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ON LEFT ϕ-BIPROJECTIVITY AND LEFT ϕ-BIFLATNESS OF CERTAIN BANACH ALGEBRAS

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In this paper, we study left ϕ-biflatness and left ϕ-biprojectivity of some Banach algebras, where ϕ is a non-zero multiplicative linear function. We show that if the Banach algebra A^{**} is left ϕ-biprojective, then A is left ϕ-biflat. Using this tool we study left ϕ-biflatness of some matrix algebras. We also study left ϕ-biflatness and left ϕ-biprojectivity of the projective tensor product of some Banach algebras related to a locally compact group. We prove that for a locally compact group G, $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is finite. We show that $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is compact.

Keywords: Left ϕ-biflatness, Left ϕ-biprojectivity, Banach algebras, Locally ompact groups.

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1. Introduction and Preliminaries

Banach homology theory has two important notions, biflatness and biprojectivity which these notions have key role in studying the structure of Banach algebras. A Banach algebra A is called biflat (biprojective), if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ ($\rho : A \to A \otimes_p A$) such that $\pi_A^{**} \circ \rho$ is the canonical embedding of A into A^{**} (ρ is a right inverse for π_A), respectively. It is well-known that for a locally compact group G, the group algebra $L^1(G)$ is biflat (biprojective) if and only if G is amenable (compact), respectively. We have to mention that a biflat Banach algebra A with a bounded approximate identity is amenable and vise versa, see [13].

A Banach algebra A is called left ϕ-amenable, if there exists a bounded net (a_α) in A such that $aa_\alpha - \phi(a)a_\alpha \to 0$ and $\phi(a_\alpha) \to 1$ for all $a \in A$, where $\phi \in \Delta(A)$. For a locally compact group G, the Fourier algebra $A(G)$ is always left ϕ-amenable. Also the group algebra $L^1(G)$ is left ϕ-amenable if and only if G is amenable, for further information see [8] and [1].

Following this course, Essmaili et. al. in [3] introduced and studied a biflat-like property related to a multiplicative linear functional, they called it condition W (which we call it here right ϕ-biflatness). The Banach algebra A is called left ϕ-biflat, if there exists a bounded linear map $\rho : A \to (A \otimes_p A)^{**}$ such that

$$\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b)$$

and

$$\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a),$$

for each $a, b \in A$. We followed their work and showed that the Segal algebra $S(G)$ is left ϕ-biflat if and only if G is amenable see [15]. Also we defined a notion of left ϕ-biprojectivity

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for Banach algebras. In fact A Banach algebra is left ϕ-biprojective if there exists a bounded linear map $\rho : A \to A \otimes_p A$ such that

$$\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a), \quad \phi \circ \pi_A \circ \rho(a) = \phi(a), \quad (a, b \in A).$$

We showed that the Lebesgue-Fourier algebra $LA(G)$ is left ϕ-biprojective if and only if G is compact. Also the Fourier algebra $A(G)$ is left ϕ-biprojective if and only if G is discrete, see [16].

In this paper, we show that if the Banach algebra A^{**} is left ϕ-biprojective, then A is left ϕ-biflat. Using this tool we study left ϕ-biflatness of some matrix algebras. We also study left ϕ-biflatness and left ϕ-biprojectivity of the projective tensor product of some Banach algebras. We prove that for a locally compact group G, $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is finite. We show that $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is compact.

We remark some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. If X is a Banach A-bimodule, then X^* is also a Banach A-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*).$$

Throughout, the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let A be a Banach algebra. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \ (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

For Banach algebras A and B with $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$, we denote $\phi \otimes \psi$ for a multiplicative linear functional on $A \otimes_p B$ given by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for each $a \in A$ and $b \in B$. The product morphism $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$, for every $a, b \in A$. Let X and Y be Banach A-bimodules. The map $T : X \to Y$ is called A-bimodule morphism, if

$$T(a \cdot x) = a \cdot T(x), \quad T(x \cdot a) = T(x) \cdot a, \qquad (a \in A, x \in X).$$

For the Banach spaces E and F, the weak star operator topology on $B(E, F^*)$ (the set of all bounded linear operators from E into F^*) is the locally convex topology given by the seminorms $\{|| \cdot ||_{e,f} : e \in E, f \in F\}$, where $||T||_{e,f} = | < f, T(e) > |$ and $T \in B(E, F^*)$. We have to remind that the weak star operator topology on $B(E, F^*)$ is exactly the w^*-topology on $B(E, F^*)$ when identified with $(E \otimes_p F)^*$. Note that every bounded net in $B(E, F^*)$ has a weak star operator topology-limit point in $B(E, F^*)$.

