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A NEW VERSION OF COUPLED FIXED POINT RESULTS IN ORDERED METRIC SPACES WITH APPLICATIONS

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The purpose of the present paper is to give more general results than many results in literature without mixed monotone property and use a new method of reducing coupled common fixed point results in ordered metric spaces to the respective results for mappings with one variable. In addition, an example and an application to integral equations are given to verify the effectiveness of the obtained results.

Keywords: Coupled common fixed point, ordered metric spaces, weakly increasing mappings, integral equations.

MSC2010: 47H10, 54H25.

1. Introduction and Preliminaries

The notion of a coupled fixed point was introduced and studied by Opoitsev [12, 13] and then by Guo and Lakhsmikantham [8]. Bhaskar and Lakhsmikantham [5] were the first to study coupled fixed points in connection to contractive type conditions. They applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. Since then, coupled fixed point theory have been a subject of interest by many authors regarding the application potential of it, for example see [1–4, 7, 9–11, 14–17] and references therein.

We begin with giving some notation and preliminaries that we shall need to state our results.

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ will denote the set of all real numbers, the set of all non-negative real numbers and the set of all natural numbers, respectively. Also, the triple $(X, \preceq, d)$ denotes an ordered metric space where $\preceq$ is a partial order on the set X and d is a metric on X.

Definition 1.1 ([5,8]). *Let $(X, \preceq)$ be an ordered set and $F : X^2 \to X$. We say that F has the mixed monotone property in X if, for any $x, y \in X$, $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y. An element $(x, y) \in X^2$ is said to be a coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.*

Definition 1.2 ([6,7]). *Let $(X, \preceq)$ be an ordered set, $f, g : X \to X$ and $F, G : X^2 \to X$ be given mappings.*

(1) *The pair (f, g) is called a weakly increasing with respect to $\preceq$, if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.*
(2) *The pair (F, G) is said to be weakly increasing with two variables with respect to $\preceq$ if, for all $(x, y) \in X^2$*

$$F(x, y) \preceq G(F(x, y), F(y, x))$$

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and

$$G(x,y) \preceq F(G(x,y), G(y,x)).$$

(3) *An element $x \in X$ is called a common fixed point of f and g, if $x = fx = gx$. We will indicate by $\mathcal{F}(f,g)$ the set of all common fixed points of f and g.*

(4) *An element $(x,y) \in X^2$ is said to be coupled common fixed point of F and G, if $x = F(x,y) = G(x,y)$ and $y = F(y,x) = G(y,x)$. We will indicate by $\mathcal{F}(F,G)$ the set of all coupled common fixed points of F and G.*

Definition 1.3. *Let $(X, \preceq, d)$ be an ordered metric space. We say that $(X, \preceq, d)$ is regular if for non-decreasing sequence $\{x_n\}$ holds $d(x_n, x) \to 0$, then $x_n \preceq x$ for all n.*

The partial order $\preceq$ on X can be induced on X^2 as follows:

$$(x,y), (u,v) \in X^2, \quad (x,y) \sqsubseteq (u,v) \Longleftrightarrow x \preceq u \wedge y \preceq v.$$

We call that (x,y) is comparable to (u,v), if $(x,y) \sqsubseteq (u,v)$ or $(u,v) \sqsubseteq (x,y)$. Also x is comparable to y if $x \preceq y$ or $y \preceq x$.

If d is a metric on X, then $\delta : X^2 \times X^2 \to \mathbb{R}^+$ defined by

$$\delta(U,V) = d(x,u) + d(y,v), \quad U = (x,y), \quad V = (u,v) \in X^2,$$

a metric on X^2.

The proof of following lemma can be easily shown.

Lemma 1.1. *The following conditions are satisfied.*

(1) *(X^2, δ) is complete iff (X,d) is complete. Also, $(X^2, \sqsubseteq, \delta)$ is regular iff $(X, \preceq, d)$ is a such.*

(2) *If $F, G : X^2 \to X$ and the pair (F,G) is weakly increasing with two variables with respect to $\preceq$, then the mappings $T_F, T_G : X^2 \to X^2$ given by*

$$T_F(U) = (F(x,y), F(y,x)), \ T_G(U) = (G(x,y), G(y,x)), \quad U = (x,y)$$

are weakly increasing with respect to $\sqsubseteq$.

