GLOBAL OPTIMAL SOLUTIONS OF NON-SELF MAPPINGS

Moosa Gabeleh[1]

In this paper we introduce a notion of the WP-property and study the existence of best proximity points for non-self-mappings. Our result generalizes the Mizoguchi and Takahashi's fixed point theorem for single valued mappings. Also by using the Schauder's fixed point theorem, we establish a best proximity point theorem in Banach spaces.

Keywords: Best proximity point; fixed point; uniformly convex Banach space; P-property; property UC.

MSC2010: 47H10, 47H09.

1. Introduction and preliminaries

Banach contraction principle is a very important tools in nonlinear analysis and there are many extensions of this principle; see, e.g.,[11] and the references cited therein. One of the interesting generalizations was given by Mizoguchi and Takahashi [13] as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and suppose that $T : X \longrightarrow CB(X)$, where $CB(X)$ denotes the class of all nonempty bounded closed subsets of X. Assume that*

$$H(Tx, Ty) \le \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, where H is a Hausdorff metric on $CB(X)$ and α is a function from $[0,\infty)$ to $[0, 1)$ such that $\limsup_{r\longrightarrow t^+} \alpha(r) < 1$ for all $t \in (0,\infty)$. Then T has a fixed point.

Now, consider the non-self-mapping $T : A \to X$ which A is a nonempty subset of a metric space (X, d). Clearly, the fixed point equation $Tx = x$ has not solution necessary. Hence it is contemplated to find an approximate $x \in A$ such that the error $d(x, Tx)$ is minimum. The following well-known best approximation theorem due to Kay Fan.

Theorem 1.2. *([12]) Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous mapping. Then there exists $x \in A$ such that $\|x - Tx\| = dist(Tx, A) := \inf\{\|Tx - a\| : a \in A\}$.*

A point $x \in A$ in the above theorem is called a best approximant point of T in A. The notion of best proximity point for non-self-mappings can be defined in the following way.

[1]Corresponding author, Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran, e-mail: gab.moo@gmail.com, gabelehmoosa@yahoo.com

Definition 1.1. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a non-self-mapping. A point $x^* \in A$ is called a best proximity point of T if $d(x^*, Tx^*) = dist(A, B)$, where*

$$dist(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}.$$

In fact best proximity point theorems are studied to find necessary conditions such that the minimization problem

$$\min_{x \in A} d(x, Tx), \tag{1.1}$$

has at least one solution.

Some of interesting results regarding best proximity points, can be found in [1, 4, 5, 6, 7, 8, 9, 10, 14, 17].

Let A and B be two nonempty subsets of a metric space (X, d). In this work, we adopt the following notations and definitions.

$$A_0 = \{x \in A : d(x, y) = dist(A, B), \quad \text{for some} \quad y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = dist(A, B), \quad \text{for some} \quad x \in A\}.$$

The following notion of a geometric property in metric spaces was introduced by Sankar Raj in [16].

Definition 1.2. *([16]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to has the P-property if and only if*

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 1.1.([16]) Let A, B be two nonempty closed convex subsets of a Hilbert space $\mathbb{H}$. Then (A, B) has the P-property.

Example 1.2. Let A, B be two nonempty subsets of a metric space (X, d) such that $A_0 \neq \emptyset$ and $dist(A, B) = 0$. Then (A, B) has the P-property.

Example 1.3. ([2]) Let A, B be two nonempty bounded, closed and convex subsets of a uniformly convex Banach space X. Then (A, B) has the P-property.

The following theorem establishes the existence and uniqueness of best proximity point for weakly contractive non-self-mappings in metric spaces.

Theorem 1.3. *([16]) Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping, that is $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$, for all $x, y \in A$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty), \psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. Assume that the pair (A, B) has the P-property. If $T(A_0) \subseteq B_0$, then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.*

In the next section, we introduce the notion of the WP-property which is weaker than the P-property. We then prove a version of Theorem 1.1 for single valued non-self-mappings in order to study of existence of best proximity points.

2. **Main result**

To establish our results, we introduce the following geometric property in metric spaces.

Definition 2.1. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to has the WP-property if and only if*

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that if (A, B) has the P-property, then (A, B) has the WP-property. Indeed, (A, B) has the P-property if and only both (A, B) and (B, A) have the WP-property.

Let us illustrate the above notion with the following example.

