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# SOME ITERATIVE SCHEMES FOR OBTAINING APPROXIMATE SOLUTION OF NONLINEAR EQUATIONS

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*In this paper, we suggest and analyze some new iterative methods for solving nonlinear equation $f(x) = 0$ by using the variational iteration technique. We also give several examples to illustrate the efficiency of these methods. Comparison with other similar methods is also given. These new methods can be considered as alternative to the developed fourth order methods. This technique can be used to suggest a wide class of new iterative methods for solving nonlinear equations.*

**Keywords:** Variational iteration technique; Iterative method; Convergence; Newton's method; Taylor series.

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## 1. **Introduction**

Finding roots of nonlinear equations efficiently has widespread applications in numerical mathematics. Due to their importance and significant applications in various branches of science, several methods are being developed for solving $f(x) = 0$ using different techniques such as Taylor series, quadrature formulas, homotopy perturbation method, Adomian decomposition and variational iteration technique [1-20]. Newton method for a single nonlinear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3, ...$$

This is an important and basic method [19], which converges quadratically. To improve the local order of convergence, many modified methods have been proposed. See [2, 3] and [9-14].

In this paper, we use the variational iteration technique to suggest and analyze some new iterative methods for solving the nonlinear equations. We would like to mention that the variational iteration technique was developed by He [6] and has been used to solve a wide class of problems arising in various branches of pure and

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applied sciences. The variational iteration technique is very reliable and efficient. See also Noor and Mohyud-Din [15] and the references therein. Essentially using the idea and technique of He [6], Noor and Shah [16] has suggested and analyzed some iterative methods for solving the nonlinear equations. Now we have used this technique to gain higher order convergent iterative methods. We show that the new methods include only 1st derivative of the functions and these are free from higher order derivatives. Several examples are given to illustrate the efficiency and performance of these new methods and their comparison with other iterative methods. These new methods can be considered as alternative to the existing higher order methods.

## 2. **Construction of iterative methods**

In this section, we construct some fourth order iterative methods for solving nonlinear equations. We use He's variational iteration technique to develop the main reoccurrence relation which will generate the iterative methods for the approximate solution of nonlinear equation. Comparison of Traub's method [19] and Ostrowski method [17] will allow us to remove $f'(y_n)$ and diversify the relation with better efficiency index.

Consider the nonlinear equation of the type

$$f(x) = 0. \tag{1}$$

We assume that $p$ is a simple root and $\gamma$ is an initial guess sufficiently close to $p$. Let $g(x)$ be any arbitrary function and $\lambda$ be a parameter which is usually called the Lagrange's multiplier and can be identified by the optimality condition. Consider the auxiliary function

$$H\,(x) = \phi(x) + \lambda[f(\phi(x))g(\phi(x))], \tag{2}$$

where $\phi(x)$ is the arbitrary auxiliary function of order $p \geq 1$.
Note that, if $\phi(x) = I$ and $p = 1$, then (2) reduces to the following

$$H\,(x) = x + \lambda[f(x)g(x)], \tag{3}$$

which was considered and analyzed by He [6]. See also Noor [10, 11].
Thus we conclude that our scheme (2) includes the He's scheme as a special case. For $p = 2$, the relation (2) is studied in [16].

In this paper, our aim is to analyze the relation (2) for obtaining higher order methods.

Using the optimality criteria, we can obtain the value of $\lambda$ from (2) as

$$\lambda = -\frac{\phi(x)}{[g'(\phi(x))f(\phi(x)) + g(\phi(x))f'(\phi(x))]} \tag{4}$$

From (3) and (4), we have

$$H(x) = \phi(x) - \frac{f(\phi(x))g(\phi(x))}{[g'(\phi(x))f(\phi(x)) + g(\phi(x))f'(\phi(x))]} \tag{5}$$

