n\to\infty}\phi_{\gamma(n)+1} = 0$. Applying Lemma 2.4, we obtain $0 \leq \phi_n \leq \max\{\phi_{\gamma(n)}, \phi_{\gamma(n)+1}\}$. Therefore, $\phi_n \to 0$. That is, $\varphi(x_n) \to \varphi(\tilde{p})$ and thus $x_n \to \tilde{p}$. This completes the proof. □

Setting $S = I$ in Algorithm 3.1 and Theorem 3.1, we have the following algorithm and corollary.

Algorithm 3.2. *Let $x_0 \in C$ be a fixed point. Let $\{x_n\}$ be a sequence generated by the following iterative format*

$$\begin{cases} s_n = \alpha_n h(x_n) + (1-\alpha_n)P_C[\varphi(x_n) - \beta_n f(x_n)], \\ t_n = P_C[s_n - \tau_n g(s_n)], \\ \varphi(x_{n+1}) = (1-\eta)\varphi(x_n) + \eta t_n, \ n \geq 0. \end{cases}$$

Corollary 3.1. *Suppose that $\Delta_1 := Sol(C, f, \varphi)\bigcap\varphi^{-1}(Sol(C, g)) \neq \emptyset$. Suppose that the following restrictions hold:*

(r1): $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*

(r2): $0 \leq \kappa < \sigma < 2\alpha$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 2\alpha$*;*

(r3): $\varrho > \gamma L_1^2 + \frac{1}{2}$ *and* $0 < a_1 \le \tau_n \le a_2 < \frac{2(\varrho - \gamma L_1^2)}{L_1^2}$ *for all* $n \ge 0$.
Then the sequence $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to* $\tilde{p_1} \in \Delta_1$ *which solves the following VI*

$$\langle h(\tilde{p_1}) - \varphi(\tilde{p_1}), \varphi(x^\dagger) - \varphi(\tilde{p_1})\rangle \le 0, \quad \forall x^\dagger \in \Delta_1.$$

Setting $\varphi = I$ and f being α-inverse strongly monotone, from Algorithm 3.1 and Theorem 3.1, we have the following algorithm and corollary.

Algorithm 3.3. *Let* $x_0 \in C$ *be a fixed point. Let* $\{x_n\}$ *be a sequence generated by the following iterative format*

$$\begin{cases} s_n = \alpha_n h(x_n) + (1-\alpha_n)P_C[x_n - \beta_n f(x_n)], \\ t_n = P_C[s_n - \tau_n g(s_n)], \\ w_n = (1-\vartheta_n)t_n + \vartheta_n S^n[(1-\zeta_n)t_n + \zeta_n S^n(t_n)], \\ x_{n+1} = (1-\eta)x_n + \eta w_n, \ n \ge 0. \end{cases}$$

Corollary 3.2. *Suppose that* $\Delta_2 := Sol(C, f) \cap Sol(C, g) \cap Fix(S)$. *Suppose that the following restrictions hold:*
(r1): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(r2): $0 < \kappa < 2\alpha$ *and* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 2\alpha$*;*
(r3): $\varrho > \gamma L_1^2 + \frac{1}{2}$ *and* $0 < a_1 \le \tau_n \le a_2 < \frac{2(\varrho-\gamma L_1^2)}{L_1^2}$ *for all* $n \ge 0$*;*
(r4): $L_2 > 1$ *and* $0 < b_1 < \vartheta_n < b_2 < \zeta_n < \frac{1}{2+\sqrt{L_2^2+4}}$ *for all* $n \ge 0$*;*
(r5): $1 \le k_n \le 2$, $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$.
Then the sequence $\{x_n\}$ *generated by Algorithm 3.3 converges strongly to* $\tilde{p_2} \in \Delta_2$ *which solves the following VI*

$$\langle h(\tilde{p_2}) - \tilde{p_2}, x^\dagger - \tilde{p_2}\rangle \le 0, \quad \forall x^\dagger \in \Delta_2.$$

4. Acknowledgments

Zhichuan Zhu was supported by the Education Department Foundation of Jilin province [grant No. JJKH20190742SK] and Advanced Talents Research Fund of Liaoning University.

