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New types of fuzzy ideals of ternary semigroups

Noor Rehman[1, 2], Muhammad Shabir[2]

The concepts of $\left(\overline{\in},\overline{\in}\vee\overline{q_k}\right)$-fuzzy left (right, lateral) ideal, $\left(\overline{\in},\overline{\in}\vee\overline{q_k}\right)$-fuzzy quasi-ideals, $\left(\overline{\in},\overline{\in}\vee\overline{q_k}\right)$-fuzzy bi-ideals in ternary semigroups are introduced and several related properties are investigated.

Key words: Ternary semigroup, $\left(\overline{\in},\overline{\in}\vee\overline{q_k}\right)$-fuzzy ideals, $\left(\overline{\in},\overline{\in}\vee\overline{q_k}\right)$-fuzzy quasi-ideals, $\left(\overline{\in},\overline{\in}\vee\overline{q_k}\right)$-fuzzy bi-ideals

1. Introduction

The fundamental concept of ternary semigroup was introduced by Lehmer [14] in 1932. In [24] Sioson studied the ideal theory of ternary semigroups. Fuzzy sets originated in a seminal paper by Zadeh published in 1965 [25]. His paper has opened up new insights and applications in wide range of scientific fields. Extensive applications of fuzzy set theory have been found in diverse fields such as economics, computer science, expert system, information sciences, control theory, decision making, pattern recognition, coding theory, operation research, robotics and many other. Rosenfeld in 1971, used the notion of fuzzy subset of a set to introduce the notion of fuzzy subgroup of a group (see [16]). Rosenfeld's paper inspired the development of fuzzy abstract algebra. The literature on fuzzy set theory has been growing very rapidly. Kuroki initiated the study of fuzzy semigroups (see [9-13]). Pu and Liu in [15] gave the concept of quasi-coincidence of a fuzzy point with a fuzzy set. It is worth mentioning that Bhakat and Das ([2-6]) introduced the concept of (α,β)-fuzzy subgroup by using belongs to and quasi-coincident with between a fuzzy point and a fuzzy subset and introduced the concept of $(\in\vee q)$-level subset, $(\in,\in\vee q)$-fuzzy normal, quasinormal, maximal subgroup and $(\in,\in\vee q)$-fuzzy subgroup. In fact $(\in,\in\vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. Many researchers used these concepts to generalize some concepts of algebra (see [1, 18 23]). The present authors in [22] introduced (α,β)-fuzzy ideals in ternary semigroups, where they characterized regular ternary semigroups by the properties of these ideals. Jun in [7] has generalized the concept of quasi-coincidence of a fuzzy point with a fuzzy set while defining $(\in,\in\vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras. In [8] Jun et al. discussed $(\in,\in\vee q_k)$-fuzzy h-ideals and $(\in,\in\vee q_k)$-fuzzy k-ideals of a hemiring. Shabir et al. in [19] characterized different classes of semigroups by $(\in,\in\vee q_k)$-fuzzy ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals. Shabir and Mahmood in [20] defined $(\in,\in\vee q_k)$-fuzzy h-subhemirings, $(\in,\in\vee q_k)$-fuzzy h-bi-ideals of a hemiring. Shabir and Rehman initiated the study of $(\in,\in\vee q_k)$-fuzzy ideals of ternary semigroups [21]. Recently Shabir and Ali in [17] characterized semigoups by the properties of their $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideals.

The purpose of this paper is to initiate the study of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ternary subsemigroups, $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (right, lateral) ideals, $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideals and $(\overline{\in},\overline{\in}\vee\overline{q_k})$-

[1] Corresponding author, Department of Basic Sciences, Riphah International University Islamabad, Pakistan, Email Address: noorrehman@qau.edu.pk; noorrehman82@yahoo.com
[2] Department of Mathematics Quaid-i-Azam University Islamabad, Pakistan
Email Address: mshabirbhatti@yahoo.co.uk

fuzzy bi-(generalized bi-) ideals in ternary semigroups and several related properties are investigated.

2. Preliminaries

In this section we review some definitions and basic results.

