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AN EXAMPLE ON SOME CONDITIONS IN THE CALCULUS OF VARIATIONS

Dan Tiba

We give an example related to certain extensions of the necessary conditions in the classical calculus of variations.

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1. **Introduction.**

In the paper [6], S.J.G. Gift discusses the problem:

$$\text{Minimize}\{J = \int_a^b F(x, y(x), \overset{\bullet}{y}(x))dx\} \tag{1}$$

where $a < b$ are fixed real numbers, y is a continuous function with piecewise continuous derivative $\overset{\bullet}{y}(x)$, $y(a) = y_a$ and $y(b) = y_b$ are fixed and $F(x, y, \overset{\bullet}{y})$ has continuous partial derivatives up to order three in $(x, y, \overset{\bullet}{y})$.

This is known as the simplest problem of the calculus of variations and the author gives elementary proofs of the wellknown results concerning the theory of the first and of the second variation. The investigation is continued in [7] and [8].

In the article [9], a counterexample for the results in [8] is provided, showing that they are not correct. Here, we extend this discussion to the results of [6] and [7] via a relevant new example.

For modern presentations of the calculus of variations with various extensions and relevant applications, we quote [2], [3], [4], [5], [1].

In the next section, we briefly recall several basic elements of this theory, while the last section is devoted to our comments on the extensions discussed in [6], [7].

[1]Institute of Mathematics (Romanian Academy) and Academy of Romanian Scientists, e-mail: dan.tiba@imar.ro

2. **Classical conditions**

We give a very short overview of the necessary conditions in the problem (1), in order to increase the readability of the article. We follow the exposition in [4].

Proposition 2.1. *If J has a minimum at y^*, piecewise C^1 function such that $y^*(a) = y_a$ and $y^*(b) = y_b$, then*

$$-\int_a^t F_y(\tau, y^*(\tau), \dot{y}^*(\tau))d\tau + F_{\dot{y}}(t, y^*(t), \dot{y}^*(t)) \tag{2}$$

is constant on $[a, b]$.

The relation (2) says that y^* is an *extremal* for the problem (1). It gives the Euler - Lagrange equation:

$$F_y(t, y^*(t), \dot{y}^*(t)) = \frac{d}{dt} F_{\dot{y}}(t, y^*(t), \dot{y}^*(t)) \text{ in } [a, b] \tag{3}$$

(if $\dot{y}^*$ has some discontinuity point, then (3) is satisfied by the lateral derivatives).

In particular, we obtain that in any such discontinuity point $t' \in (a, b)$, the Weierstrass - Erdmann corner condition is valid:

$$F_{\dot{y}}(t', y^*(t'), \dot{y}^*(t'_-)) = F_{\dot{y}}(t', y^*(t'), \dot{y}^*(t'_+)) \tag{4}$$

(with obvious notations for the lateral derivatives).

If the integrand F is *regular* in the sense that $F_{\dot{y}\dot{y}} > 0$, situations as in (4) cannot occur, i.e. any extremal arc y^* has no corners (no discontinuity of $\dot{y}^*$).

Concerning higher order necessary conditions, under supplementary smoothness (F should be in C^4) and the above regularity assumptions, one has to study second order derivatives of J at extremals y^*, for admissible variations.

Proposition 2.2.

$$\begin{gathered}\int_a^b [F_{yy}(t, y^*(t), \dot{y}^*(t))y^2(t) + 2F_{y\dot{y}}(t, y^*(t), \dot{y}^*(t))y(t)\dot{y}(t) + \\ + F_{\dot{y}\dot{y}}(t, y^*(t)\dot{y}^*(t))\dot{y}^2(t)] \geq 0\end{gathered} \tag{5}$$

for any admissible variation y.

