SECOND ORDER (σ, τ)-COHOMOLOGY OF TRIANGULAR BANACH ALGEBRAS

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Let $\mathcal{A}$ be a Banach algebra. In this paper, we introduce (σ, τ)-2-cocycle and (σ, τ)-coboundary maps on $\mathcal{A}$, where σ and τ are homomorphisms on $\mathcal{A}$. By applying these definitions, we introduce the second (σ, τ)-cohomology of triangular Banach algebras.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A derivation is a linear map $D : \mathcal{A} \longrightarrow X$ such that

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathcal{A}).$$

For $x \in X$, set $ad_x : a \mapsto a.x - x.a$, $\mathcal{A} \longrightarrow X$. Then ad_x is the inner derivation induced by x.

The linear space of bounded derivations from $\mathcal{A}$ into X denoted by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations denoted by $N^1(\mathcal{A}, X)$. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with units $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Suppose that $\mathcal{M}$ is a unital Banach $\mathcal{A}, \mathcal{B}$-bimodule. We define triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix},$$

with the sum and product being giving by the usual 2×2 matrix operations and internal module actions. The norm on $\mathcal{T}$ is

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

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The Banach algebra $\mathcal{T}$ as a Banach space is isomorphic to the ℓ^1-direct sum of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Forrest and Marcoux introduced and studied derivation of triangular Banach algebras in [1]. The first Hochschild cohomology group of triangular Banach algebras was studied in [2, 5].

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. By $B^n(\mathcal{A}, X)$, we mean that the space of bounded n-linear maps from $\mathcal{A}$ into X. A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called *2-cocycle* if it satisfies the following equation

$$a.\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b).c = 0,$$

for every $a, b, c \in \mathcal{A}$. The space of 2-cocycles is a linear subspace of $B^2(\mathcal{A}, X)$, which is denoted by $Z^2(\mathcal{A}, X)$. Let $\varphi \in \Delta(\mathcal{A})$, where $\Delta(\mathcal{A})$ is the carrier space of $\mathcal{A}$, then a 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called *point 2-cocycle* at φ if it satisfies the following equation

$$\varphi(a).\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b).\varphi(c) = 0,$$

for every $a, b, c \in \mathcal{A}$. For given $T \in B(\mathcal{A}, X)$, we let

$$(\delta^1 T)(a, b) = a.T(b) - T(ab) + T(a).b,$$

for every $a, b \in \mathcal{A}$ and $\delta^1 : B(\mathcal{A}, X) \longrightarrow B^2(\mathcal{A}, X)$. Then $\{\delta^1 T : T \in B(\mathcal{A}, X)\}$ is a linear subspace of $Z^2(\mathcal{A}, X)$. These maps are called *2-coboundaries*. The collection of all 2-coboundaries is denoted by $N^2(\mathcal{A}, X)$. Similar to the first Hochschild cohomology group, the *second Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X is denoted by $H^2(\mathcal{A}, X) = Z^2(\mathcal{A}, X)/N^2(\mathcal{A}, X)$. The second Hochschild cohomology group of triangular Banach algebras is studied in [3], and higher cohomology of these algebras are studied by Moslehian in [9].

Let $\mathcal{T}$ be a triangular Banach algebras defined as above, and let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$. Let $\gamma_1 : \mathcal{T} \longrightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \longrightarrow \mathcal{M}$ and $\gamma_4 : \mathcal{T} \longrightarrow \mathcal{B}$ denote the coordinate functions associated to γ. That is

$$\gamma(T_1, T_2) = \begin{bmatrix} \gamma_1(T_1, T_2) & \gamma_2(T_1, T_2) \\ & \gamma_3(T_1, T_2) \end{bmatrix},$$

for $T_1, T_2 \in \mathcal{T}$. Let $\gamma : \mathcal{T} \longrightarrow \mathcal{T}$ be a 2-cocycle. The coordinate function γ_1 is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{A}$ if there exists a 2-cocycle (2-coboundary) $\tau_{\mathcal{A}}$ on $\mathcal{A}$ such that $\gamma_1(T_1, T_2) = \tau_{\mathcal{A}}(a_1, a_2)$, where $\mathcal{T}_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$.

Similarly, γ_4 is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{B}$ if there exists a 2-cocycle (2-coboundary) $\tau_{\mathcal{B}}$ on $\mathcal{B}$ such that $\gamma_4(T_1, T_2) = \tau_{\mathcal{B}}(b_1, b_2)$.