Let us consider

$$\phi(x) = y = x - \frac{f(x)}{f'(x)}. \tag{6}$$

Using (6) in (5), we obtain the following iterative relation for solving the nonlinear equations as:

$$H(x) = y - \frac{f(y)g(y)}{[g'(y)f(y) + g(y)f'(y)]}. \tag{7}$$

We observe that, if $p$ is the root of $f(x)$, then for $x = p$,

$$\frac{g'(y)}{g(y)} = \frac{g'(p - \frac{f(p)}{f'(p)})}{g(p - \frac{f(p)}{f'(p)})} = \frac{g'(p)}{g(p)}. \tag{8}$$

Also we have

$$\frac{g'(x)}{g(x)} = \frac{g'(p)}{g(p)}. \tag{9}$$

Combining (8) and (9), we obtain

$$\frac{g'(y)}{g(y)} = \frac{g'(x)}{g(x)}. \tag{10}$$

We now replace $\frac{g'(y)}{g(y)}$ by $\frac{g'(x)}{g(x)}$ in (7) and obtain the following

$$H(x) = y - \frac{f(y)g(x)}{g'(x)f(y) + g(x)f'(y)}.$$

This fixed point formulation enables to define the following iterative method

$$x_{n+1} = y_n - \frac{f(y_n)g(x_n)}{g'(x_n)f(y_n) + g(x_n)f'(y_n)} \tag{11}$$

From the above scheme, for different values of the auxiliary function $g(x_n)$ one can obtain several iterative methods of fourth order convergence for solving nonlinear equations. Here our aim is to improve the efficiency of the above iterative scheme by removing $f'(y_n)$. For this we consider the two well known iterative methods i.e. Traub's method and Ostrowski method as follows.

**Algorithm2.1**.[**19**]. For a given $x_0$, find the approximation solution $x_{n+1}$ by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}, \qquad n = 0, 1, 2, 3, ... \end{aligned}$$

Algorithm 2.1 is known as Traub's method[19] and also as double Newton method. This method is also the special case of the main iteration scheme (11), for some constant value of the auxiliary function $g$.

**Algorithm2.2**.[**17**]. For a given $x_0$, find the approximation solution $x_{n+1}$ by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n)}{f(x_n) - 2f(y_n)} \qquad n = 0, 1, 2, 3, ... \end{aligned}$$

Algorithm 2.2 is known as fourth order convergent Ostrowski method, see [16, 17].

We compare the Algorithm 2.1 and Algorithm 2.2 as:

$$y_n - \frac{f(y_n)}{f'(y_n)} \approx y_n - \frac{f'(x_n)}{f(x_n)} \frac{f(y_n)}{f(x_n) - 2f(y_n)}.$$

Simplifying we obtain

$$f'(y_n) \approx \frac{f'(x_n)}{f(x_n)}[f(x_n) - 2f(y_n))] \tag{12}$$

Replacing (12) in (11), we get

$$x_{n+1} = y_n - \frac{f(x_n)f(y_n)g(x_n)}{f'(x_n)[f(x_n) - 2f(y_n)]g(x_n) + f(x_n)f(y_n)g'(x_n)}. \tag{13}$$

This is the main iterative scheme. We will use some special values of $g(x_n)$ and get the iterative methods for implimentation as:

**1**. Let $g(x_n) = e^{\alpha x_n}$. Then from (13), we obtain the following iterative method for solving the nonlinear equation (1).

**Algorithm2.3**. For a given $x_0$, find the approximation solution $x_{n+1}$ by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)f(y_n)}{f'(x_n)[f(x_n) - 2f(y_n)] + \alpha f(x_n)f(y_n)}, n = 0, 1, 2, ... \end{aligned}$$

If $\alpha = 0$, then Algorithm 2.3 reduces to the well known Ostrowski method.

**2**. Let $g(x_n) = e^{\alpha f(x_n)}$. Then from (13), we obtain the following iterative method for solving the nonlinear equation (1).