2. **Some general properties**

Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called approximate left ϕ-biprojective if there exists a net of bounded linear maps from A into $A \otimes_p A$, say $(\rho_\alpha)_{\alpha \in I}$, such that

(i) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \xrightarrow{||\cdot||} 0$,
(ii) $\rho_\alpha(ba) - \phi(a)\rho_\alpha(b) \xrightarrow{||\cdot||} 0$,
(iii) $\phi \circ \pi_A \circ \rho_\alpha(a) - \phi(a) \to 0$,

for every $a, b \in A$, see [14].

Proposition 2.1. *Let A be a left ϕ-biflat Banach algebra. Then A is approximate left ϕ-biprojective.*

Proof. Since A is left ϕ-biflat, there exists a bounded linear map $\rho : A \to (A \otimes_p A)^{**}$ such that $\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a)$ and $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$. Since $\rho \in B(A, (A \otimes_p A)^{**})$, there exists a net $\rho_\alpha \in B(A, A \otimes_p A)$ such that $\rho_\alpha \xrightarrow{W^*OT} \rho$. Thus for each $a \in A$ we have $\rho_\alpha(a) \xrightarrow{w^*} \rho(a)$. Then

$$a \cdot \rho_\alpha(b) \xrightarrow{w^*} a \cdot \rho(b) = \rho(ab), \quad \rho_\alpha(ab) \xrightarrow{w^*} \rho(ab), \quad \phi(b)\rho_\alpha(a) \xrightarrow{w^*} \phi(b)\rho(a) = \rho(ab).$$

On the other hand, the map π_A^{**} is a w^*-continuous map, so $\pi_A^{**} \circ \rho_\alpha(a) \xrightarrow{w^*} \pi_A^{**} \circ \rho(a)$, for each $a \in A$. Then

$$\phi \circ \pi_A \circ \rho_\alpha(a) = \tilde{\phi} \circ \pi_A^{**} \circ \rho_\alpha(a) = \pi_A^{**} \circ \rho_\alpha(a)(\phi) \to \pi_A^{**} \circ \rho(a)(\phi) = \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a).$$

Also for each $a, b \in A$, we have

$$a \cdot \rho_\alpha(b) \xrightarrow{w^*} a \cdot \rho(b) = \rho(ab), \quad \rho_\alpha(ab) \xrightarrow{w^*} \rho(ab), \quad \phi(b)\rho_\alpha(a) \xrightarrow{w^*} \phi(b)\rho(a).$$

So

$$a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \xrightarrow{w^*} 0, \quad \phi(b)\rho_\alpha(a) - \phi(b)\rho_\alpha(a) \xrightarrow{w^*} 0.$$

Put $F = \{a_1, a_2, ..., a_n\}$ and $G = \{b_1, b_2, ..., b_n\}$ for finite subsets of A. Define

$$M = \{(a_1 \cdot T(b_1) - T(a_1 b_1), a_2 \cdot T(b_2) - T(a_2 b_2), ..., a_n \cdot T(b_n) - T(a_n b_n)) : T \in B(A, A \otimes_p A)\}.$$

It is easy to see that M is a convex subset of $\prod_{i=1}^n (A \otimes_p A) \oplus_1 \prod_{i=1}^n \mathbb{C}$ and $(0, 0, ..., 0) \in \overline{M}^w = \overline{M}^{||\cdot||}$. It follows that, there exists a net $\xi_{(\epsilon,F,G)} \in B(A, A \otimes_p A)$ such that

$$||a_i \cdot \xi_{(\epsilon,F,G)}(b_i) - \xi_{(\epsilon,F,G)}(a_i b_i)|| < \epsilon, \quad ||\xi_{(\epsilon,F,G)}(a_i b_i) - \phi(b_i)\xi_{(\epsilon,F,G)}(a_i)|| < \epsilon$$

and $|\phi \circ \pi_A \circ \xi_{(\epsilon,F,G)}(a_i) - \phi(a_i)| < \epsilon$, for each $i \in \{1, 2, ..., n\}$. It follow that the net $(\xi_{(\epsilon,F,G)})$, for each $a, b \in A$, satisfies

$$a \cdot \xi_{(\epsilon,F,G)} - \xi_{(\epsilon,F,G)}(ab) \to 0, \quad \phi(b)\xi_{(\epsilon,F,G)}(a) - \xi_{(\epsilon,F,G)}(ab) \to 0$$

and

$$\phi \circ \pi_A \circ \xi_{(\epsilon,F,G)}(a) - \phi(a) \to 0.$$

Therefore A is approximately left $\phi-$biprojective. □

The converse of the above proposition is partially valid:

Lemma 2.1. *If A is an approximately left ϕ-biprojective with bounded net ρ_α, then A is left ϕ-biflat.*

Proof. Let A be approximately left ϕ-biprojective with bounded net ρ_α. So $\rho_\alpha \in B(A, (A \otimes_p A)^{**}) \cong (A \otimes_p (A \otimes_p A)^*)^*$ has a w^*-limit-point, say ρ. Since

$$a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \to 0, \quad \phi(b)\rho_\alpha(a) - \rho_\alpha(ab) \to 0, \quad \phi \circ \pi_A \circ \rho_\alpha(a) - \phi(a) \to 0.$$

Note that for each $a \in A$, $\rho_\alpha(a) \xrightarrow{w^*} \rho(a)$. It follows that

$$a \cdot \rho(b) = \rho(ab) = \phi(b)\rho(a), \quad \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a),$$

for each $a \in A$. □

Example 2.1. *We give a Banach algebra which is not left ϕ-biflat but it is approximate left ϕ-biprojective. So the converse of Proposition 2.1 is not always true. Let denote ℓ^1 for the set of all sequences $a = ((a_n))$ of complex numbers equipped with $||a|| = \sum_{n=1}^\infty |a_n| < \infty$ as its norm. With the following product:*

$$(a * b)(n) = \begin{cases} a(1)b(1) & \textit{if} \quad n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \textit{if} \quad n > 1, \end{cases}$$

$A = (\ell^1, ||\cdot||)$ becomes a Banach algebra. Clearly $\Delta(\ell^1) = \{\phi_1, \phi_1 + \phi_n\}$, where $\phi_n(a) = a(n)$ for every $a \in \ell^1$. We claim that ℓ^1 is not left ϕ_1-biflat but ℓ^1 is approximately left $\tilde{\phi}_1$-biprojective for some $\phi \in \Delta(\ell^1)$. We assume conversely that ℓ^1 is left ϕ_1-biflat. One can see that $(1, 0, 0, ...)$ is a unit for ℓ^1. Therefore by [15, Lemma 2.1] *left ϕ_1-biflatness of ℓ^1 implies that ℓ^1 is left ϕ_1-amenable. On the other hand by* [9, Example 2.9] *ℓ^1 is not left ϕ_1-amenable which is a contradiction.*

Applying [9, Example 2.9], *gives that ℓ^1 is approximate left ϕ_1-amenable. So* [14, Proposition 2.4] *follows that that ℓ^1 is approximate left ϕ_1-biprojective.*

Proposition 2.2. *Let A be a Banach algebra with an approximate identity and let $\phi \in \Delta(A)$. If A^{**} is approximately biflat, then A is left ϕ-biflat.*

Proof. Since A has an approximate identity $\overline{A\ker\phi} = \ker\phi$. Thus by [11, Theorem 3.3] A is left $\phi-$amenable. So there exists an element $m \in A^{**}$ such that $am = \phi(a)m$ and $\tilde{\phi}(m) = 1$ for every $a \in A$. Define $\rho : A \to A^{**} \otimes_p A^{**}$ by $\rho(a) = \phi(a)m \otimes m$. Clearly ρ is a bounded linear map such that

$$a \cdot \rho(b) = \rho(ab) = \phi(b)\rho(a), \quad \tilde{\phi} \circ \pi_{A^{**}} \circ \rho(a) = \phi(a), \quad (a \in A).$$

There exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m)$,

see [4, Lemma 1.7]. Set $\eta = \psi \circ \rho : A \to (A \otimes_p A)^{**}$. It is easy to see that $a \cdot \eta(b) = \eta(ab) = \phi(b)\eta(a)$

$$\tilde{\phi} \circ \pi_A^{**} \circ \eta(a) = \tilde{\phi} \circ \pi_{A^{**}} \circ \psi \circ \rho(a) = \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a), \quad (a \in A).$$

So A is left ϕ-biflat. □

Proposition 2.3. *Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. Let A^{**} be left $\tilde{\phi}$-biprojective. Then A is left ϕ-biflat.*