Let $\mathcal{A}$ be a Banach algebra, and let σ, τ be continuous homomorphisms on $\mathcal{A}$. Suppose that X is a Banach $\mathcal{A}$-bimodule. A linear mapping $D : \mathcal{A} \longrightarrow X$ is called a (σ, τ)-derivation if

$$D(ab) = \tau(a).D(b) + D(a).\sigma(b) \qquad (a, b \in \mathcal{A}).$$

It is clear that if $\sigma = \tau = id_{\mathcal{A}}$, where $id_{\mathcal{A}}$ is the identity mapping on $\mathcal{A}$, then a (σ, τ)-derivation is nothing else as an ordinary derivation which was defined as above. A linear mapping $d : \mathcal{A} \longrightarrow X$ is called a (σ, τ)-inner derivation if there exists $x \in X$ such that $d(a) = \tau(a).x - x.\sigma(a)$ for every $a \in \mathcal{A}$. These derivations on Banach algebras are studied by Mirzavaziri and Moslehian in [6, 7, 8].

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. The set of all continuous (σ,τ)-derivations from $\mathcal{A}$ into X is denoted by $Z^1_{(\sigma,\tau)}(\mathcal{A},X)$ and the set of all (σ,τ)-inner derivations is denoted by $N^1_{(\sigma,\tau)}(\mathcal{A},X)$. The quotient space $H^1_{(\sigma,\tau)}(\mathcal{A},X) = Z^1_{(\sigma,\tau)}(\mathcal{A},X)/N^1_{(\sigma,\tau)}(\mathcal{A},X)$, called the *first* (σ,τ)*-cohomology group* of $\mathcal{A}$ with coefficients in X.

Let $\mathcal{T}$ be a triangular Banach algebra and let $\mathcal{X}$ be a unital Banach $\mathcal{T}$-bimodule, then we use these notations in this paper: $\mathcal{X}_{\mathcal{A}\mathcal{A}} = e_{\mathcal{A}}.\mathcal{X}.e_{\mathcal{A}}, \mathcal{X}_{\mathcal{B}\mathcal{B}} = e_{\mathcal{B}}.\mathcal{X}.e_{\mathcal{B}}, \mathcal{X}_{\mathcal{A}\mathcal{B}} = e_{\mathcal{A}}.\mathcal{X}.e_{\mathcal{B}}$, and $\mathcal{X}_{\mathcal{B}\mathcal{A}} = e_{\mathcal{B}}.\mathcal{X}.e_{\mathcal{A}}$. If $\mathcal{X}$ replaced by $\mathcal{T}$, we have $\mathcal{X}_{\mathcal{A}\mathcal{A}} = \mathcal{A}, \mathcal{X}_{\mathcal{B}\mathcal{B}} = \mathcal{B}, \mathcal{X}_{\mathcal{A}\mathcal{B}} = \mathcal{M}$, and $\mathcal{X}_{\mathcal{B}\mathcal{A}} = 0$. Many results concerning the first (σ,τ)-cohomology of triangular Banach algebras are considered in [4].

Let σ and τ be two homomorphisms on $\mathcal{T}$ with the following properties (see [4]):

$$\sigma(e_{\mathcal{A}} \oplus 0) = e_{\mathcal{A}} \oplus 0, \quad \tau(e_{\mathcal{A}} \oplus 0) = e_{\mathcal{A}} \oplus 0;$$
$$\sigma(0 \oplus e_{\mathcal{B}}) = 0 \oplus e_{\mathcal{B}} \quad \tau(0 \oplus e_{\mathcal{B}}) = 0 \oplus e_{\mathcal{B}}.$$

The above relations imply that $\sigma(\mathcal{A}) \subseteq \mathcal{A}$, $\sigma(\mathcal{B}) \subseteq \mathcal{B}$ and $\sigma(\begin{bmatrix} 0 & \mathcal{M} \\ & 0 \end{bmatrix}) \subseteq \mathcal{M}$. Then $\sigma(\begin{bmatrix} a & m \\ & b \end{bmatrix}) = \begin{bmatrix} \sigma(a) & \sigma(m) \\ & \sigma(b) \end{bmatrix}$.