**Algorithm2.4**. For a given $x_0$, find the approximation solution $x_{n+1}$ by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)f(y_n)}{f'(x_n)([f(x_n) - 2f(y_n)] + \alpha f(x_n)f(y_n))}, n = 0, 1, 2, ... \end{aligned}$$

If $\alpha = 0$, then Algorithm 2.4 reduces to the well known Ostrowski method.

**3**. Let $g(x_n) = e^{-\frac{\alpha}{f'(x_n)}}$. Then $g'(x_n) = e^{-\frac{\alpha}{f'(x_n)}}(\frac{\alpha f''(x_n)}{[f'(x_n)]^2})$.

Using the Taylor series technique as used in [16], we have

$$f(y_n) \approx f(x_n) + (y_n - x_n)f'(x_n) + \frac{(y_n - x_n)^2}{2} f''(x_n) = \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^2} \tag{14}$$

Now from (13) and (14), we have the following iterative method.

**Algorithm2.5**. For a given $x_0$, find the approximation solution $x_{n+1}$ by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{[f(x_n)]^2 f(y_n)}{f(x_n)f'(x_n)[f(x_n) - 2f(y_n)] + 2\alpha f(y_n)^2}, n = 0, 1, 2, ... \end{aligned}$$

**4**. Let $g(x_n) = e^{-\frac{\alpha f(x_n)}{f'(x_n)}}$. Then from (13), we have the following iterative scheme after combining with (14) for solving the nonlinear Eq. (1).

**Algorithm2.6**. For a given $x_0$, find the approximation solution $x_{n+1}$ by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)f(y_n)}{[f(x_n) - 2f(y_n)][f'(x_n) + 2\alpha f(y_n)]}, n = 0, 1, 2, ... \end{aligned}$$

It is important to say that never choose such a value of $\alpha$ which makes the denominator zero. It is necessary that sign of $\alpha$ should be chosen so as to keep the denominator largest in magnitude in above Algorithms.

## 3. **Convergence analysis**

In this section, we consider the convergence criteria of the main iterative scheme (13) developed in section 2.

**Theorem 3.1.** *Assume that the function $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ for an open interval in $\mathcal{D}$ with simple root $p \in \mathcal{D}$. Let $f(x)$ be a smooth sufficiently in some neighborhood of root and then (13) has fourth order convergence.*

*Proof.* Let $p$ ba a simple root of the nonlinear equation $f(x)$. Since $f$ is sufficiently differentiable. Expanding $f(x)$ and $f'(x)$ in Taylor's series at $p$, we obtain

$$f(x_n) = f'(p)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)]. \tag{15}$$

and

$$f'(x_n) = f'(p)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^7)]. \tag{16}$$

where

$$e_n = x_n - p, c_k = \frac{f^k(p)}{k!f'(p)} \quad \text{and} \quad k = 2, 3, \cdots$$

From (15) and (16), we get

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 \\ &+ (8c_2^4 - 20c_3c_2^2 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + (13c_2c_5 - 28c_2^2c_4 \\ &- 5c_6 - 16c_2^5 + 52c_2^3c_3 + 17c_3c_4 - 33c_2c_3^2)e_n^6 + O(e_n^7).\end{aligned} \tag{17}$$

Using (17), we have

$$\begin{aligned}y_n &= p + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 - (8c_2^4 \\ &- 20c_3c_2^2 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 - (13c_2c_5 - 28c_2^2c_4 \\ &- 5c_6 - 16c_2^5 + 52c_2^3c_3 + 17c_3c_4 - 33c_2c_3^2)e_n^6 + O(e_n^7).\end{aligned} \tag{18}$$

From (18), we obtain

$$\begin{aligned}f(y_n) &= f'(p)[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 5c_2^3 - 3c_4)e_n^4 \\ &- (12c_2^4 - 24c_3c_2^2 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + O(e_n^7)]\end{aligned} \tag{19}$$