A ternary semigroup is an algebraic structure $(T,[\ \ ])$ such that T is a non-empty set and $[\ \ ]: T^3 \to T$ a ternary operation satisfying the associative law: $[[xyz]uv]=[x[yzu]v]=[xy[zuv]]$ for all $x,y,z,u,v \in T.$ For the sake of simplicity we write $[xyz]$ as ` xyz ' and consider the ternary operation as multiplication. A non-empty subset A of a ternary semigroup T is called a ternary subsemigroup of T if $AAA \subseteq A.$ By a left (right, lateral) ideal of a ternary semigroup T we mean a non-empty subset A of T such that $TTA \subseteq A$ $(ATT \subseteq A, TAT \subseteq A).$ If a non-empty subset A of T is a left and right ideal of $T,$ then it is called a two sided ideal of $T.$ If a non-empty subset A of a ternary semigroup T is a left, right and lateral ideal of $T,$ then it is called an ideal of $T.$ A non-empty subset A of a ternary semigroup T is called a quasi-ideal of T if $ATT \cap TAT \cap TTA \subseteq A$ and $ATT \cap TTATT \cap TTA \subseteq A.$ A non-empty subset A of a ternary semigroup T is called a generalized bi-ideal of T if $ATATA \subseteq A.$ A ternary subsemigroup of a ternary semigroup T is called bi-ideal if A is a generalized bi-ideal of $T.$ It is clear that every left (right, lateral) ideal of a ternary semigroup T is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal of $T.$ A non-empty subset A of a ternary semigroup T is called an interior ideal of T if $TTATT \subseteq A.$ An element s of a ternary semigroup T is called regular if there exists an element $x \in T$ such that $sxs = s.$ A ternary semigroup T is called regular if every element of T is regular.

Example 2.1. *Let $\mathbb{Z}^-$ be the set of all negative integers. Then with usual ternary multiplication, $\mathbb{Z}^-$ forms a ternary semigroup.*

Example 2.2 *The set of all odd permutations under composition forms a ternary semigroup.*

Example 2.3 *Let $T=\{-i,0,i\}.$ Then T is a ternary semigroup under the ternary multiplication of complex numbers.*

A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0,1],$ that is, $\lambda : X \to [0,1]$.

A fuzzy subset λ of a universe X of the form

$$\lambda(y)=\begin{cases} t\ \ (\neq 0) & \text{if } y=x \\ 0 & \text{if } y\neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t . For a fuzzy point x_t and fuzzy subset λ in a set X , Pu and Liu in [15], gave meaning to the symbol $x_t\alpha\lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set λ written $x_t \in \lambda$ (resp. $x_t q\lambda$) if $\lambda(x)\geq t$ (resp. $\lambda(x)+t>1$) and in this case, $x_t \in\vee q\lambda$ (resp. $x_t \in\wedge q\lambda$) means that $x_t \in \lambda$ or $x_t q\lambda$ (resp. $x_t \in \lambda$ and $x_t q\lambda$). By $x_t\overline{\alpha}\lambda$ we mean that $x_t\alpha\lambda$ does not hold.

For any two fuzzy subsets λ and μ of $T,$ $\lambda \leq \mu$ means that, for all $x\in T,$ $\lambda(x)\leq\mu(x).$ The symbols $\lambda\wedge\mu$ and $\lambda\vee\mu$ will mean the following fuzzy subsets of $T.$

$$(\lambda\wedge\mu)(x)=\lambda(x)\wedge\mu(x)$$

$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$ for all $x \in T$.

Let λ, μ and ν be three fuzzy subsets of a ternary semigroup T. The product $\lambda \circ \mu \circ \nu$ is a fuzzy subset of T defined by:

$$(\lambda \circ \mu \circ \nu)(a) = \begin{cases} \bigvee_{a=xyz} \{\lambda(x) \wedge \mu(y) \wedge \nu(z)\} & \text{if there exist } x, y, z \in T \text{ such that } a = xyz \\ 0 & \text{otherwise.} \end{cases}$$

Let λ be a fuzzy subset of a ternary semigroup T. Then the set

$U(\lambda; t) = \{x \in T \ : \ \lambda(x) \geq t\}$,

where $t \in [0,1]$, is called a level subset of λ.