REFERENCES

[1] T.Q. Bao and P.Q. Khanh, *A Projection-type algorithm for pseudomonotone nonlipschitzian multivalued variational inequalities*, Nonconvex Optim. Appl., **77**(2005), 113–129.

[2] R.I. Bot and E.R. Csetnek, *A hybrid proximal-extragradient algorithm with inertial effects*, Numer. Funct. Anal. Optim., **36**(2015), 951–963.

[3] R.L. Bot, E.R. Csetnek and P.T. Vuong, *The forward-backward-forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces*, Eur. J. Oper. Res., **287**(2020), 49–60.

[4] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed Points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19**(2018), 487–502.

[5] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20**(2019), 113–133.

[6] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., **148**(2011), 318–335.

[7] Y. Censor, A. Gibali and S. Reich, *Extensions of Korpelevichs extragradient method for the variational inequality problem in Euclidean space*, Optim., **61**(2012), 1119–1132.

[8] Y. Censor, A. Gibali, S. Reich and S. Sabach, *Common solutions to variational inequalities*, Set-Valued Var. Anal., **20**(2012), 229–247.

[9] S.S. Chang, H.W. Joseph Lee and C.K. Chan, *Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces*, Appl. Math. Lett., **20**(2007), 329–334.

[10] C. Chen, S. Ma and J. Yang, *A general inertial proximal point algorithm for mixed variational inequality problem*, SIAM J. Optim., **25**(2014), 2120–2142.

[11] G. Cohen, *Auxiliary problem principle extended to variational inequalities*, J. Optim. Theory Appl., **59**(1988), 325–333.

[12] J.Y. Cruz Bello and A.N. Iusem, *A strongly convergent direct method for monotone variational inequalities in Hilbert space*, Numer. Funct. Anal. Optim., **30**(2009), 23–36.

[13] V. Dadashi and M. Postolache, *Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators*, Arab. J. Math., **9**(2020), 89–99.

[14] Q.L. Dong, Y. Peng and Y. Yao, *Alternated inertial Projection methods for the split equality problem*, J. Nonlinear Convex Anal., **22**(2021), 53–67.

[15] R. Glowinski, Numerical methods for nonlinear variational problems, Springer, New York, 1984.

[16] A. Kaplan and R. Tichatschke, *Extended auxiliary problem principle to variational inequalities involving multi-valued operators*, Optim., **53**(2004), 223–252.

[17] P.E. Maingé, *Strong convergence of projected reflected gradient methods for variational inequalities*, Fixed Point Theory, **19**(2018), 659–680.

[18] Y. Malitsky, *Proximal extrapolated gradient methods for variational inequalities*, Optim. Meth. Soft., **33**(2018), 140–164.

[19] A. Moslemipour and M. Roohi, *A Krasnoselskii-Mann type iteration for nonexpansive mappings in Hadamard spaces*, J. Adv. Math. Stud., **14**(2021), 85–93.

[20] M.A. Noor, *General variational inequalities*, Appl. Math. Lett., **1**(1988), 119–121.

[21] M.A. Noor, *Differentiable nonconvex functions and general variational inequalities*, Appl. Math. Comput., **199**(2008), 623–630.

[22] M.A. Noor and Z. Huang, *Wiener-Hopf equation technique for variational inequalities and nonexpansive mappings*, Appl. Math. Comput., **191**(2007), 504–510.

[23] D.R. Sahu, A. Pitea and M. Verma, *A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems*, Numer. Algor., **83**(2020), 421–449.

[24] G.S. Saluja, *Three-step iterative scheme for a pair of multivalued nonexpansive mappings in Banach spaces*, J. Adv. Math. Stud., **11**(2018), 528–536.

[25] P. Shi, *Equivalence of variational inequalities with Wiener-Hopf equations*, Proc. Amer. Math. Soc., **111**(1991), 339–346.