Proof. Let A^{**} be $\tilde{\phi}$-biprojective. So we have a bounded linear map $\rho : A^{**} \to A^{**} \otimes_p A^{**}$ such that $\rho(ab) = a \cdot \rho(b) = \tilde{\phi}(b)\rho(a)$ and $\tilde{\phi} \circ \pi_{A^{**}} \circ \rho(a) = \tilde{\phi}(a)$, for each $a, b \in A^{**}$. Let ψ be a bounded linear map as in the proof of previous proposition. Set $\eta = \psi \circ \rho|_A : A \to (A \otimes_p A)^{**}$. Clearly η is a bounded linear map which satisfies

$$\eta(ab) = \psi \circ \rho|_A(ab) = \psi(a \cdot \rho|_A(b)) = a \cdot \psi \circ \rho|_A(b)$$

and

$$\phi(b)\eta(a) = \phi(b)\psi \circ \rho|_A(a) = \psi(\phi(b)\rho|_A) = \psi \circ \rho|_A(ab) = \eta(ab).$$

Also we have

$$\tilde{\phi} \circ \pi_A^{**} \circ \eta(a) = \tilde{\phi} \circ \pi_A^{**} \circ \psi \circ \rho|_A(a) = \tilde{\phi} \circ \pi_{A^{**}} \circ \rho|_A(a) = \phi(a),$$

for each $a \in A$. It follows that A is left ϕ-biflat. □

Let A be a Banach algebra and I be a totally ordered set. By $UP_I(A)$ we denote the set of $I \times I$ upper triangular matrices which its entries come from A and

$$||(a_{i,j})_{i,j \in I}|| = \sum_{i,j \in I} ||a_{i,j}|| < \infty.$$

With matrix operations and $||\cdot||$ as a norm, $UP_I(A)$ becomes a Banach algebra. Let $\phi\in\Delta(A)$ and i_0 be the greatest element of I. Define $\psi_\phi(a_{i,j})=\phi(a_{i_0,i_0})$. Clearly ψ_ϕ is a character on $UP_I(A)$.

Proposition 2.4. *Let I be a totally ordered set with the greatest element. Also let A be a Banach algebra with left identity and $\phi\in\Delta(A)$. Then $UP_I(A)^{**}$ is left ψ_ϕ-biflat if and only if $|I|=1$ and A is left ϕ-biflat.*

Proof. Suppose $UP_I(A)^{**}$ is left ψ_ϕ-biflat. Let $i_0\in I$ be the greatest element of I with respect to $\geq$. Since A has a left unit, $UP_I(A)$ has a left approximate identity. By [15, Lemma 2.1] left ψ_ϕ-amenability of $UP_I(A)^{**}$ implies that $UP_I(A)$ is left ψ_ϕ-amenable. Define

$$J=\{(a_{i,j})_{i,j\in I}\in UP_I(A)|a_{i,j}=0\quad\text{for}\quad j\neq i_0\}.$$

Clearly J is a closed ideal of $UP_I(A)$ with $\psi_\phi|_J\neq 0$. Applying [6, Lemma 3.1] gives that J is left ψ_ϕ-amenable. So by [6, Theorem 1.4] there exists a bounded net (j_α) in J which satisfies

$$jj_\alpha-\psi_\phi(j)j_\alpha\to 0,\quad \psi_\phi(j_\alpha)=1\quad (j\in J).\tag{1}$$

Suppose in contradiction that I has at least two elements. Let a_0 be an element in A such that $\phi(a_0)=1$. Set $j=\begin{pmatrix}\cdots&0&\cdots&0&a_0\\ \cdots&0&\cdots&0&a_0\\ :&:&:&:&:\\ \cdots&0&\cdots&0&a_0\\ :&:&:&:&0\end{pmatrix}$. Clearly for each α the net j_α has a form $\begin{pmatrix}\cdots&0&\cdots&0&j_i^\alpha\\ \cdots&0&\cdots&0&\cdots\\ :&:&:&:&:\\ \cdots&0&\cdots&0&j_k^\alpha\\ :&:&:&:&j_{i_0}^\alpha\end{pmatrix}$, where $(j_i^\alpha),(j_k^\alpha)$ and $(j_{i_0}^\alpha)$ are some nets in A. Put j and j_α in (1) we have $j_{i_0}^\alpha a_0\to 0$. Since ϕ is continuous, we have $\phi(j_{i_0}^\alpha)\to 0$. On the other hand $\psi_\phi(j_\alpha)=\phi(j_{i_0}^\alpha)=1$ which is a contradiction. So I must be singleton and the proof is complete. □

Corollary 2.1. *Let I be a totally ordered set with the greatest element. Also let A be a Banach algebra with left identity and $\phi\in\Delta(A)$. If $UP_I(A)^{**}$ is approximately biflat, then $|I|=1$ and A is approximately biflat.*