(3) *The mappings F and G are continuous iff T_F and T_G are continuous.*

(4) *The mappings F and G have a coupled common fixed point iff T_F and T_G have a common fixed point in X^2.*

Recently, Isik and Turkoglu [9] proved following coupled fixed point theorem.

Theorem 1.1. *([9]) Let $(X, \preceq, d)$ be a complete ordered metric space and $F : X^2 \to X$ be a mixed monotone mapping such that*

$$\varphi(d(F(x,y), F(u,v))) \leq \frac{1}{2}\psi(d(x,u) + d(y,v)) \tag{1}$$

for all $(x,y), (u,v) \in X^2$ with $x \succeq u$, $y \preceq v$, where $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy φ is continuous and non-decreasing, $\varphi(t) = 0$ iff $t = 0$, $\varphi(t+s) \leq \varphi(t) + \varphi(s)$, for all $t, s \in \mathbb{R}^+$ and ψ is continuous with the condition $\varphi(t) > \psi(t)$ for all $t > 0$. Suppose either

(a) *F is continuous or*

(b) *X satisfies the following property:*

if (x_n) is a nondecreasing sequence with $x_n \to x$, then $x_n \preceq x$ for all n,

if (y_n) is a nonincreasing sequence with $y_n \to y$, then $y \preceq y_n$ for all n.

If there exist $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

We denote set $\Phi = \{\ \varphi : \mathbb{R}^+ \to \mathbb{R}^+$: φ satisfies $(i) - (ii)\}$, where

(i) φ is nondecreasing,

(ii) $\varphi(t) = 0$ if and only if $t = 0$, and $\Psi = \{ \psi : \mathbb{R}^+ \to \mathbb{R}^+ \colon \psi$ is a right upper semi-continuous with the condition $\varphi(t) > \psi(t)$ for all $t > 0$ where $\varphi \in \Phi\}$.

Note that, by the properties of φ and ψ, we have $\psi(0) = 0$.

The aim of the present paper is to give more general results than obtained in [9] without mixed monotone property and use a new method of reducing coupled common fixed point results in ordered metric spaces to the respective results for mappings with one variable. Our results generalize and modify several comparable results in the literature. An example as well as an application to integral equations are also given in order to illustrate the effectiveness of the results proved herein.

2. **Main Results**

Before proceeding to our results, let us give the following lemma which will be used efficiently in the proof of main results.

Lemma 2.1. *Let $(X, \preceq, d)$ be a complete ordered metric space, f and g be selfmaps on X and the pair (f, g) be weakly increasing with respect to $\preceq$ such that*

$$\varphi\left(d\left(fx, gy\right)\right) \leq \psi\left(d\left(x, y\right)\right), \tag{2}$$

for all comparable $x, y \in X$ where $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose either

(a) f (or g) are continuous or

(b) $(X, \preceq, d)$ is regular.

Then f and g have a common fixed point. Moreover, if x^* and y^* are comparable whenever x^*, $y^* \in \mathcal{F}(f, g)$, then f and g have a unique common fixed point.

Proof. Firstly, we prove that z is a fixed point of f if and only if z is a fixed point of g. Suppose that z is a fixed point of f. Then, since $z \preceq z$, from (2), we have

$$\begin{aligned} \varphi\left(d\left(z, gz\right)\right) &= \varphi\left(d\left(fz, gz\right)\right) \\ &\leq \psi\left(d\left(z, z\right)\right) = \psi\left(0\right) = 0, \end{aligned}$$

which implies $\varphi\left(d\left(z, gz\right)\right) = 0$. Therefore $d\left(z, gz\right) = 0$ and so $z = gz$. Similarly, it is easy to show that if z is a fixed point of g, then z is a fixed point of f.

Let $x_0 \in X$. Define the sequence $\{x_n\}$ in X by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X.