[26] K. Sitthithakerngkiet, J. Deepho and P. Kumam, *A hybrid viscocity algorithm via modify the hybrid steepest descent method for solving the split variational inclusion in image reconstruction and fixed point problems*, Appl. Math. Comput., **250**(2015), 986–1001.

[27] T.M.M. Sow, *General viscosity methods for solving equilibrium problems, variational inequality problems and fixed point problems involving a finite family of multivalued strictly pseudo-contractive mappings*, J. Adv. Math. Stud., **13**(2020), 275–293.

[28] G. Stampacchi, *Formes bilineaires coercivites surles ensembles convexes*, C. R. Acad. Sciences, **258**(1964), 4413–4416.

[29] B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., **275**(2016), 147–155.

[30] B.S. Thakur, D. Thakur and M. Postolache, *A new iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat, **30**(2016), 2711–2720.

[31] D. Thakur, B.S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings*, J. Inequal. Appl., **2014**(2014), Art. No. 328.

[32] R.U. Verma, *Generalized system for relaxed cocoercive variational inequalities and its Projection methods*, J. Optim. Theory Appl., **121**(2004), 203–210.

[33] P.T. Vuong, J.J. Strodiot and V.H. Nguyen, *Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems*, J. Optim. Theory Appl., **155**(2012), 605–627.

[34] X. Wang, S. Li and X. Kou, *An extension of subgradient method for variational inequality problems in Hilbert space*, Abstr. Appl. Anal., **2013**(213), Art. No. 531912.

[35] F.Q. Xia and N.J. Huang, *A Projection-proximal point algorithm for solving generalized variational inequalities*, J. Optim. Theory Appl., **150**(2011), 98–117.

[36] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **2**(2002), 1–17.

[37] Y. Yao, Ravi P. Agarwal, M. Postolache and Y.C. Liou, *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem*, Fixed Point Theory Appl., **2014**(2014), Art. No. 183.

[38] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18**(2017), 875–882.

[39] Y. Yao, H. Li and M. Postolache, *Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions*, Optim., in press, DOI: 10.1080/02331934.2020.1857757.

[40] Y. Yao, Y.C. Liou and M. Postolache, *Self-adaptive algorithms for the split problem of the demicontractive operators*, Optim., **67**(2018), 1309–1319.

[41] Y. Yao, Y.C. Liou and J.C. Yao, *Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction*, Fixed Point Theory Appl., (2015)2015, Article No. 127, 19 pages.

[42] Y. Yao, Y.C. Liou and J.C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843–854.

[43] Y. Yao, M. Postolache and Y.C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013**(2013), Art. No. 201.

[44] Y. Yao, M. Postolache and J.C. Yao, *Iterative algorithms for generalized varational inequalities*, U.P.B. Sci. Bull., Series A, **81**(2019), 3–16.

[45] Y. Yao, M. Postolache and J.C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7**(2019), Art. No. 61.

[46] Y. Yao, M. Postolache and J.C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U.P.B. Sci. Bull., Series A, **82(1)**(2020), 3–12.

[47] Y. Yao, X. Qin and J.C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19**(2018), 407–415.

[48] Y. Yao, N. Shahzad and J.C. Yao, *Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems*, Carpathian J. Math., in press.

[49] Y. Yao, Y. Shehu, X. Li and Q. Dong, *A method with inertial extrapolation step for split monotone inclusion problems*, Optim., 70(2021), 741–761.

[50] Y. Yao, J.C. Yao, Y.C. Liou and M. Postolache, *Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms*, Carpathian J. Math., **34**(2018), 459–466.

[51] M.L. Ye and Y.R. He, *A double Projection method for solving variational inequalities without mononicity*, Comput. Optim. Appl., **60**(2015), 141–150.

[52] H. Zegeye, N. Shahzad and Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64**(2015), 453–471.

[53] X.P. Zhao, J.C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U.P.B. Sci. Bull., Series A, **82**(2020), 43–52.

[54] X.P. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., **69**(2020), 1987–2002.

[55] H. Zhou, *Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70**(2009), 3140–3145.

[56] L.J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, **83(1)**(2021), 3–14.