From (15) and (19), we have

$$\begin{aligned}f(x_n) - 2f(y_n) = f'(p)[e_n - c_2e_n^2 + (4c_2^2 - 3c_3)e_n^3 + (14c_2c_3 \\ -10c_2^3 - 5c_4)e_n^4 + O(e_n^5)].\end{aligned} \tag{20}$$

From (15) and (19), we also obtain

$$\begin{aligned}f(x_n)f(y_n)g(x_n) = f'^2[g(p)c_2e_n^3 + (g'(p)c_2 - g(p)c_2^2 \\ +2g(p)c_3)e_n^4 + O(e_n^5)]\end{aligned} \tag{21}$$

Now using (15), (16), (20) and (21), we get

$$\begin{aligned}f'(x_n)[f(x_n) - 2f(y_n)]g(x_n) + f(x_n)f(y_n)g'(x_n) = f'(p)^2\Bigg[g(p)e_n \\ +(g(p)c_2 + g'(p))e_n^2 + \left(\frac{1}{2}g''(p) + 2g(p)c_2^2\right)e^3 + O(e_n^4)\Bigg]\end{aligned} \tag{22}$$

$$\frac{f(x_n)f(y_n)g(x_n)}{f'(x_n)[f(x_n) - 2f(y_n)]g(x_n) + f(x_n)f(y_n)g'(x_n)}$$

$$= \left[c_2e_n^2 + (2c_3 - c_2^2)e_n^3 + \left(\frac{g'(p)}{g(p)}c_2^2 + 3c_2^3 + c_4 - 6c_2c_3\right)e_n^4 + O(e_n^5)\right] \tag{23}$$

Now from (18) and (23), we get

$$x_{n+1} = p + \left(\frac{g'(p)}{g(p)}c_2^2 + 3c_2^3 + c_4 - 6c_2c_3\right)e_n^4 + O(e_n^5) \tag{24}$$

Finally, the error equation is

$$e_{n+1} = \left(\frac{g'(p)}{g(p)}c_2^2 + 3c_2^3 + c_4 - 6c_2c_3)\right)e_n^4 + O(e_n^5) \tag{25}$$

Thus we conclude that the main recurrence relation (13) has fourth-order convergence and all the methods derived from (13) has also fourth order convergence. □

## 4. **Numerical results**

We now present some examples to illustrate the efficiency of the new developed two-step iterative methods (see Tables 4.1-4.6). We compare the Newton method (NM) [19], Turab's method (TM) [19], Ostrowski method (OM) [17] Algorithm 2.3 (NR1), Algorithm 2.4 (NR2), Algorithm 2.5 (NR3) and Algorithm 2.6 (NR4) which are introduced here in this paper. We also note that these methods do not require the computation of second derivative to carry out the iterations. All computations are done using the MAPLE using 60 digits floating point arithmetics (Digits: =60). We will use $\varepsilon = 10^{-32}$. The following stopping criteria are used for computer programs.

$$(i)\ |x_{n+1} - x_n| \le \varepsilon, \qquad (ii)\ |f(x_n)| \le \varepsilon.$$

The computational order of convergence $p$ approximated for all the examples in Tables 4.1-4.6, (see [19]) by means of

$$p = \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}$$

along with the total number of functional evaluations (TNFE) as required for the iterations during the computation.
We consider the following nonlinear equations as test problems which are same as Noor and Noor [14].

$$f_1(x) = \sin^2 x - x^2 + 1,$$
$$f_2(x) = x^2 - e^{-x}3x + 2,$$
$$f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$$
$$f_4(x) = e^{x^2-7x-30} - 1$$

**Table4.1. Comparison of iterative schemes.**$(\mathbf{f_1}, \mathbf{x_0} = \mathbf{1.0}, \alpha = \mathbf{1})$

| Method | IT | TNFE | $x_n$ | $\lvert f(x_n)\rvert$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 7 | 14 | 1.4044916482153 | 1.04e-50 | 7.33e-26 | 2.00003 |
| TM | 4 | 16 | 1.4044916482153 | 0.00e-01 | 7.33e-26 | 4.29576 |
| OM | 4 | 12 | 1.4044916482153 | 0.00e-01 | 5.64e-28 | 4.24367 |
| NR1 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 2.14e-27 | 4.27401 |
| NR2 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 4.20e-18 | 3.97200 |
| NR3 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 3.48e-17 | 4.29898 |
| NR4 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 2.00e-16 | 4.52494 |