3. Left ϕ-biprojectivity of the projective tensor product Banach algebras

Theorem 3.1. *Let A and B be Banach algebras which $\phi\in\Delta(A)$ and $\psi\in\Delta(B)$. Suppose that A has a unit and B has an idempotent x_0 such that $\psi(x_0)=1$. If $A\otimes_p B$ is left $\phi\otimes\psi$-biflat, then A is left ϕ-amenable.*

Proof. Let $\rho:A\otimes_p B\to((A\otimes_p B)\otimes_p(A\otimes_p B))^{**}$ be a bounded linear map such that

$$\rho(xy)=x\cdot\rho(y)=\widetilde{\phi\otimes\psi}(y)\rho(x),\quad \widetilde{\phi\otimes\psi}\circ\pi^{**}_{A\otimes_p B}\circ\rho(x)=\phi\otimes\psi(x)\quad (x,y\in A\otimes_p B).$$

For idempotent $x_0\in B$ and elements $a_1,a_2\in A$ we have

$$a_1a_2\otimes x_0=a_1a_2\otimes x_0=a_1a_2\otimes x_0^2=(a_1\otimes x_0)(a_2\otimes x_0).$$

We denote e for the unit of A. So we have

$$\begin{aligned}\rho(a_1a_2\otimes x_0)=\rho((a_1\otimes x_0)(a_2\otimes x_0))&=(a_1\otimes x_0)\cdot\rho(a_2\otimes x_0)\\&=a_1(e\otimes x_0)\cdot\rho(a_2\otimes x_0)\\&=a_1\rho(ea_2\otimes x_0^2),\end{aligned}$$

also

$$\rho(a_1a_2\otimes x_0)=\rho((a_1\otimes x_0)(a_2\otimes x_0))=\phi\otimes\psi(a_2\otimes x_0)\rho(a_1\otimes x_0)=\phi(a_2)\rho(a_1\otimes x_0)$$

and

$$\widetilde{\phi\otimes\psi}\circ\pi^{**}_{A\otimes_p B}\circ\rho(a_1\otimes x_0)=\phi\otimes\psi(a_1\otimes x_0)=\phi(a_1),$$

for each $a_1,a_2\in A$. Put $\xi:(A\otimes_p B)\otimes_p(A\otimes_p B)\to A\otimes_p A$ for a bounded linear map which is given by $\xi((a\otimes b)\otimes(c\otimes d)=\psi(bd)a\otimes c$, for each $a,c\in A$ and $b,d\in B$. Clearly

$$\pi^{**}_A\circ\xi^{**}=(id_A\otimes\psi)^{**}\circ\pi^{**}_{A\otimes_p B}.$$

Define $\theta:A\to(A\otimes_p A)^{**}$ by $\theta(a)=\xi^{**}\circ\rho(a\otimes x_0)$. Clearly θ is a bounded linear map. We have

$$a\cdot\theta(b)=a\cdot\xi^{**}\circ\rho(b)=\xi^{**}\circ\rho(ab)=\phi(b)\xi^{**}\circ\rho(a)=\phi(b)\theta(a),\quad (a,b\in A).$$

Also

$$\begin{aligned}\tilde{\phi}\circ\pi^{**}_A\circ\theta(a)=\tilde{\phi}\circ\pi^{**}_A\circ\xi^{**}\circ\rho(a\otimes x_0)&=\tilde{\phi}\circ(id_A\otimes\psi)^{**}\circ\pi^{**}_{A\otimes_p B}\circ\rho(a\otimes x_0)\\&=\widetilde{\phi\otimes\psi}\circ\pi^{**}_{A\otimes_p B}\circ\rho(a\otimes x_0)\\&=\phi\otimes\psi(a\otimes x_0)=\phi(a),\end{aligned}$$

for each $a\in A$. It follows that A is left ϕ-biflat. Since A has a unit by [15, Lemma 2.1] A is left ϕ-amenable. □

Note that the previous theorem is also valid in the left ϕ-biprojective case. In fact we have the following corollary which we omit its proof.

Corollary 3.1. *Let A and B be Banach algebras which $\phi\in\Delta(A)$ and $\psi\in\Delta(B)$. Suppose that A has a unit and B has an idempotent x_0 such that $\psi(x_0)=1$. If $A\otimes_p B$ is left $\phi\otimes\psi$-biprojective, then A is left ϕ-contractible.*

Theorem 3.2. *Let A and B be Banach algebra with $\phi\in\Delta(A)$ and $\psi\in\Delta(B)$. If A left ϕ-biprojective and B is ψ-biprojective, then $A\otimes_p B$ is left $\phi\otimes\psi$-biprojective.*