Definition 2.1 *A fuzzy subset* λ *of a ternary semigroup* T *is a fuzzy ternary subsemigroup of* T *if* $\lambda(xyz) \geq \lambda(x) \wedge \lambda(y) \wedge \lambda(z)$ *for all* $x, y, z \in T$.

Definition 2.2 *A fuzzy subset* λ *of a ternary semigroup* T *is a fuzzy left (right, lateral) ideal of* T *if* $\lambda(xyz) \geq \lambda(z)$ $(\lambda(xyz) \geq \lambda(x), \lambda(xyz) \geq \lambda(y))$ *for all* $x, y, z \in T$.

Definition 2.3 *A fuzzy subset* λ *of a ternary semigroup* T *is called a fuzzy generalized bi-ideal of* T *if* $\lambda(xuyvz) \geq \lambda(x) \wedge \lambda(y) \wedge \lambda(z)$ *for all* $x, y, z, u, v \in T$ *, and is called a fuzzy bi-ideal of* T *if it is both a fuzzy ternary subsemigroup and a fuzzy generalized bi-ideal of* T.

If $A \subseteq T$, then the characteristic function of A is a function C_A of T into $\{0,1\}$ defined by:

$$C_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

3. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals

Throughout this paper T will denote a ternary semigroup and $k \in [0,1)$ unless stated otherwise.

In [7], Jun defined $x_t q_k \lambda$ if $\lambda(x) + t + k > 1$, $x_t \in \vee q_k \lambda$ if $x_t \in \lambda$ or $x_t q_k \lambda$.

Definition 3.1 *A fuzzy subset* λ *of a ternary semigroup* T *is called an* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-fuzzy ternary subsemigroup of* T *if for all* $x, y, z \in T$ *and* $t, r, s \in (0,1]$ *the following condition hold:*

$(T1a)$ $(xyz)_{\min\{t,r,s\}} \overline{\in} \lambda$ implies $x_t \overline{\in} \vee \overline{q_k} \lambda$ or $y_r \overline{\in} \vee \overline{q_k} \lambda$ or $z_s \overline{\in} \vee \overline{q_k} \lambda$.

Theorem 3.1 *Let* λ *be a fuzzy subset of a ternary semigroup* T, $x, y, z \in T$ *and* $t, r, s \in (0,1]$. *Then the following conditions are equivalent:*

$(T1a)$ $(xyz)_{\min\{t,r,s\}} \overline{\in} \lambda$ implies $x_t \overline{\in} \vee \overline{q_k} \lambda$ or $y_r \overline{\in} \vee \overline{q_k} \lambda$ or $z_s \overline{\in} \vee \overline{q_k} \lambda$.

$(T1b)$ $\max\{\lambda(xyz), \frac{1-k}{2}\} \geq \min\{\lambda(x), \lambda(y), \lambda(z)\}$.

Proof. $(T1a) \Rightarrow (T1b)$: Let $x, y, z \in T$ be such that $\max\{\lambda(xyz), \frac{1-k}{2}\} < \min\{\lambda(x), \lambda(y), \lambda(z)\}$. Choose $t \in (\frac{1-k}{2}, 1]$ such that $\max\{\lambda(xyz), \frac{1-k}{2}\} < t = \min\{\lambda(x), \lambda(y), \lambda(z)\}$. Then $(xyz)_t \overline{\in} \lambda$ but $x_t \in \wedge q_k \lambda$, $y_t \in \wedge q_k \lambda$ and $z_t \in \wedge q_k \lambda$, which is a contradiction. Hence $\max\{\lambda(xyz), \frac{1-k}{2}\} \geq \min\{\lambda(x), \lambda(y), \lambda(z)\}$.