Using (5) we can introduce the secondary minimum problem (for the corresponding quadratic form) on the space of admissible variations and look for possible nonzero secondary minimizers. The points $(t', y^*(t'))$ are called *conjugate* to (a, y_a) if there is a nonzero secondary extremal z^* such that $z^*(a) = z^*(t') = 0$, $t' > a$ (secondary extremals are defined similarly as extremals, but for the above secondary minimum problem). The Jacobi condition is:

Proposition 2.3. *If J has a minimum at y^*, there are no conjugate points to (a, y_a) for $a < t' < b$.*

Notice that if y^* is the unique minimizer of J (assumed to be strict convex), then the secondary minimum problem has the unique minimizer $z^* = 0$ and no conjugate points exist.

Extensions to higher dimension, discussion on the corresponding sufficient conditions, relaxation of smoothness assumptions are investigated in [4], [3], [5]. A useful recent account on sufficient conditions may be found in [1].

3. **The Example**

Theoretical developments on the strong second variation (defined below) are described in [6] in order to derive new necessary and sufficient conditions for the problem (1).

The following notations are used (see [6]):

$$\delta J = \int_a^b (F_y \delta y + F_{\dot{y}} \delta \dot{y}) dx, \tag{6}$$

where $F_y, F_{\dot{y}}$ are evaluated along the admissible arc $(x, \bar{y}(x), \dot{\bar{y}}(x))$ and $(\delta y(x), \delta \dot{y}(x))$ is an admissible variation, i.e. $\delta y(a) = \delta y(b) = 0$; the curves $\bar{y}$ satisfying $\delta J = 0$ are the extremals for the problem (1).

We recall more notation:

$$\Delta J = J(y) - J(\bar{y}), \tag{7}$$

where $y(x) = \bar{y}(x) + \delta y(x)$;

$$\overline{\overline{\delta}}^{\,2} J = \Delta J - \delta J, \text{that is} \tag{8}$$

$$\bar{\bar{\delta}}^2 J = \frac{1}{2}\int_a^b (\bar{\bar{F}}_{yy}\delta y^2 + 2\,\bar{\bar{F}}_{y\dot{y}}\delta y \delta \dot{y} + \bar{\bar{F}}_{\dot{y}\dot{y}}\delta \dot{y}^2)dx \tag{9}$$

and the two overbars on the derivatives of F indicate evaluation along $(x, \bar{y} + \theta\delta y, \dot{\bar{y}} + \theta\delta\dot{y})$ with $\theta \in (0,1)$ obtained from a Taylor series expansion and depending on $x, \delta y$:

$$\bar{\delta}^2 J = \frac{1}{2}\int_a^b (\bar{F}_{yy}\delta y^2 + 2\bar{F}_{y\dot{y}}\delta y \delta \dot{y} + \bar{F}_{\dot{y}\dot{y}}\delta \dot{y}^2)dx, \tag{10}$$

where the overbar on the derivatives of F indicates evaluation along $(x, \bar{y}, \dot{\bar{y}} + \theta\delta\dot{y})$, θ as above.

We remark that this definition of $\bar{\delta}^2 J$ (the strong second variation), given as formula (29) in [6], is unclear since θ obtained in (9) may be not unique (for all fixed δy or x).

The fact that θ which appears in (10) should be precised in more detail is shown by **Example 3.1** below, which contradicts the statement of **Theorem 4.6** from [6]: *A necessary condition for a strong relative minimum along an extremal* $\bar{y}(x)$ *is* $\bar{\delta}^2 J \geq 0$ *for all admissible* y *in a sufficiently restricted strong neighbourhood of* $\bar{y}$.

Here the strong neighbourhood of y is defined by the topology of $C(a,b)$.

Example 3.1. *We take* $a = 0, b = 1$,

$$J = \int_0^1 f(y + \dot{y})dx \tag{11}$$

and $f : R \to R$ is the C^∞ mapping given by:

$$f(x) = \begin{cases} e^{\frac{2}{(x-1)^2-1}} & 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

It has two inflexion points in $x_1 = 1 - 1/\sqrt{3}$ and $x_2 = 1 + 1/\sqrt{3}$, supp$f = [0,2]$ and f is positive. We have $f'' < 0$ for $x \in (x_1, x_2)$, $f'' > 0$ for $x \in (0, x_1) \cup (x_2, 2)$.