2. **Main Results**

In this section, at first we introduce some new definitions and after those we will consider the second (σ,τ)-cohomology of triangular Banach algebra. Note that in the whole of this paper, by $\mathcal{T}$ and $\mathcal{X}$ we mean that triangular Banach algebra as defined in Section one and a Banach $\mathcal{T}$-bimodule, respectively.

Definition 2.1. *Let $\mathcal{A}$ be a Banach algebra, let X be a Banach $\mathcal{A}$-bimodule, and let σ,τ be continuous homomorphisms from $\mathcal{A}$ into X. We say that $f \in B^2(\mathcal{A},X)$ is a* (σ,τ)-2-cocycle *if it is satisfies the following equation:*

$$\tau(a).f(b,c) - f(ab,c) + f(a,bc) - f(a,b).\sigma(c) = 0,$$

for every $a,b,c \in \mathcal{A}$. We denote all of these mappings by $Z^2_{(\sigma,\tau)}(\mathcal{A},X)$. For given $T \in B(\mathcal{A},X)$ let

$$(\delta^1 T)(a,b) = \tau(a).T(b) - T(ab) - T(a).\sigma(b),$$

for every $a,b \in \mathcal{A}$. The maps $\{\delta^1 T : T \in B(\mathcal{A},X)\}$ is a linear subspace of $Z^2_{(\sigma,\tau)}(\mathcal{A},X)$. We call these maps (σ,τ)-2-coboundries, and we denote all of these maps by $N^2_{(\sigma,\tau)}(\mathcal{A},X)$. Finally, we define the second (σ,τ)-cohomology group of $\mathcal{A}$ with coefficient in X to be the linear space

$$H^2_{(\sigma,\tau)}(\mathcal{A},X) = Z^2_{(\sigma,\tau)}(\mathcal{A},X)/N^2_{(\sigma,\tau)}(\mathcal{A},X).$$

By the following Example we show that the space of (σ,τ)-2-cocycle ((σ,τ)-2-coboundries) maps is wider than 2-cocycle (2-coboundries) maps.

Example 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule.*

(i) *Let $\mathcal{A} = X$. Then every 2-cocycle is an $(id_{\mathcal{A}}, id_{\mathcal{A}})$-2-cocycle, where $id_{\mathcal{A}}$ is the identity map on the algebra $\mathcal{A}$.*

(ii) *Let $\mathcal{A} = \mathbb{C}$ and $\varphi \in \Delta(\mathcal{A})$. A (φ,φ)-2-cocycle is nothing than a point 2-cocycle at φ.*

Definition 2.2. *Let $\gamma \in B^2(\mathcal{T},\mathcal{T})$, $\gamma_1 : \mathcal{T} \longrightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \longrightarrow \mathcal{M}$ and $\gamma_4 : \mathcal{T} \longrightarrow \mathcal{B}$ denote the coordinate functions associated to γ. That is*

$$\gamma(T_1,T_2) = \begin{bmatrix} \gamma_1(T_1,T_2) & \gamma_2(T_1,T_2) \\ & \gamma_3(T_1,T_2) \end{bmatrix},$$

for $T_1, T_2 \in \mathcal{T}$. Let $\gamma \in B^2(\mathcal{T},\mathcal{T})$ be a (σ,τ)-2-cocycle ((σ,τ)-2-coboundries). We say that γ_1 corresponds to a (σ,τ)-2-cocycle ((σ,τ)-2-coboundries) on $\mathcal{A}$ if there exists a (σ,τ)-2-cocycle ((σ,τ)-2-coboundries) $\tau_{\mathcal{A}}$ on $\mathcal{A}$ such that $\gamma_1(T_1,T_2) = \tau_{\mathcal{A}}(a_1,a_2)$, where $\mathcal{T}_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$.

Similarly, we say that γ_4 corresponds to a (σ,τ)-2-cocycle ((σ,τ)-2-coboundries) on $\mathcal{B}$ if there exists a (σ,τ)-2-cocycle ((σ,τ)-2-coboundries) $\tau_{\mathcal{B}}$ on $\mathcal{B}$ such that $\gamma_4(T_1,T_2) = \tau_{\mathcal{B}}(b_1,b_2)$.