**Table4.2. Comparison of iterative schemes.**$(\mathbf{f_1}, \mathbf{x_0} = \mathbf{1.0}, \alpha = \mathbf{0.5})$

| Method | IT | TNFE | $x_n$ | $\lvert f(x_n)\rvert$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 7 | 14 | 1.4044916482153 | 1.04e-50 | 7.33e-26 | 2.00003 |
| TM | 4 | 16 | 1.4044916482153 | 0.00e-01 | 7.33e-26 | 4.29576 |
| OM | 4 | 12 | 1.4044916482153 | 0.00e-01 | 5.64e-28 | 4.24367 |
| NR1 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 9.57e-23 | 4.25801 |
| NR2 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 1.80e-43 | 3.87150 |
| NR3 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 2.33e-22 | 4.34817 |
| NR4 | 4 | 12 | 1.4044916482153 | 0.00e-01 | 2.83e-24 | 4.09265 |

**Table4.3. Comparison of iterative schemes.**$(\mathbf{f_2}, \mathbf{x_0} = \mathbf{2.0}, \alpha = \mathbf{1})$

| Method | IT | TNFE | $x_n$ | $\lvert f(x_n)\rvert$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 6 | 12 | 0.2575302854398 | 2.93e-55 | 9.10e-28 | 2.00050 |
| TM | 4 | 16 | 0.2575302854398 | 1.00e-59 | 7.74e-56 | 3.86670 |
| OM | 4 | 12 | 0.2575302854398 | 0.00e-01 | 2.70e-23 | 4.15500 |
| NR1 | 4 | 12 | 0.2575302854398 | 2.00e-59 | 4.23e-24 | 4.29911 |
| NR2 | 4 | 12 | 0.2575302854398 | 1.00e-59 | 2.26e-16 | 4.10939 |
| NR3 | 4 | 12 | 0.2575302854398 | 0.00e-01 | 2.60e-25 | 4.49624 |
| NR4 | 4 | 12 | 0.2575302854398 | 0.00e-01 | 1.35e-40 | 3.92927 |

**Table4.4. Comparison of iterative schemes.** $(\mathbf{f_2}, \mathbf{x_0} = \mathbf{2.0}, \alpha = \mathbf{0.5})$

| Method | IT | TNFE | $x_n$ | $\|f(x_n)\|$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 6 | 12 | 0.2575302854398 | 2.93e-55 | 9.10e-28 | 2.00050 |
| TM | 4 | 16 | 0.2575302854398 | 1.00e-59 | 7.74e-56 | 3.86670 |
| OM | 4 | 12 | 0.2575302854398 | 0.00e-01 | 2.70e-23 | 4.15500 |
| NR1 | 4 | 12 | 0.2575302854398 | 0.00e-01 | 3.37e-40 | 3.85293 |
| NR2 | 4 | 12 | 0.2575302854398 | 0.00e-01 | 5.62e-18 | 4.91925 |
| NR3 | 4 | 12 | 0.2575302854398 | 1.00e-59 | 2.83e-24 | 4.52741 |
| NR4 | 4 | 12 | 0.2575302854398 | 1.00e-59 | 5.76e-27 | 3.96131 |