Since the pair (f, g) is weakly increasing with respect to $\preceq$, we get

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 \preceq fgx_1 = x_3 \preceq \cdots,$$

and hence for all $n \in \mathbb{N}$

$$x_n \preceq x_{n+1}. \tag{3}$$

If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$, then $x_{2n} = fx_{2n}$. Thus x_{2n} is a fixed point of f and so x_{2n} is a fixed point of g, that is, $x_{2n} = fx_{2n} = gx_{2n}$. Similarly, if $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$, then it is easy to see that $x_{2n+1} = fx_{2n+1} = gx_{2n+1}$.

Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, by (2) and (3), we obtain

$$\varphi\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) = \varphi\left(d\left(fx_{2n}, gx_{2n+1}\right)\right) \leq \psi\left(d\left(x_{2n}, x_{2n+1}\right)\right). \tag{4}$$

By a similar proof, one can also show that

$$\varphi\left(d\left(x_{2n}, x_{2n+1}\right)\right) \leq \psi\left(d(x_{2n-1}, x_{2n})\right). \tag{5}$$

Thus, from (4) and (5)

$$\varphi\left(d\left(x_n, x_{n+1}\right)\right) \leq \psi\left(d(x_{n-1}, x_n)\right), \qquad n \geq 1. \tag{6}$$

By the properties of φ and ψ, we deduce that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, that is, the sequence $\{d(x_n, x_{n+1}) : n \in \mathbb{N}\}$ is decreasing. Hence, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. If $r > 0$, letting $n \to \infty$ in (6), we have

$$\varphi(r) \leq \lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) \leq \lim_{n\to\infty} \psi(d(x_{n-1}, x_n)) \leq \psi(r),$$

which implies $r = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

To prove that $\{x_n\}$ is a Cauchy sequence, it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence in X. Suppose, to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that n_k is the smallest index satisfying $n_k > m_k > k$ and

$$d(x_{2n_k}, x_{2m_k}) \geq \varepsilon \quad \text{and} \quad d(x_{2n_k-1}, x_{2m_k}) < \varepsilon. \tag{8}$$

By the triangle inequality and (8), we get

$$\begin{aligned} \varepsilon \leq d(x_{2n_k}, x_{2m_k}) &\leq d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2m_k}) \\ &< d(x_{2n_k}, x_{2n_k-1}) + \varepsilon. \end{aligned}$$

Taking $k \to \infty$ on both sides of the above inequality, we obtain

$$\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k}) = \varepsilon. \tag{9}$$

Again, from the triangular inequality

$$\begin{aligned} d(x_{2n_k}, x_{2m_k}) &\leq d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2n_k+2}) \\ &\quad + d(x_{2n_k+2}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k}) \end{aligned}$$

and

$$\begin{aligned} d(x_{2m_k+1}, x_{2n_k+2}) &\leq d(x_{2m_k+1}, x_{2m_k}) + d(x_{2m_k}, x_{2n_k}) \\ &\quad + d(x_{2n_k}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k+2}). \end{aligned}$$

Letting $k \to \infty$ in the above two inequalities and using (7) and (9), we get

$$\lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k+2}) = \varepsilon. \tag{10}$$

In a similar way, one can also prove that

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \varepsilon. \tag{11}$$

Since $x_{2m_k} \preceq x_{2n_k+1}$ with $n_k > m_k$, by (2), we deduce

$$\begin{aligned} \varphi(d(x_{2m_k+1}, x_{2n_k+2})) &= \varphi(d(fx_{2m_k}, gx_{2n_k+1})) \\ &\leq \psi(d(x_{2m_k}, x_{2n_k+1})). \end{aligned} \tag{12}$$

Using (7), (9), (10) and (11) as $k \to \infty$ in (12), we have

$$\begin{aligned} \varphi(\varepsilon) &\leq \lim_{k\to\infty} \varphi(d(x_{2m_k+1}, x_{2n_k+2})) \\ &\leq \lim_{k\to\infty} \psi(d(x_{2m_k}, x_{2n_k+1})) \leq \psi(\varepsilon), \end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus $\{x_n\}$ is a Cauchy sequence in X. By the completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{13}$$

Suppose that the assumption (a) is satisfied. Without loss of generality, we assume that f is continuous. Then

$$z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = f\left(\lim_{n\to\infty} x_{2n}\right) = fz.$$

Thus, z is a fixed point of f, and so z is also a fixed point of g.