Example 2.1. Let l^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of l^∞. Suppose that

$$A := \{2e_1 + xe_3 : \frac{1}{4} \leq x \leq \frac{1}{2}\}, \quad B := \{2e_2 + ye_3 : \frac{1}{2} \leq y \leq 1\}.$$

It is clear that $dist(A, B) = 2$ and $A_0 = A$ and $B_0 = B$. Also for each $(X, Y) \in A \times B$, we have $\|X - Y\|_\infty = 2 = dist(A, B)$. Now, if $\|X - Y\|_\infty = \|\acute{X} - \acute{Y}\|_\infty = dist(A, B)$ for $(X, Y), (\acute{X}, \acute{Y}) \in A \times B$ then it is easy to see that

$$\|X - \acute{X}\|_\infty = \frac{1}{4} \text{ and } \|Y - \acute{Y}\|_\infty = \frac{1}{2}.$$

That is $\|X - \acute{X}\|_\infty < \|Y - \acute{Y}\|_\infty$, and hence (A, B) has the WP-property. Obviously, the pair (A, B) has not P-property.

The following theorem state our main result of this section.

Theorem 2.1. *Let (A, B) be a pair of two nonempty closed subset of a complete metric space (X, d) such that A_0 is nonempty. Assume that $T : A \longrightarrow B$ is a non-self-mapping such that*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \quad \text{for all } x, y \in A,$$

where α is a function from $[0, \infty)$ to $[0, 1)$ such that $\limsup_{r \longrightarrow t^+} \alpha(r) < 1$ for all $t \in (0, \infty)$. Suppose that (A, B) has the WP-property and $T(A_0) \subseteq B_0$. Then T has a unique best proximity point in A.

Proof. Choose $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. Again, since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A, B)$. Continuing this process, we can find a sequence $\{x_n\}$ in A such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad \text{for all } n \in \mathbb{N}.$$

Since (A, B) has the WP-property, we obtain

$$d(x_{n+1}, x_n) \leq d(Tx_n, Tx_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

We now have

$$d(x_{n+1}, x_n) \leq d(Tx_n, Tx_{n-1}) \leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}).$$

Since $\alpha(t) < 1$ for all $t \in [0, \infty)$, the sequence $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence in $\mathbb{R}$. Hence $\{d(x_n, x_{n+1})\}$ converges to some nonnegative real number η. Since $\limsup_{s\longrightarrow\eta^+} \alpha(s) < 1$, there exist $\delta \in [0, 1)$ and $\varepsilon > 0$ such that $\alpha(s) \leq \delta$ for all $s \in [\eta, \eta + \varepsilon]$. Let $N \in \mathbb{N}$ be such that

$$\eta \leq d(x_n, x_{n+1}) \leq \eta + \varepsilon, \quad \forall n \geq N.$$

Thus for all $n \geq N$ we must have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \\ &\delta d(x_n, x_{n-1}) \leq \delta d(Tx_{n-1}, Tx_{n-2}) \\ &\leq \delta\alpha(d(x_{n-1}, x_{n-2}))d(x_{n-1}, x_{n-2}) \\ &\leq \delta^2 d(x_{n-1}, x_{n-2}) \leq ... \leq \delta^{n-N} d(x_{n-N+1}, x_{n-N}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_n, x_{n+1}) &= \sum_{n=1}^{N} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} d(x_{N+n+1}, x_{n+N}) \\ &\leq \sum_{n=1}^{N} d(x_n, x_{n+1}) + \sum_{n=1}^{N} \delta^n d(x_{N+1}, x_N) < \infty. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in A. Since X is complete, there exists $x^* \in A$ such that $x_n \to x^*$. By the continuity of T, we conclude that $Tx_n \to Tx^*$. Hence

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = dist(A, B).$$

That is, $x^* \in A$ is a best proximity point of the mapping T. We now claim that the best proximity point of the mapping T is unique. Let $y^* \in A$ be such that $d(y^*, Ty^*) = dist(A, B)$ and $y^* \neq x^*$. Since (A, B) has the WP-property,

$$d(x^*, y^*) \leq d(Tx^*, Ty^*) \leq \alpha(d(x^*, y^*))d(x^*, y^*) < d(x^*, y^*),$$

which is a contradiction. Therefore, $y^* = x^*$. Hence the best proximity point of T is unique. □

If we consider $A = B = X$ in above theorem, then the following corollary, which is a single valued version of Mizoguchi and Takahashi's fixed point theorem, is obtained.

Corollary 2.1. *Let (X, d) be a complete metric space. Suppose $T : X \longrightarrow X$ is a mapping such that*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, where α is a function from $[0, \infty)$ to $[0, 1)$ such that $\limsup_{r\longrightarrow t^+} \alpha(r) < 1$ for all $t \in (0, \infty)$. Then T has a unique fixed point.