Proof. Since A left ϕ-biprojective and B is ψ-biprojective, there exist bounded linear map $\rho_A:A\to A\otimes_p A$ and $\rho_B:B\to B\otimes_p B$ such that

$$\rho_A(a_1a_2)=a_1\cdot\rho_A(a_2)=\phi(a_2)\rho_A(a_1),\quad \phi\circ\pi_A\circ\rho_A=\phi,\quad (a_1,a_2\in A)$$

and

$$\rho_B(b_1b_2)=b_1\cdot\rho_B(b_2)=\phi(b_2)\rho_B(b_1),\quad \psi\circ\pi_B\circ\rho_B=\psi,\quad (b_1,b_2\in B).$$

Let θ be an isometrical isomorphism from $(A\otimes_p A)\otimes_p(B\otimes_p B)$ into $(A\otimes_p B)\otimes_p(A\otimes_p B)$ which is given by $\theta(a_1\otimes a_2\otimes b_1\otimes b_2)=a_1\otimes b_1\otimes a_2\otimes b_2$ for each $a_1,a_2\in A$ and $b_1,b_2\in B$. Define $\rho=\theta\circ(\rho_A\otimes\rho_B)$. So

$$\begin{aligned}\rho((a_1\otimes b_1)(a_2\otimes b_2))&=\theta\circ(\rho_A\otimes\rho_B)((a_1\otimes b_1)(a_2\otimes b_2))\\&=\theta(\rho_A(a_1a_2)\otimes\rho_B(b_1b_2)\\&=\theta(a_1\cdot\rho_A(a_2)\otimes b_1\cdot\rho_B(b_2))\\&=\theta((a_1\otimes b_1)\cdot(\rho_A(a_2)\otimes\rho_B(b_2))\\&=(a_1\otimes b_1)\cdot\theta\circ(\rho_A\otimes\rho_B)(a_2\otimes b_2),\end{aligned}$$

for each $a_1, a_2 \in A$ and $b_1, b_2 \in B$. It follows that $\rho(xy) = x \cdot \rho(y)$ for each $x, y \in A \otimes_p B$. Also we have

$$\begin{aligned}\phi \otimes \psi(a_1 \otimes b_1)\rho(a_2 \otimes b_2) &= \phi(a_1)\psi(b_1)\theta \circ (\rho_A(a_2) \otimes \rho_B(b_2)) \\ &= \theta \circ (\phi(a_1)\rho_A(a_2) \otimes \psi(b_1)\rho_B(b_2)) \\ &= \theta \circ (\rho_A(a_2a_1) \otimes \rho_B(b_2b_1)) \\ &= \rho((a_2 \otimes b_2)(a_1 \otimes b_1),\end{aligned}$$

for each $a_1, a_2 \in A$ and $b_1, b_2 \in B$. So for each $x, y \in A \otimes_p B$, we have $\phi \otimes \psi(x)\rho(y) = \rho(yx)$. Note that

$$\pi_{A\otimes_p B} \circ \theta(a_1 \otimes a_2 \otimes b_1 \otimes b_2) = \pi_{A\otimes_p B}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = \pi_A(a_1 \otimes a_2)\pi_B(b_1 \otimes b_2),$$

it implies that $\pi_{A\otimes_p B} \circ \theta = \pi_A \otimes \pi_B$. Then

$$\begin{aligned}(\phi \otimes \psi) \circ \pi_{A\otimes_p B} \circ \rho(a \otimes b) &= (\phi \otimes \psi) \circ \pi_{A\otimes_p B} \circ \theta \circ (\rho_A \otimes \rho_B)(a \otimes b) \\ &= (\phi \otimes \psi) \circ (\pi_A \otimes \pi_B) \circ (\rho_A \otimes \rho_B)(a \otimes b) \\ &= \phi \circ \pi_A \circ \rho_A(a)\psi \circ \pi_B \circ \rho_B(b) \\ &= \phi(a)\psi(b) = \phi \otimes \psi(a \otimes b),\end{aligned}$$

for each $a \in A$ and $b \in B$. Therefore $(\phi \otimes \psi) \circ \pi_{A\otimes_p B} \circ \rho(x) = \phi \otimes \psi(x)$ for every $x \in A \otimes_p B$. It follows that $A \otimes_p B$ is left $\phi \otimes \psi$-biprojective. □

Let $\hat{G}$ be the dual group of G which consists of all non-zero continuous homomorphism $\rho : G \to \mathbb{T}$. It is well-known that every character (multiplicative linear functional) $\phi \in \Delta(L^1(G))$ has the form $\phi_\rho(f) = \int_G \overline{\rho(x)} f(x)dx$, where dx is the normalized Haar measure and $\rho \in \hat{G}$, for more details see [5, Theorem 23.7]. Note that, since $L^1(G)$ is a closed ideal of the measure algebra $M(G)$, each character on $L^1(G)$ can be extended to $M(G)$. For a locally compact group G, we denote $A(G)$ for the Fourier algebra. The character space $\Delta(A(G))$ consists of all point evaluations ϕ_x for each $x \in G$, where

$$\phi_x(f) = f(x), \quad (f \in A(G)),$$

see[6, Example 2.6].