We now suppose that the assumption (b) holds. Then, by the regularity of $(X, \preceq, d)$ and (13), we deduce that $x_{2n} \preceq z$.

Hence, by (2)

$$\varphi\left(d\left(x_{2n+1}, gz\right)\right) = \varphi\left(d\left(fx_{2n}, gz\right)\right) \leq \psi\left(d\left(x_{2n}, z\right)\right). \tag{14}$$

Letting $n \to \infty$ in (14), we get

$$\begin{aligned}\varphi\left(d\left(z, gz\right)\right) &\leq \lim_{n\to\infty} \varphi\left(d\left(x_{2n+1}, gz\right)\right) \\ &\leq \lim_{n\to\infty} \psi\left(d\left(x_{2n}, z\right)\right) \leq \psi\left(0\right),\end{aligned}$$

implies that $z = gz$. Therefore we conclude that $z = gz = fz$.

To prove the uniqueness of common fixed point, suppose that z^* is common fixed point of f and g, too. Then, by hypothesis, z and z^* are comparable. Thus, applying (2), we obtain

$$\varphi\left(d\left(z, z^*\right)\right) = \varphi\left(d\left(fz, gz^*\right)\right) \leq \psi\left(d\left(z, z^*\right)\right), \tag{15}$$

which implies $d\left(z, z^*\right) = 0$, that is, $z = z^*$. □

Theorem 2.1. *Let $(X, \preceq, d)$ be a complete ordered metric space, $F, G : X^2 \to X$ be given mappings and the pair (F, G) be weakly increasing with two variables with respect to $\preceq$ such that*

$$\varphi\left(d\left(F\left(x, y\right), G\left(u, v\right)\right) + d\left(F\left(y, x\right), G\left(v, u\right)\right)\right) \leq \psi\left(d\left(x, u\right) + d\left(y, v\right)\right), \tag{16}$$

for all comparable $(x, y), (u, v) \in X^2$ where $\varphi \in \Phi$ and $\psi \in \Psi$. If the following conditions hold:

(a) *F (or G) are continuous or*

(b) *$(X, \preceq, d)$ is regular.*

Then F and G have a coupled common fixed point. Moreover, if (x_1, y_1) and (x_2, y_2) are comparable whenever $(x_1, y_1), (x_2, y_2) \in \mathcal{F}(F, G)$, then F and G have a unique coupled common fixed point.

Proof. According to Lemma 1.1 (1)-(3) and the inequality (16) implies

$$\varphi\left(\delta\left(T_F\left(U\right), T_G\left(V\right)\right)\right) \leq \psi\left(\delta\left(U, V\right)\right),$$

for all comparable $U, V \in X^2$ where $U = (x, y)$, $V = (u, v)$. The rest of proof follows from Lemma 2.1 and the condition (4) of Lemma 1.1 □

Remark 2.1. *Theorem 2.1 is more general than Theorem 1.1, since the contractive condition (16) is weaker than (1), a fact which is clearly illustrated by the next example.*

Example 2.1. *Let $X = \mathbb{R}^+$ be equipped with the usual metric and the partial order defined by*

$$x \preceq y \Longleftrightarrow y \leq x.$$

Define $F, G : X^2 \to X$ by $F(x, y) = G(x, y) = \frac{x+3y}{9}$. Then, it is easy to see that F and G are weakly increasing with respect to $\preceq$.

The inequality (1) does not satisfy. Indeed, assume that there exist $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t$, $\psi(t) = \frac{4t}{9}$ such that (1) holds. This means

$$\left|\frac{x+3y}{9} - \frac{u+3v}{9}\right| \leq \frac{1}{2} \cdot \frac{4}{9}\left(|x-u| + |y-v|\right), \qquad x \geq u,\ y \geq v,$$

by which, for $x = u$, we have

$$\frac{3}{9}|y-v| \leq \frac{2}{9}|y-v|, \quad y \geq v,$$

which is a contradiction for $y > v$. Hence, Theorem 1.1 can not be applied to this example.