Example 2.2. Suppose that $X = \mathbb{R}^2$ and consider the metric d on X by $d((x, y), (\acute{x}, \acute{y})) = \max\{|x - \acute{x}|, |y - \acute{y}|\}$. Put

$$A := \{(1, 0), (1, 1), (8, 0)\}, \quad B := \{(0, -\frac{1}{2}), (0, 2), (-\frac{1}{2}, 3)\}.$$

Clearly, $dist(A, B) = 1$. Also, $A_0 = \{(1, 0), (1, 1)\}$ and $B_0 = \{(0, -\frac{1}{2}), (0, 2)\}$. Moreover, it is easy to see that the pair (A, B) has the WP-property and has not the P-property. Let $T : A \longrightarrow B$ be a mapping defined by

$$T(1, 0) = T(1, 1) = (0, 2), \quad T(8, 0) = (0, -\frac{1}{2}).$$

Then $T(A_0) \subseteq B_0$ and we note that the mapping T is a non-self-contraction. Hence T has a unique best proximity point and this point is $x^* = (1, 1)$.

The following theorem shows the existence and uniqueness of a best proximity point for expansive non-self-mappings.

Theorem 2.2. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and (B, A) satisfies the WP-property. Assume that $T : A \to B$ is an expansive non-self-mapping, that is*

$$d(Tx, Ty) \geq \alpha d(x, y),$$

for some $\alpha > 1$ and for all $x, y \in A$. If T is onto and $T(A_0) \supseteq B_0$, then T has a unique best proximity point in A.

Proof. Since T is an expansive mapping and onto, the inverse of $T : A \to B$ exists. Moreover, $T^{-1} : B \to A$ is a contraction non-self-mapping. Therefore by Theorem 2.2, T^{-1} has a unique best proximity point $y^* \in B$. Thus $d(T^{-1}y^*, y^*) = dist(A, B)$. Since T is a bijection, there exists a unique $x^* \in A$ such that $y^* = Tx^*$. We now have

$$d(x^*, Tx^*) = d(T^{-1}y^*, y^*) = dist(A, B).$$

□

Here, we can deduce the following result due to Wang et al. [18], as a corollary from the above theorem.

Corollary 2.2. *([18]) Let (X, d) be a complete metric space and $T : X \to X$ be an expansive mapping such that T is onto. Then T has a unique fixed point.*

3. Additional results

Property UC was defined in [17] in the following way.

Definition 3.1. ([17]) Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC provided if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_n d(x_n, y_n) = dist(A, B)$ and $\lim_n d(z_n, y_n) = dist(A, B)$, then $\lim_n d(x_n, z_n) = 0$.

Example 3.1.([7]) Let A and B be nonempty subsets of a uniformly convex Banach space X. Assume that A is convex. Then (A, B) satisfies the property UC.

Other examples of pairs having the property UC can be found in [17].

Very recently, Abkar and Gabeleh established the next theorem which ensures that the existence of best proximity points for nonexpansive non-self-mappings in Banach spaces.

Theorem 3.1. *([3]) Let (A, B) be a pair of nonempty closed and convex subsets of a Banach space X such that A_0 is nonempty. Let $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. If the pair (A, B) has the P-property and A is compact, then T has a best proximity point in A.*

The following theorem is another version of Theorem 3.2 in uniformly convex Banach spaces.

Theorem 3.2. *Let (A, B) be a nonempty convex pair of a uniformly convex Banach space X such that A is bounded, closed and B is compact. Assume that $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then T has a best proximity point in A.*

Proof. By Examples 1.3 and 3.1 we note that (A, B) has the P-property and satisfies the property UC. Also by Theorem 2.3 of [3] we conclude that there exists a sequence $\{x_n\}$ in A such that $\|x_n - Tx_n\| \to dist(A, B)$. Since A is bounded, closed and convex in a uniformly convex Banach space X and B is compact, we may assume that $x_n \rightharpoonup p \in A$ and $Tx_n \to q \in B$, where "$\rightharpoonup$" denotes the weak convergence. Hence,

$$\|p - q\| \leq \liminf_{n\to\infty} \|x_n - Tx_n\| = dist(A, B).$$

Then $\|p - q\| = dist(A, B)$. Also, for all $n \in \mathbb{N}$ we have

$$dist(A, B) \leq \|x_n - q\| \leq \|q - Tx_n\| + \|x_n - Tx_n\|,$$

which implies that $\|x_n - q\| \to dist(A, B)$. Thus

$$\begin{cases} \|p - q\| = dist(A, B), \\ \|x_n - q\| \to dist(A, B). \end{cases}$$

Since (A, B) satisfies the property UC, we must have $\|x_n - p\| \to 0$, or $x_n \to p$. Hence, the continuity of the mapping T implies that $Tx_n \to Tp$ and thus $Tp = q$, i.e., p is a best proximity point of T. □

Here, we recall the other geometric properties in Banach spaces as follows.