Theorem 3.3. *Let G be a locally compact group. Then $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is finite, where $\phi \in \Delta(L^1(G))$ and $\psi \in \Delta(A(G))$.*

Proof. Let $M(G) \otimes_p A(G)$ be left $\phi \otimes \psi$-biprojective. Also let e be the unit of $M(G)$ and a_0 be the element of $A(G)$ such that $\psi(a_0) = 1$. Put $x_0 = e \otimes a_0$. Clearly $xx_0 = x_0x$ and $\phi \otimes \psi(x_0) = 1$, for every $x \in M(G) \otimes_p A(G)$. Now applying [16, Lemma 2.2] $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-contractible. Now using [10, Theorem 3.14] $M(G)$ is left ϕ-contractible, so by [10, Theorem 6.2] G is compact. Also by [10, Theorem 3.14] $A(G)$ is left ψ-contractible. Thus by [10, Proposition 6.6] G is discrete. Therefore G is finite.

Converse is clear. □

Theorem 3.4. *Let G be a locally compact group. Then $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is compact, where $\phi, \psi \in \Delta(L^1(G))$.*

Proof. Suppose that $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective. Let e be the unit of $M(G)$ and e_α be a bounded approximate identity of $L^1(G)$. Clearly $e \otimes e_\alpha$ is a bounded approximate identity. Thus by [16, Lemma 2.2] $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-contractible. So [10, Theorem 3.14] $L^1(G)$ is left ψ-contractible. Then by [10, Theorem 6.2] G is compact.

For converse, suppose that G is compact. Then by [10, Theorem 3.14] M(G) is left ϕ-contractible and by [10, Theorem 3.14] $L^1(G)$ is left ψ-contractible. Applying [10, Theorem

3.14] $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-contractible. So by [16, Lemma 2.1] $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective. □

A Banach algebra A is called left character biprojective (left character biflat) if A is left ϕ-biprojective (if A is left ϕ-biflat) for each $\phi \in \Delta(A)$, respectively.

Theorem 3.5. *Let G be a locally compact group. Then $M(G) \otimes_p L^1(G)$ is left character biprojective if and only if G is finite.*

Proof. Let $M(G) \otimes_p L^1(G)$ be left character biprojective. So $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective for each $\phi \in \Delta(M(G))$ and $\psi \in \Delta(L^1(G))$. So by similar arguments as in previous theorem, $M(G)$ left ϕ-contractible for each $\phi \in \Delta(M(G))$. Since $M(G)$ is unital, by [10, Corollary 6.2] G is finite.

Converse is clear. □

A Banach algebra A is amenable if and only if A has a bounded virtual diagonal, that is there exists a bounded net $m_\alpha \in (A \otimes_p A)$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$ for each $a \in A$, see [13].

Theorem 3.6. *Let G be a locally compact group. Then $M(G) \otimes_p L^1(G)$ is left character biflat if and only if G is a discrete amenable group.*

Proof. Since $M(G)$ is unital and $L^1(G)$ has a bounded approximate identity, $M(G) \otimes_p L^1(G)$ has a bounded approximate identity. Thus by [15, Lemma 2.1] $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-amenable for each $\phi \in \Delta(M(G))$ and $\psi \in \Delta(L^1(G))$. So by [6, Theorem 3.3] $M(G)$ is left ϕ-amenable for each $\phi \in \Delta(M(G))$. Since $M(G)$ is unital, $M(G)$ character amenable. Therefore by the main result of [8], G is discrete and amenable.