Lemma 2.2. *Let $\delta \in B^2(\mathcal{T},\mathcal{T})$ be a (σ,τ)-2-cocycle ((σ,τ)-2-coboundary). Then there are corresponding (σ,τ)-2-cocycles ((σ,τ)-2-coboundries) on $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Define $\delta_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\delta_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ as follows

$$\delta_{\mathcal{A}}(a_1,a_2) = e_{\mathcal{A}}\delta\left(\begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix}\right)e_{\mathcal{A}},$$

and

$$\delta_{\mathcal{B}}(b_1,b_2) = e_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0 \\ & b_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b_2 \end{bmatrix}\right)e_{\mathcal{B}}.$$

It is easy to check that $\delta_{\mathcal{A}}$ and δ_B are (σ,τ)-2-cocycle ((σ,τ)-2-coboundary). □

Lemma 2.3. *Let $\mathcal{X}$ be $\mathcal{T}$-bimodule, $\delta_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}_{\mathcal{A}\mathcal{A}}$, $\delta_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{X}_{\mathcal{B}\mathcal{B}}$ be (σ,τ)-2-cocycles, and $\mathcal{X}_{\mathcal{A}\mathcal{B}} = 0$. Then there exists a (σ,τ)-2-cocycle mapping from $\mathcal{T}$ into $\mathcal{X}$.*

Proof. For every $\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \in \mathcal{T}$, define $D : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{X}$ by

$$D\left(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\right) = \delta_{\mathcal{A}}(a_1,a_2) + \delta_{\mathcal{B}}(b_1,b_2).$$

We claim that D is a (σ,τ)-2-cocycle. Because for every $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}, T_3 = \begin{bmatrix} a_3 & m_3 \\ & b_3 \end{bmatrix} \in \mathcal{T}$, we have

$$\begin{aligned} &\tau(T_1).D(T_2,T_3) - D(T_1T_2,T_3) + D(T_1,T_2T_3) - D(T_1,T_2).\sigma(T_3) \\ &\quad= \tau(T_1).(\delta_{\mathcal{A}}(a_2,a_3) + \delta_{\mathcal{B}}(b_2,b_3)) - \delta_{\mathcal{A}}(a_1a_2,a_3) - \delta_{\mathcal{B}}(b_1b_2,b_3) \\ &\qquad+\delta_{\mathcal{A}}(a_1,a_2a_3) + \delta_{\mathcal{B}}(b_1,b_2b_3) - (\delta_{\mathcal{A}}(a_1,a_2) + \delta_{\mathcal{B}}(b_1,b_2))\sigma(T_3) \\ &\quad= \tau(T_1)\tau(e_{\mathcal{A}})\delta_{\mathcal{A}}(a_2,a_3) - \delta_{\mathcal{A}}(a_1a_2,a_3) + \delta_{\mathcal{A}}(a_1,a_2a_3) - \delta_{\mathcal{A}}(a_1,a_2)\tau(e_{\mathcal{A}})\sigma(T_3) \\ &\qquad+\tau(T_1)\tau(e_{\mathcal{B}})\delta_{\mathcal{B}}(b_2,b_3) - \delta_{\mathcal{B}}(b_1b_2,b_3) + \delta_{\mathcal{B}}(b_1,b_2b_3) - \delta_{\mathcal{B}}(b_1,b_2)\sigma(e_{\mathcal{B}})\sigma(T_3) \\ &\quad= \tau(a_1)\delta_{\mathcal{A}}(a_2,a_3) - \delta_{\mathcal{A}}(a_1a_2,a_3) + \delta_{\mathcal{A}}(a_1,a_2a_3) - \delta_{\mathcal{A}}(a_1,a_2)\sigma(a_3) \\ &\qquad+\tau(b_1)\delta_{\mathcal{B}}(b_2,b_3) - \delta_{\mathcal{B}}(b_1b_2,b_3) + \delta_{\mathcal{B}}(b_1,b_2b_3) - \delta_{\mathcal{B}}(b_1,b_2)\sigma(b_3) \\ &\quad= 0. \end{aligned}$$

This proves our claim. □

Lemma 2.4. *Let $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ be (σ,τ)-2-coboundaries on $\mathcal{A}$ and $\mathcal{B}$, respectively. Then there exists a (σ,τ)-2-coboundaries δ on $\mathcal{T}$ such that δ_1 corresponds to $\delta_{\mathcal{A}}$ and δ_2 corresponds to $\delta_{\mathcal{B}}$, where δ_1 and δ_2 are coordinate functions associated to δ.*