Definition 3.2. *([8]) A pair (A, B) of subsets of a metric space is said to be proximinal if for each $(x, y) \in A \times B$ there exists $(x_0, y_0) \in A \times B$ such that*

$$d(x, y_0) = d(x_0, y) = dist(A, B).$$

If, additionally, we impose the condition that the pair of points $(x_0, y_0) \in A \times B$ is unique for each $(x, y) \in A \times B$, then we say that the pair (A, B) is a sharp proximinal pair.

It was shown in [8] that when a pair of subsets (A, B) of a strictly convex Banach space is proximinal then the sets A and B also satisfy the following definition.

Definition 3.3. *([9]) Let A and B be nonempty subsets of a Banach space X. We say that A and B are proximinal parallel sets if the following two conditions are fulfilled:*
(1) (A, B) is a sharp proximinal pair.
(2) $B = A + h$ for a certain $h \in X$ such that $h = dist(A, B)$.

Using the Schauder's fixed point theorem, we prove a best proximity point theorem in Banach spaces.

Theorem 3.3. *Let (A, B) be a nonempty compact convex pair in a strictly convex Banach space X, such that (A, B) is proximinal . Suppose that $T : A \longrightarrow B$ is a continuous mapping. Then T has at least one best proximity point in A.*

Proof. Since (A, B) is proximinal and X is a strictly convex Banach space, we conclude that A and B are proximinal parallel sets. Thus $B = A + h$ for certain $h \in X$ where $\|h\| = dist(A, B)$. Now, define $S : A \longrightarrow A$ by $Sx = Tx - h$. Then S is a continuous self-map on a compact set. Hence by Schauder's fixed point theorem, S has at least one fixed point, that is, there exits $x^* \in A$ such that $Sx^* = x^*$. This implies that $Tx^* - h = x^*$ and then

$$\|Tx^* - x^*\| = \|h\| = dist(A, B).$$

Therefore, T has a best proximity point in A. $\square$

Corollary 3.1. *Let (A, B) be a nonempty bounded, closed, convex and proximinal pair in $\mathbb{R}^n$. Suppose that $T : A \longrightarrow B$ is a continuous non-self-mapping. Then T has a best proximity point in A.*

The following theorem is subsumed in Theorem 3.6.

Theorem 3.4. *Let (A, B) be a nonempty compact, convex and proximinal pair in a uniformly convex Banach space X. Suppose that $T : A \longrightarrow B$ and $S : B \longrightarrow A$ are two continuous non-self-mappings such that $\|Tx - Sy\| < \|x - y\|$ whenever $\|x - y\| > dist(A, B)$ for $x \in A$, $y \in B$. Then there exists $(x^*, y^*) \in A \times B$ such that*

$$\|x^* - Tx^*\| = \|y^* - Sy^*\| = dist(A, B), \quad \|x^* - y^*\| = dist(A, B).$$

Proof. By Theorem 3.6 there exists $(x^*, y^*) \in A \times B$ such that

$$\|x^* - Tx^*\| = \|y^* - Sy^*\| = dist(A, B).$$

Since (A, B) has the P-property, we conclude that $\|x^* - y^*\| = \|Tx^* - Sy^*\|$. Now if $\|x^* - y^*\| > dist(A, B)$, then

$$\|x^* - y^*\| = \|Tx^* - Sy^*\| < \|x^* - y^*\|,$$

which is a contradiction. Thus $\|x^* - y^*\| = dist(A, B)$. $\square$

The following common fixed point theorem for *non-commuting* continuous mappings is obtained from Theorem 3.8.

Corollary 3.2. *Let A be a nonempty convex compact set in a Banach space X. Suppose that $T : A \longrightarrow A$ and $S : A \longrightarrow A$ are two continuous self-mappings such that $\|Tx - Sy\| < \|x - y\|$, whenever x and y are distinct elements in A. Then T, S have a unique common fixed point.*

Recently Sadiq Basha established the following theorem as an application of best proximity point theory to analytic functions of a complex variable.