$(T1b) \Rightarrow (T1a)$: Let $(xyz)_{\min\{t,r,s\}} \overline{\in} \lambda$. Then $\lambda(xyz) < \min\{t,r,s\}$. If $\max\{\lambda(xyz), \frac{1-k}{2}\} = \lambda(xyz)$, then $\min\{\lambda(x),\lambda(y),\lambda(z)\} \le \lambda(xyz) < \min\{t,r,s\}$. This implies that $\lambda(x) < t$ or $\lambda(y) < r$ or $\lambda(z) < s$. Thus $x_t \overline{\in} \lambda$ or $y_r \overline{\in} \lambda$ or $z_s \overline{\in} \lambda$. Thus $x_t \overline{\in \vee q_k} \lambda$ or $y_r \overline{\in \vee q_k} \lambda$ or $z_s \overline{\in \vee q_k} \lambda$.

If $\max\{\lambda(xyz), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\min\{\lambda(x),\lambda(y),\lambda(z)\} \le \frac{1-k}{2}$. Suppose $x_t \in \lambda$, $y_r \in \lambda$ and $z_s \in \lambda$. Then $t \le \lambda(x) \le \frac{1-k}{2}$ or $r \le \lambda(y) \le \frac{1-k}{2}$ or $s \le \lambda(z) \le \frac{1-k}{2}$. This implies that $x_t \overline{q_k} \lambda$ or $y_r \overline{q_k} \lambda$ or $z_s \overline{q_k} \lambda$. Hence $x_t \overline{\in} \vee \overline{q_k} \lambda$ or $y_r \overline{\in} \vee \overline{q_k} \lambda$ or $z_s \overline{\in} \vee \overline{q_k} \lambda$.

Corollary 3.1 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of T if it satisfies condition $(T1b)$.*

In the next theorem we describe the relationship among $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup and the crisp ternary subsemigroup of T.

Theorem 3.2 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of T if and only if $U(\lambda;t)(\ne \phi)$ is a ternary subsemigroup of T for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Let λ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of T and $x, y, z \in U(\lambda;t)$ for some $t \in (\frac{1-k}{2}, 1]$. Then $\lambda(x) \ge t$, $\lambda(y) \ge t$ and $\lambda(z) \ge t$. Hence $\frac{1-k}{2} < t \le \min\{\lambda(x),\lambda(y),\lambda(z)\} \le \max\{\lambda(xyz), \frac{1-k}{2}\}$. Thus $\lambda(xyz) \ge t$ and so $xyz \in U(\lambda;t)$. Consequently $U(\lambda;t)$ is a ternary subsemigroup of T.

Conversely, assume that $U(\lambda;t)(\ne \phi)$ is a ternary subsemigroup of T for all $t \in (\frac{1-k}{2}, 1]$. Suppose that there exist $x, y, z \in T$ such that $\max\{\lambda(xyz), \frac{1-k}{2}\} < \min\{\lambda(x),\lambda(y),\lambda(z)\}$. Choose $t \in (\frac{1-k}{2}, 1]$ such that $\max\{\lambda(xyz), \frac{1-k}{2}\} < t \le \min\{\lambda(x),\lambda(y),\lambda(z)\}$. Then $x, y, z \in U(\lambda;t)$, but $xyz \notin U(\lambda;t)$, which is a contradiction. Therefore, $\max\{\lambda(xyz), \frac{1-k}{2}\} \ge \min\{\lambda(x),\lambda(y),\lambda(z)\}$. Hence λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of T.

Theorem 3.3 *Let A be a non-empty subset of a ternary semigroup T. Define a fuzzy subset λ of T by:*

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in A \\ \le \dfrac{1-k}{2} & \text{otherwise.} \end{cases}$$

Then A is a ternary subsemigroup of T if and only if λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of T.