**Table4.5. Comparison of iterative schemes.** $(\mathbf{f_3}, \mathbf{x_0} = -\mathbf{2.0}, \alpha = \mathbf{1})$

| Method | IT | TNFE | $x_n$ | $\|f(x_n)\|$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 9 | 18 | -1.207647827130918 | 2.27e-40 | 2.73e-21 | 2.00085 |
| TM | 5 | 20 | -1.207647827130918 | 1.10e-58 | 2.73e-21 | 4.00484 |
| OM | 5 | 15 | -1.207647827130918 | 8.00e-59 | 3.71e-43 | 4.13695 |
| NR1 | 5 | 15 | -1.207647827130918 | 8.00e-59 | 3.06e-24 | 4.01277 |
| NR2 | 6 | 18 | -1.207647827130918 | 1.10e-58 | 2.00e-22 | 4.02808 |
| NR3 | 5 | 15 | -1.207647827130918 | 8.00e-59 | 1.08e-48 | 4.52071 |
| NR4 | 5 | 15 | -1.207647827130918 | 8.00e-59 | 3.29e-48 | 3.97513 |

**Table4.6. Comparison of iterative schemes.** $(\mathbf{f_3}, \mathbf{x_0} = -\mathbf{2.0}, \alpha = \mathbf{0.5})$

| Method | IT | TNFE | $x_n$ | $\|f(x_n)\|$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 9 | 18 | -1.207647827130918 | 2.27e-40 | 2.73e-21 | 2.00085 |
| TM | 5 | 20 | -1.207647827130918 | 1.10e-58 | 2.73e-21 | 4.00484 |
| OM | 5 | 15 | -1.207647827130918 | 8.00e-59 | 3.71e-43 | 4.13695 |
| NR1 | 5 | 15 | -1.207647827130918 | 8.00e-59 | 1.36e-30 | 4.03378 |
| NR2 | 7 | 21 | -1.207647827130918 | 8.00e-59 | 4.13e-30 | 4.05656 |
| NR3 | 5 | 15 | -1.207647827130918 | 8.00e-59 | 5.78e-45 | 4.24036 |
| NR4 | 5 | 15 | -1.207647827130918 | 8.00e-59 | 4.57e-51 | 3.98835 |

**Table4.7. Comparison of iterative schemes.** $(\mathbf{f_4}, \mathbf{x_0} = \mathbf{3.5}, \alpha = \mathbf{1})$

| Method | IT | TNFE | $x_n$ | $\lvert f(x_n)\rvert$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 13 | 26 | 3.00000000000000 | 1.52e-47 | 4.21e-25 | 2.00023 |
| TM | 7 | 28 | 3.00000000000000 | 0.00e-01 | 4.21e-25 | 3.83827 |
| OM | 6 | 18 | 3.00000000000000 | 0.00e-01 | 6.93e-17 | 3.97578 |
| NR1 | 6 | 18 | 3.00000000000000 | 0.00e-01 | 5.25e-23 | 4.03116 |
| NR2 | 6 | 18 | 3.00000000000000 | 0.00e-01 | 5.14e-19 | 4.04841 |
| NR3 | 6 | 18 | 3.00000000000000 | 0.00e-01 | 7.08e-18 | 4.15183 |
| NR4 | 6 | 18 | 3.00000000000000 | 0.00e-01 | 5.14e-19 | 4.12008 |

**Table4.8. Comparison of iterative schemes.** $(\mathbf{f_4}, \mathbf{x_0} = \mathbf{3.5}, \alpha = \mathbf{0.5})$

| Method | IT | TNFE | $x_n$ | $\lvert f(x_n)\rvert$ | $\delta$ | $p$ |
|---|---|---|---|---|---|---|
| NM | 13 | 26 | 3.00000000000000 | 1.52e-47 | 4.21e-25 | 2.00023 |
| TM | 7 | 28 | 3.00000000000000 | 0.00e-01 | 4.21e-25 | 3.83827 |
| OM | 6 | 18 | 3.00000000000000 | 0.00e-01 | 6.93e-17 | 3.97578 |
| NR1 | 6 | 18 | 3.00000000000000 | 0.00e-01 | 1.54e-19 | 3.98918 |
| NR2 | 6 | 18 | 3.00000000000000 | 2.00e-58 | 1.13e-30 | 4.04841 |
| NR3 | 5 | 15 | 3.00000000000000 | 2.00e-58 | 2.47e-17 | 4.05576 |
| NR4 | 6 | 18 | 3.00000000000000 | 2.00e-58 | 4.96e-16 | 3.92802 |