Theorem 3.5. *([15]) Let A and B be nonempty compact and convex subsets of a domain D of the complex plane. Let $f(z)$ and $g(z)$ be analytic functions in D. Suppose that*

(a) $f(A) \subseteq B$ and $g(B) \subseteq A$,
(b) $|f'(z)| < 1$, for all $z \in A$,
(c) $|g'(z)| < 1$, for all $z \in B$,
(d) $|f(z_1) - g(z_2)| < |z_1 - z_2|$,

whenever $|z_1 - z_2| > dist(A, B)$ for $z_1 \in A$ and $z_2 \in B$. Then there exist $z_1^* \in A$ and $z_2^* \in B$ such that

$$|z_1^* - f(z_1^*)| = |z_2^* - g(z_2^*)| = dist(A, B), \quad and \quad |z_1^* - z_2^*| = dist(A, B).$$

The following result is another version of Theorem 3.10.

Theorem 3.6. *Let (A, B) be a nonempty compact, convex and proximinal pair of a domain D of the complex plane. Let $f(z)$ and $g(z)$ be continuous functions in D. Suppose that*

(a) $f(A) \subseteq B$ and $g(B) \subseteq A$,

(*b*) $|f(z_1) - g(z_2)| < |z_1 - z_2|$,
whenever $|z_1 - z_2| > dist(A, B)$ *for* $z_1 \in A$ *and* $z_2 \in B$*. Then there exist* $z_1^* \in A$ *and* $z_2^* \in B$ *such that*

$$|z_1^* - f(z_1^*)| = |z_2^* - g(z_2^*)| = dist(A, B), \quad and \quad |z_1^* - z_2^*| = dist(A, B).$$

Proof. It is easy to see that the result follows by invoking Theorem 3.8. □

REFERENCES

[1] A. Abkar, M. Gabeleh, *Generalized cyclic contractions in partially ordered metric spaces*, Optim Lett., **In Press** DOI DOI 10.1007/s11590-011-0379-y.

[2] A. Abkar, M. Gabeleh, *Global optimal solutions of noncyclic mappings in metric spaces*, J. Optim. Theory Appl., **153** (2012), 298-305.

[3] A. Abkar, M. Gabeleh, *Best proximity points of non-self mappings*, Top, **In Press** DOI 10.1007/s11750-012-0255-7.

[4] M.A. Al-Thagafi, N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Analysis, **70** (2009), 3665-3671.

[5] M. Derafshpour, S. Rezapour, N. Shahzad, *Best Proximity Points of cyclic φ-contractions in ordered metric spaces*, Topological Methods in Nonlinear Analysis, **37** (2011), 193-202.

[6] C. Di Bari, T. Suzuki, C. Vetro, *Best proximity points for cyclic Meir- Keeler contractions*, Nonlinear Anal., **69** (2008), 3790-3794.

[7] A. Eldred, P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001-1006.

[8] R. Espinola, *A new approach to relativelt nonexpansive mappings*, Proc. Amer. Math. Soc., **136** (2008), 1987-1996.

[9] R. Espinola, A. Fernandez-Leon, *On best proximity points in metric and Banach spaces*, Canad. J. Math. **63** (2011), 533-550.

[10] F. Giannessi, A. Maugeri, P.M. Pardalos, *Equilibrium Problems: nonsmooth optimization and variational inequality models*, Nonconvex Optim. Appl., **58** (2004), DOI 10.1007/b101835.

[11] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics,Wiley-Interscience, New York, USA, 2001.

[12] K. Fan, *Extensions of two fixed point theorems of F.E. Browder*, Math. Z. **122** (1969), 234-240.

[13] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141** (1989), 177-188.

[14] P.M. Pardalos, T.M. Rassias, A.A. Khan, *Equilibrium problems: nonlinear analysis and variational problems in honor of George Isac*, Springer Optim. Appl., **38** (2010), DOI 10.1007/978-1-4419-0158-3.

[15] S. Sadiq Basha, *Best proximity points: global optimal approximate solutions*, J. Optim. Theory Appl., **49** (2011), 15-21.

[16] V. Sankar Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., **74** (2011), 4804-4808.

[17] T. Suzuki, M. Kikkawa, C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71** (2009), 2918-2926.

[18] S.Z. Wang, B.Y. Li, Z.M. Gao, *Some fixed point theorems on expansion mappings*, Adv. Math. (China), **11**(2) (1982), 149-153.