Proof. According to our assumption $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ are (σ,τ)-2-coboundaries, therefore there are bounded linear maps $R : \mathcal{A} \longrightarrow \mathcal{A}$ and $S : \mathcal{B} \longrightarrow \mathcal{B}$ such that $\delta_{\mathcal{A}} = \delta^1 R$ and $\delta_A = \delta^1 S$. Consider the map $\delta : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ defined by

$$\delta\left(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\right) = \begin{bmatrix} \delta_{\mathcal{A}}(a_1,a_2) & \tau(m_1)S(b_2) + R(a_1)\sigma(m_2) \\ & \delta_{\mathcal{B}}(b_1,b_2) \end{bmatrix}.$$

We will show it is the required map. Let $F : \mathcal{T} \longrightarrow \mathcal{T}$ such that

$$F\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} R(a) & 0 \\ & S(b) \end{bmatrix}.$$

Then F is a both linear and bounded map. Also, for every $T_1, T_2 \in \mathcal{T}$ we have

$$\begin{aligned}
(\delta^1 F)(T_1,T_2) &= (\delta^1 F)\left(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\right) \\
&= \tau(T_1)F(T_2) - F\left(\begin{bmatrix} a_1a_2 & a_1m_2 + m_2b_2 \\ & b_1b_2 \end{bmatrix}\right) + F(T_1)\sigma(T_2) \\
&= \begin{bmatrix} \tau(a_1) & \tau(m_1) \\ & \tau(b_1) \end{bmatrix} \begin{bmatrix} R(a_2) & 0 \\ & S(b_2) \end{bmatrix} - \begin{bmatrix} R(a_1a_2) & 0 \\ & S(b_1b_2) \end{bmatrix} \\
&\quad + \begin{bmatrix} R(a_1) & 0 \\ & S(b_1) \end{bmatrix} \begin{bmatrix} \sigma(a_2) & \sigma(m_2) \\ & \sigma(b_2) \end{bmatrix} \\
&= \begin{bmatrix} \tau(a_1)R(a_2) - R(a_1a_2) + R(a_1)\sigma(a_2) & \tau(m_1)S(b_2) + R(a_1)\sigma(m_2) \\ & \tau(b_1)S(b_2) - S(b_1b_2) + R(b_1)\sigma(b_2) \end{bmatrix} \\
&= \begin{bmatrix} (\delta^1 R)(a_1,a_2) & \tau(m_1)S(b_2) + R(a_1)\sigma(m_2) \\ & (\delta^1 S)(b_1,b_2) \end{bmatrix} \\
&= \delta\left(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\right) = \delta(T_1,T_2).
\end{aligned}$$

Thus, there exists a linear map F such that $\delta^1 F = \delta$. This means that δ is a (σ,τ)-2-coboundary. □

Now, we are ready to prove our main Theorem.

Theorem 2.5. *Let $\mathcal{M} = 0$. Then*

$$H^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T}) = H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) \oplus H^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B}). \tag{2.1}$$

Proof. Consider the map $\alpha : Z^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T}) \longrightarrow H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) \oplus H^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B})$ defined by

$$\delta \mapsto \left(\delta_{\mathcal{A}} + N^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}), \delta_{\mathcal{B}} + N^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B})\right), \tag{2.2}$$

where $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ are (σ,τ)-2-cocycles that obtained from Lemma 2.2.

For given $\delta_{\mathcal{A}} \in Z^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})$ and $\delta_{\mathcal{B}} \in Z^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B})$, by Lemma 2.3, $D : \mathcal{T}\times\mathcal{T} \longrightarrow \mathcal{T}$ defined as follows is (σ,τ)-2-cocycle:

$$D\left(\begin{bmatrix} a_1 & 0 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ & b_2 \end{bmatrix}\right) = \delta_{\mathcal{A}}(a_1,a_2) + \delta_{\mathcal{B}}(b_1,b_2).$$

Then

$$\begin{aligned}
\alpha(D) &= \left(D_{\mathcal{A}} + N^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}), D_{\mathcal{B}} + N^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B})\right) \\
&= \left(\delta_1 + N^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}), \delta_1 + N^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B})\right),
\end{aligned} \tag{2.3}$$

where

$$D_{\mathcal{A}}(a_1,a_2) = e_{\mathcal{A}}(\delta_1(a_1,a_2) + \delta_2(0,0))e_{\mathcal{A}} = \delta_1(a_1,a_2),$$

and

$$D_{\mathcal{B}}(b_1,b_2) = e_{\mathcal{B}}(\delta_1(0,0) + \delta_2 b_1, b_2))e_{\mathcal{B}} = \delta_2(b_1,b_2).$$