If we fix $y_a = y_0 = y(0) = 1$ and $y_b = y_1 = y(1) = e^{-1}$, then one (global) solution of the problem (11) is obtained by

$$y + \overset{\bullet}{y} = 0,\ y(0) = 1,\ y(1) = e^{-1}, \tag{12}$$

that is $\bar{y}(x) = e^{-x}$, $x \in [0, 1]$.

We consider the following strong admissible variation δy, $\frac{1}{2} > \lambda > 0$:

$$\delta y = \begin{cases} -3x & x \in [0, \lambda], \\ -6\lambda + 3x & x \in [\lambda, 2\lambda], \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, for $\lambda \to 0$, $\bar{y} + \delta y$ enters any strong neighbourhood of y. By a Taylor expansion around $\bar{y} + \overset{\bullet}{\bar{y}} = 0$ we have

$$f(\delta y + \delta \overset{\bullet}{y}) = \frac{1}{2} f''(\theta_x(\delta y + \delta \overset{\bullet}{y}))(\delta y + \delta \overset{\bullet}{y})^2, \tag{13}$$

where $\theta_x \in (0, 1)$ is as in (9). We remark that

$$\delta y + \delta \overset{\bullet}{y} = \begin{cases} 0 & x \in (2\lambda, 1], \\ 3 + \delta y & x \in (\lambda, 2\lambda), \\ -3 + \delta y & x \in [0, \lambda). \end{cases} \tag{14}$$

Then $f(\delta y + \delta \overset{\bullet}{y}) = 0$ on $[0, 1]$, except at $x = \lambda$, $x = 2\lambda$, since $3 + \delta y > 2$ for $\lambda < \frac{1}{3}$. By (13) and the properties of f'', we can choose $\theta_x(\delta y + \delta \overset{\bullet}{y}) = x_1$, $x \in (\lambda, 2\lambda)$ and arbitrary in $(0, 1)$ otherwise.

We infer from (10) and this choice that

$$\bar{\delta}^2 J = \frac{1}{2} \int_0^1 f''(\theta_x \delta \overset{\bullet}{y})(\delta y + \delta \overset{\bullet}{y})^2 dx =$$

$$= \frac{1}{2} \int_\lambda^{2\lambda} f'' \left(\frac{3x_1}{3 + \delta y(x)} \right) (\delta y(x) + \delta \overset{\bullet}{y}(x))^2 dx < 0,$$

because $x_1 < \dfrac{3x_1}{3 + \delta y(x)} < x_2$ for $x \in (\lambda, 2\lambda)$, λ small.

Remark 3.1. The gap in the proof of **Theorem 4.6**, consists in an inconsistent continuity argument. Namely since $\bar{\delta}^2 J$ is not fixed when δy varies, it is possible that $\bar{\bar{\delta}}^2 J \geq 0$ and $\bar{\delta}^2 J \leq 0$ and $\bar{\delta}^2 J < 0$, although $\bar{\bar{\delta}}^2 J - \bar{\delta}^2 J \to 0$ when $\delta y \to 0$ in $C(a, b)$.

We note that in this proof $\theta \in (0, 1)$ may be arbitrary, therefore **Example 3.1** also contradicts the proof of **Theorem 4.6**.

So it is not possible to "adapt" the statement of the result and to preserve proof.

Remark 3.2. The same argument appears in the proofs given by the author to the wellknown results from **Theorem 3.1, Theorem 4.1** on necessary conditions.

Remark 3.3. In the proofs of sufficient conditions, another vicious continuity argument is used. There are two small parameters, η' and ε_T, and ε_T is made small enough such that $\eta' - \varepsilon_T > 0$ (see (46) in [6]). However, this may be impossible since the two parameters are not independent.

Remark 3.4. Similar considerations may be done in connection with the work [7] by the same author.

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