For converse, let G be discrete and amenable. Then $M(G) \otimes_p L^1(G) = \ell^1(G) \otimes_p \ell^1(G)$. Applying Johnson's theorem (see [13, Theorem 2.1.18]) that $\ell^1(G)$ is an amenable Banach algebra. Then $\ell^1(G) \otimes_p \ell^1(G)$ is amenable. Therefore $\ell^1(G) \otimes_p \ell^1(G)$ is left ϕ-amenable for all $\phi \in \Delta(\ell^1(G) \otimes_p \ell^1(G))$. Using similar arguments as in the proof of [15, Theorem 2.2] $\ell^1(G) \otimes_p \ell^1(G)$ is left ϕ-biflat for every $\phi \in \Delta(\ell^1(G) \otimes_p \ell^1(G))$. Then $\ell^1(G) \otimes_p \ell^1(G)$ is left character biflat. □

Proposition 3.1. *Let G be an amenable group. Then $A(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is finite.*

Proof. Since G is amenable, Leptin's Theorem [13, Theorem 7.1.3] gives that $A(G)$ has a bounded approximate identity. It is well-known that $L^1(G)$ has a bounded approximate identity. So $A(G) \otimes_p L^1(G)$ has a bounded approximate identity. Then by [16, Proposition 2.4], left $\phi \otimes \psi$-biprojectivity of $A(G) \otimes_p L^1(G)$ implies that $A(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-contractible. So using [10, Theorem 3.14] gives that $A(G)$ is left ϕ-contractible. Then by [10, Proposition 6.6] G is discrete. Also [10, Theorem 3.14] gives that $L^1(G)$ is left ψ-contractible. Then [10, Theorem 6.1] implies that G is compact. It follows that G is finite.

Converse is clear. □

Proposition 3.2. *Let G be a locally compact group. Then $A(G) \oplus_1 L^1(G)$ is left character biprojective if and only if G is finite.*

Proof. Suppose that $A(G) \oplus_1 L^1(G)$ is left character biprojective. Let $\phi \in \Delta(A(G))$. Choose an element $a_0 \in A(G)$ such that $\phi(a_0) = 1$. Clearly the element $x_0 = (a_0, 0)$ belongs to $A(G) \oplus_1 L^1(G)$ which $xx_0 = x_0x$ and $\phi(x_0) = 1$. Using [16, Lemma 2.2], left character biprojectivity of $A(G) \oplus_1 L^1(G)$ implies that $A(G) \oplus_1 L^1(G)$ is left ϕ-contractible. Since $A(G)$ is a closed ideal in $A(G) \oplus_1 L^1(G)$ and $\phi|_{A(G)} \neq 0$, by [10, Proposition 3.8]$A(G)$

is left ϕ-contractible. So by [10, Proposition 6.6] G is discrete. Thus $A(G) \oplus_1 L^1(G) = A(G) \oplus_1 \ell^1(G)$. We know that $\ell^1(G)$ has an identity e. Replacing e with a_0 and ψ with ϕ (for some $\psi \in \Delta(L^1(G))$) and following the same argument as above, we can see that $\ell^1(G)$ is left ψ-contractible. Thus by [10, Theorem 6.1] G is compact. Therefore G must be finite.

Converse is clear. □

A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions

(i) $S(G)$ is dense in $L^1(G)$,

(ii) $S(G)$ with a norm $||\cdot||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,

(iii) for $f \in S(G)$ and $y \in G$, we have $L_y(f) \in S(G)$ the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,

(iv) $||L_y(f)||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $y \in G$.

For various examples of Segal algebras, we refer the reader to [12].

A locally compact group G is called SIN, if it contains a fundamental family of compact invariant neighborhoods of the identity, see [2, p. 86].

Proposition 3.3. *Let G be a SIN group. Then $S(G) \otimes_p S(G)$ is left $\phi \otimes \psi$-biprojective if and only if G is compact, for some $\phi \in \Delta(S(G))$.*

Proof. Let $S(G) \otimes_p S(G)$ be left $\phi \otimes \phi$-biprojective. Since G is a SIN group, the main result of [7] gives that $S(G)$ has a central approximate identity. It follows that there exists an element $x_0 \in S(G)$ such that $xx_0 = x_0x$ and $\phi(x_0) = 1$, for each $x \in S(G)$. Set $u_0 = x_0 \otimes x_0$. It is easy to see that $uu_0 = u_0u$ and $\phi \otimes \phi(u_0) = 1$, for every $u \in S(G) \otimes_p S(G)$. Using [16, Lemma 2.2] left $\phi \otimes \phi$-biprojectivity of $S(G) \otimes_p S(G)$ follows that $S(G) \otimes_p S(G)$ is left $\phi \otimes \phi$-contractible. By [10, Theorem 3.14] $S(G)$ is left ϕ-contractible. Thus [1, Theorem 3.3] gives that G is compact.

For converse, suppose that G is compact. Then by [1, Theorem 3.3] $S(G)$ is left ϕ-contractible. So by [10, Theorem 3.14] $S(G) \otimes_p S(G)$ be left $\phi \otimes \phi$-contractible. Applying [16, Lemma 2.1] finishes the proof. □

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