Proof. Suppose A is a ternary subsemigroup of T and $x, y, z \in T$. If $x, y, z \in A$, then $\lambda(x) = \lambda(y) = \lambda(z) = 1$. Since A is a ternary subsemigroup of T, so $xyz \in A$. This implies that $\lambda(xyz) = 1$. Hence $\max\left\{\lambda(xyz), \frac{1-k}{2}\right\} = 1 = \min\left\{\lambda(x), \lambda(y), \lambda(z)\right\}$. If $x \notin A$ or $y \notin A$ or $z \notin A$, then $\lambda(x) \le \frac{1-k}{2}$ or $\lambda(y) \le \frac{1-k}{2}$ or $\lambda(z) \le \frac{1-k}{2}$. Thus $\min\left\{\lambda(x), \lambda(y), \lambda(z)\right\} \le \frac{1-k}{2} \le \max\left\{\lambda(xyz), \frac{1-k}{2}\right\}$. Hence in any case $\max\left\{\lambda(xyz), \frac{1-k}{2}\right\} \ge \min\left\{\lambda(x), \lambda(y), \lambda(z)\right\}$. This shows that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of T.

Conversely, assume that λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ternary subsemigroup of T and $x,y,z\in A.$ Then $\lambda(x)=\lambda(y)=\lambda(z)=1.$ Thus by hypothesis $\max\{\lambda(xyz),\frac{1-k}{2}\}\geq\min\{\lambda(x),\lambda(y),\lambda(z)\}=1.$ This implies that $\lambda(xyz)=1,$ that is $xyz\in A$. Hence A is a ternary subsemigroup of $T.$

Corollary 3.2 *A non-empty subset A of a ternary semigroup T is a ternary subsemigroup of T if and only if the characteristic function C_A , of $A,$ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ternary subsemigroup of $T.$*

Definition 3.2 *A fuzzy subset λ of a ternary semigroup T is called an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (resp. right, lateral) ideal of T if the following condition holds.*

$(T2a)$ $(xyz)_t\overline{\in}\lambda$ implies $z_t\overline{\in}\vee\overline{q_k}\lambda$ (resp. $(T3a)$ $(xyz)_t\overline{\in}\lambda$ implies $x_t\overline{\in}\vee\overline{q_k}\lambda,$ $(T4a)$ $(xyz)_t\overline{\in}\lambda$ implies $y_t\overline{\in}\vee\overline{q_k}\lambda$) for all $x,y,z\in T$ and $t\in(0,1].$

A fuzzy subset λ of a ternary semigroup T is called an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy two sided ideal of T if it is both an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal and $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal of $T.$ A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of T if it is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal, $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy lateral ideal and an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal of T .

Theorem 3.4 *Let λ be a fuzzy subset of a ternary semigroup $T,$ $x,y,z\in T$ and $t\in(0,1].$ Then the following conditions are equivalent:*

$(T2a)$ $(xyz)_t\overline{\in}\lambda$ implies $z_t\overline{\in}\vee\overline{q_k}\lambda;$

$(T2b)$ $\max\{\lambda(xyz),\frac{1-k}{2}\}\geq\lambda(z).$

$$\left(\begin{array}{l}\text{resp. } (T3a)\ (xyz)_t\overline{\in}\lambda \text{ implies } x_t\overline{\in}\vee\overline{q_k}\lambda.\\ (T3b)\\ \\ \max\{\lambda(xyz),\frac{1-k}{2}\}\geq\lambda(x)\end{array}\right)$$

Proof. $(T2a)\Rightarrow(T2b):$ Let $x,y,z\in T$ be such that $\max\{\lambda(xyz),\frac{1-k}{2}\}<\lambda(z).$ Choose $t\in(\frac{1-k}{2},1]$ such that $\max\{\lambda(xyz),\frac{1-k}{2}\}<t=\lambda(z).$ Then $(xyz)_t\overline{\in}\lambda$ but $z_t\in\wedge q_k\lambda,$ which is a contradiction. Hence $\max\{\lambda(xyz),\frac{1-k}{2}\}\geq\lambda(z).$