## 5. Conclusions

In this work we have presented some new fourth-order convergent iterative methods for solving nonlinear equations. Theses all methods are free from $2^{nd}$ derivative. These methods are compared with some other methods and the proposed methods have been observed to have at least better performance. If we consider the definition of efficiency index [4] as $p^{\frac{1}{m}}$ where $p$ is the order of the method and $m$ is the number of functional evaluations per iteration required by the method, we have that all of the methods obtained have the efficiency index equal to $4^{\frac{1}{3}} \approx 1.5874$ which is better than the one of Newton's method $2^{\frac{1}{2}} \approx 1.4142$.

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## REFERENCES


[1] *C. Chun*, A method for obtaining iterative formulas of order three, Appl. Math. Lett., **20**(2007), 1103–1109.

[2] *C. Chun*, On the construction of iterative methods with at least cubic convergence, Appl. Math. Comput., **189**(2007), 1384–1392.

[3] *C. Chun*, Some variant of Chebshev-Halley method free from second derivative, it Appl. Math. Comput., **191**(2007), 1384–1392.

[4] *W. Gautschi*, Numerical Analysis: An introduction, Birkhauser, 1997.

[5] *J. H. He*, Homotpy perturbation method, Comp. Math. Appl. Mech. Eng., **178**(1999), 257-262.

[6] *J. H. He*, Variational iteration method some new results and new interpretations, J. Comput. Appl. Math., **207**(2007), 3-17.

[7] *J. H. He*, Variational iteration method-a kind of non-linear analytical technique: some examples, Internet. J. Nonlinear Mech., **34(4)**(1999), 699-708.

[8] *M. Inokuti, H. Sekine and T. Mura*, General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Nasser (Ed.), Variational Methods in the Mechanics of Solids, Pergamon Press, New York (1978).

[9] *J. Kou*, The improvement of modified Newton's method, Appl. Math. Comput., **189**(2007), 602-609.

[10] *M. A. Noor*, New classes of iterative methods for nonlinear equations, Appl. Math. Comput., **191**(2007), 128–131.

[11] *K. I. Noor and M. A. Noor*, Predictor-corrector Halley method for nonlinear equations, Appl. Math. Comput., **188**(2007), 1587–1591.

[12] *K. I. Noor and M. A. Noor*, Iterative methods with fourth-order convergence for nonlinear equations, Appl. Math. Comput., **188**(2007), 221-227.

[13] *K. I. Noor, M. A. Noor and S. Momani*, Modified Householder iterative methods for nonlinear equations, Appl. Math. Comput., **190**(2007), 1534-1539.

[14] *M. A. Noor, and K. I. Noor* , Some iterative schemes for nonlinear equations, Appl. Math. Comput., **183**(2006), 774-779.

[15] *M. A. Noor and S. T. Mohyud-Din*, Variational iteration techniques for solving higher-order boundary value problems, Appl. Math. Comput., **189**(2007), 1929-1942.

[16] *M. A. Noor, F. A. Shah*, Variational iteration technique for solving nonlinear equations, J. Appl. Math. Comput., **31**(2009), 247–254.

[17] *M. A. Noor, F. A. Shah, K. I. Noor and E. Al-said*, Variational iteration technique for finding multiple roots of nonlinear equations, Sci. Res. Essays., **6(6)**(2011), 1344–1350.

[18] *M. A. Noor*, On iterative method for nonlinear equation using homotopy perturbation technique, Appl. Math. Inf. Sci., **4(2)**(2010), 227–235.

[19] *J. F. Traub*, Iterative methods for solutions of equations, Printice Hall, New York (1964).

[20] *S. Weerakoon and T.G.I. Fernando*, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett, **13**(2000), 87-93.