On the other hand, contractive condition (16) is satisfied which follows from

$$\begin{aligned} d\left(F\left(x,y\right),G\left(u,v\right)\right) &= \left|\frac{x+3y}{9}-\frac{u+3v}{9}\right| \\ &\leq \frac{1}{9}\left|x-u\right|+\frac{3}{9}\left|y-v\right| \end{aligned}$$

and

$$\begin{aligned} d\left(F\left(y,x\right),G\left(v,u\right)\right) &\leq \left|\frac{y+3x}{9}-\frac{v+3u}{9}\right| \\ &\leq \frac{1}{9}\left|y-v\right|+\frac{3}{9}\left|x-u\right| \end{aligned}$$

for all $x\geq u$, $y\geq v$. Hence

$$\varphi\left(d\left(F\left(x,y\right),G\left(u,v\right)\right)+d\left(F\left(y,x\right),G\left(v,u\right)\right)\right)\leq\psi\left(d\left(x,u\right)+d\left(y,v\right)\right),$$

for all comparable $\left(x,y\right),\left(u,v\right)\in X^{2}$. Therefore, by Theorem 2.1, there exists a unique coupled common fixed point $(0,0)$ of the mappings F and G.

3. **An Application**

Consider the following integral equations:

$$\begin{aligned} x\left(s\right) &= \textstyle\int_a^b H_1\left(s,r,x\left(r\right),y\left(r\right)\right)dr \\ y\left(s\right) &= \textstyle\int_a^b H_1\left(s,r,y\left(r\right),x\left(r\right)\right)dr \end{aligned} \tag{17}$$

and

$$\begin{aligned} x\left(s\right) &= \textstyle\int_a^b H_2\left(s,r,x\left(r\right),y\left(r\right)\right)dr \\ y\left(s\right) &= \textstyle\int_a^b H_2\left(s,r,y\left(r\right),x\left(r\right)\right)dr \end{aligned} \tag{18}$$

where $s\in I=\left[a,b\right]$, $H_1,H_2:I\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $b>a\geq 0$.

In this section, we prove the existence of a common solution to integral equations (17) and (18) that belongs to $X:=C(I,\mathbb{R})$ (the set of continuous and real-valued functions defined on I) by using the results obtained in Theorem 2.1.

We consider the operators $F,G:X\times X\to X$ given by

$$F\left(x,y\right)\left(s\right)=\int_a^b H_1\left(s,r,x\left(r\right),y\left(r\right)\right)dr,\quad x,y\in X,\quad s\in I,$$

and

$$G\left(x,y\right)\left(s\right)=\int_a^b H_2\left(s,r,x\left(r\right),y\left(r\right)\right)dr,\quad x,y\in X,\quad s\in I.$$

Then the existence of a common solution to (17) and (18) is equivalent to the existence of a coupled common fixed point of F and G.

Meanwhile, X endowed with the metric d defined by

$$d(x,y)=\sup_{s\in I}\left|x\left(s\right)-y\left(s\right)\right|,$$

for all $x,y\in X$, is a complete metric space. X can also be equipped with the partial order $\preceq$ given by

$$x,y\in X,\quad x\preceq y\Leftrightarrow x(s)\leq y(s),\quad s\in I.$$

Then the corresponding metric δ on X^2 is defined by

$$\delta((x_1,y_1),(x_2,y_2))=\sup_{s\in I}\left|x_1\left(s\right)-x_2\left(s\right)\right|+\sup_{s\in I}\left|y_1\left(s\right)-y_2\left(s\right)\right|.$$

Also consider on X^2 the partial order relation:

$$\left(x_1,y_1\right)\sqsubseteq\left(x_2,y_2\right)\Longleftrightarrow x_1\left(s\right)\leq x_2\left(s\right)\quad\text{and}\quad y_1\left(s\right)\leq y_2\left(s\right),\quad s\in I.$$

Then, since $(X, \preceq, d)$ is regular, the triple $\left(X^2, \sqsubseteq, \delta\right)$ is also regular.

Suppose that the following conditions hold.