Hence, α is onto. Now, we will show that $\ker\alpha$ is $N^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})$. Let $\delta\in\ker\alpha$, then by (2.2), $\delta_{\mathcal{A}}\in N^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})$ and $\delta_{\mathcal{B}}\in N^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B})$. Then by Lemma 2.4, there exists $D:\mathcal{T}\longrightarrow\mathcal{T}$ defined as follows

$$D\left(\begin{bmatrix} a_1 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & b_2\end{bmatrix}\right)=\delta_{\mathcal{A}}(a_1,a_2)+\delta_{\mathcal{B}}(b_1,b_2),$$

is (σ,τ)-2-coboundary, and thereupon $D\in N^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})$. We claim that $D=\delta$. For proving this assertion we use the following statements that obtaining of these relations are depend on δ, which it is (σ,τ)-2-cocycle:

(1) $\delta\left(\begin{bmatrix} e_{\mathcal{A}} & 0\\ & 0\end{bmatrix},\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix}\right)=\delta\left(\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix},\begin{bmatrix} 0 & 0\\ & e_{\mathcal{B}}\end{bmatrix}\right)=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix};$

(2) $e_{\mathcal{B}}\delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)e_{\mathcal{B}}=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix};$

(3) $e_{\mathcal{A}}\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)e_{\mathcal{A}}=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix};$

(4) $e_{\mathcal{A}}\delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)e_{\mathcal{A}}=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix};$

(5) $e_{\mathcal{B}}\delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)e_{\mathcal{B}}=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix};$

(6) $e_{\mathcal{A}}\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)e_{\mathcal{A}}=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix};$

(7) $e_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)e_{\mathcal{B}}=\begin{bmatrix} 0 & 0\\ & 0\end{bmatrix}.$

In addition for given $T_1=\begin{bmatrix} a_1 & 0\\ & b_1\end{bmatrix}, T_2=\begin{bmatrix} a_2 & 0\\ & b_2\end{bmatrix}\in\mathcal{T}$, we can write

$$\delta(T_1,T_2)=e_{\mathcal{A}}\delta(T_1,T_2)e_{\mathcal{A}}+e_{\mathcal{B}}\delta(T_1,T_2)e_{\mathcal{B}}. \tag{2.4}$$

Since δ is a (σ,τ)-2-cocycle and bilinear, so relations (1)-(7) and (2.4), imply

$$\begin{aligned}(\delta-D)(T_1,T_2) &= \delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & b_2\end{bmatrix}\right)+\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & b_2\end{bmatrix}\right)\\
&\quad -e_{\mathcal{A}}\delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)e_{\mathcal{A}}-e_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)e_{\mathcal{B}}\\
&= \delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)+\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)\\
&\quad +\delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)+\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)\\
&\quad -e_{\mathcal{A}}\delta\left(\begin{bmatrix} a_1 & 0\\ & 0\end{bmatrix},\begin{bmatrix} a_2 & 0\\ & 0\end{bmatrix}\right)e_{\mathcal{A}}-e_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0\\ & b_1\end{bmatrix},\begin{bmatrix} 0 & 0\\ & b_2\end{bmatrix}\right)e_{\mathcal{B}}\\
&= \begin{bmatrix} 0 & 0\\ & 0\end{bmatrix}.\end{aligned}$$

Therefore $\delta = D$, and this means that δ is (σ,τ)-2-coboundary. Conversely, let $\delta \in N^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})$. Then by Lemma 2.2 there are (σ,τ)-2-coboundaries $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ on $\mathcal{A}$ and $\mathcal{B}$, respectively. Therefore $\delta \in \ker\alpha$. This means that $\ker\alpha = N^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})$. Then

$$\begin{aligned} H^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T}) &= \frac{Z^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})}{N^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})} = \frac{Z^2_{(\sigma,\tau)}(\mathcal{T},\mathcal{T})}{\ker\alpha} \\ &= H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) \oplus H^2_{(\sigma,\tau)}(\mathcal{B},\mathcal{B}). \end{aligned}$$

□

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