(A) $H_1, H_2 : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous,

(B) for all $s, r \in I$ and $x, y \in X$, we have

$$H_1\left(s, r, x\left(r\right), y\left(r\right)\right) \leq H_2\left(\begin{array}{c} s, r, \int_a^b H_1\left(r, \tau, x\left(\tau\right), y\left(\tau\right)\right) d\tau, \\ \int_a^b H_1\left(r, \tau, y\left(\tau\right), x\left(\tau\right)\right) d\tau \end{array}\right)$$

and

$$H_2\left(s, r, x\left(r\right), y\left(r\right)\right) \leq H_1\left(\begin{array}{c} s, r, \int_a^b H_2\left(r, \tau, x\left(\tau\right), y\left(\tau\right)\right) d\tau, \\ \int_a^b H_2\left(r, \tau, y\left(\tau\right), x\left(\tau\right)\right) d\tau \end{array}\right),$$

(C) for all comparable $(x, y), (u, v) \in X^2$ and $s, r \in I$, we have

$$\begin{aligned} &\left|H_1\left(s, r, x\left(r\right), y\left(r\right)\right) - H_2\left(s, r, u\left(r\right), v\left(r\right)\right)\right| \\ &\qquad \leq \gamma\left(s, r\right) \ln\left(1 + \left|x\left(r\right) - u\left(r\right)\right| + \left|y\left(r\right) - v\left(r\right)\right|\right), \end{aligned}$$

where $\gamma : I \times I \to \mathbb{R}^+$ is a continuous function satisfying

$$\sup_{s \in I} \int_a^b \gamma\left(s, r\right) dr \leq \frac{1}{2}.$$

Theorem 3.1. *Assume that the conditions $(A) - (C)$ are satisfied. Then, the integral equations (17) and (18) have a common solution in X^2.*

Proof. From condition (B), the pair (F, G) is weakly increasing with respect to $\preceq$.

Let $(x, y), (u, v) \in X \times X$ be comparable. Then, by (C), for all $s \in I$, we deduce

$$\begin{aligned} &\left|F\left(x, y\right)\left(s\right) - G\left(u, v\right)\left(s\right)\right| \\ &\qquad \leq \int_a^b \left|H_1\left(s, r, x\left(r\right), y\left(r\right)\right) - H_2\left(s, r, u\left(r\right), v\left(r\right)\right)\right| dr \\ &\qquad \leq \int_a^b \gamma\left(s, r\right) \ln\left(1 + \left|x\left(r\right) - u\left(r\right)\right| + \left|y\left(r\right) - v\left(r\right)\right|\right) dr \\ &\qquad \leq \int_a^b \gamma\left(s, r\right) \ln\left(1 + d\left(x, u\right) + d\left(y, v\right)\right) dr \\ &\qquad \leq \sup_{s \in I} \left(\int_a^b \gamma\left(s, r\right) dr\right) \ln\left(1 + d\left(x, u\right) + d\left(y, v\right)\right) \\ &\qquad \leq \tfrac{1}{2} \ln\left(1 + d\left(x, u\right) + d\left(y, v\right)\right). \end{aligned}$$

Therefore, we get

$$\sup_{s \in I} \left|F\left(x, y\right)\left(s\right) - G\left(u, v\right)\left(s\right)\right| \leq \frac{1}{2} \ln\left(1 + d\left(x, u\right) + d\left(y, v\right)\right). \tag{19}$$

Similarly, one can also obtain

$$\sup_{s \in I} \left|F\left(y, x\right)\left(s\right) - G\left(v, u\right)\left(s\right)\right| \leq \frac{1}{2} \ln\left(1 + d\left(y, v\right) + d\left(x, u\right)\right). \tag{20}$$

By summing up (19) and (20), we have

$$\delta\left(\left(F\left(x, y\right), F\left(y, x\right)\right), \left(G\left(u, v\right), G\left(v, u\right)\right)\right) \leq \ln\left(1 + \delta(\left(x, y\right), \left(u, v\right))\right),$$

for all comparable $(x, y), (u, v) \in X \times X$.

Defining $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t$ and $\psi(t) = \ln(1 + t)$, and using the last inequality, we deduce that the contractive condition (16) is satisfied. Therefore F and G have a coupled common fixed point by Theorem 2.1, that is, the integral equations (17) and (18) have a common solution. □

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