$(T2b)\Rightarrow(T2a):$ Assume that $(xyz)_t\overline{\in}\lambda.$ Then $\lambda(xyz)<t.$ If $\max\{\lambda(xyz),\frac{1-k}{2}\}=\lambda(xyz),$ then $\lambda(z)\leq\lambda(xyz)<t.$ This implies that $\lambda(z)<t.$ It follows that $z_t\overline{\in}\lambda.$ Thus $z_t\overline{\in}\vee\overline{q_k}\lambda.$ If $\max\{\lambda(xyz),\frac{1-k}{2}\}=\frac{1-k}{2},$ then $\lambda(z)\leq\frac{1-k}{2}.$ Suppose $z_t\in\lambda.$ Then $t\leq\lambda(x)<\frac{1-k}{2}.$ It follows that $z_t\overline{q_k}\lambda.$ Hence $z_t\overline{\in}\vee\overline{q_k}\lambda.$

Similarly, $(T3a)\Leftrightarrow(T3b)$ and $(T4a)\Leftrightarrow(T4b).$

Corollary 3.3 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (resp. right, lateral) ideal of T if it satisfies condition $(T2b)$ (resp. $(T3b),(T4b)$).*

Corollary 3.4 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of T if it satisfies conditions $(T2b),$ $(T3b)$ and $(T4b).$*

In the next theorem we describe the relationship among $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy left (resp. right, lateral) ideal and the crisp left (resp. right, lateral) ideal of* T.

Theorem 3.5 *A fuzzy subset* λ *of a ternary semigroup* T *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy left (resp. right, lateral) ideal of* T *if and only if* $U(\lambda;t)(\neq\phi)$ *is a left (resp. right, lateral) ideal of* T *for all* $t\in(\frac{1-k}{2},1]$.

Proof. Let λ be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left ideal of T. Suppose $z\in U(\lambda;t)(\neq\phi)$ and $x,y\in T$, for some $t\in(\frac{1-k}{2},1]$. Then $\lambda(z)\geq t$. Hence $\frac{1-k}{2}<t\leq\lambda(z)\leq\max\{\lambda(xyz),\frac{1-k}{2}\}$. This implies that $\lambda(xyz)\geq\lambda(z)\geq t$. Thus $\lambda(xyz)\geq t$ and so $xyz\in U(\lambda;t)$. Consequently $U(\lambda;t)$ is a left ideal of T.

Conversely, assume that $U(\lambda;t)(\neq\phi)$ is a left ideal of T for all $t\in(\frac{1-k}{2},1]$. Suppose that there exist $x,y,z\in T$ such that $\max\{\lambda(xyz),\frac{1-k}{2}\}<\lambda(z)=t$. Then $t\in(\frac{1-k}{2},1]$ and $z\in U(\lambda;t)$ but $xyz\notin U(\lambda;t)$, which is a contradiction. Hence $\max\{\lambda(xyz),\frac{1-k}{2}\}\geq\lambda(z)$ and so λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left ideal of T.

Similarly, we can prove the case of right and lateral ideal of T.

Corollary 3.5 *A fuzzy subset* λ *of a ternary semigroup* T *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy ideal of* T *if and only if* $U(\lambda;t)(\neq\phi)$ *is an ideal of* T *for all* $t\in(\frac{1-k}{2},1]$.

Theorem 3.6 *Let* A *be a non-empty subset of a ternary semigroup* T. *Define a fuzzy subset* λ *of* T *by:*

$$\lambda(x)=\begin{cases}1 & if\ x\in A\\ \leq\dfrac{1-k}{2} & otherwise.\end{cases}$$

Then A is a left (resp. right, lateral) ideal of T if and only if λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left (resp. right, lateral) ideal of T.

Proof. The proof is similar to the proof of Theorem 3.3.

Corollary 3.6 *A non-empty subset* A *of a ternary semigroup* T *is a left (resp. right, lateral, two sided) ideal of* T *if and only if the characteristic function* C_A *, of* A *, is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy left (resp. right, lateral, two sided) ideal of* T.

Theorem 3.7 *Let* λ *be an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy left ideal,* μ *an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy lateral ideal and* ν *an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy right ideal of* T. *Then* $\lambda\circ\mu\circ\nu$ *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy two sided ideal of* T.

Proof. Suppose λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left ideal, μ an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy lateral ideal and ν an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy right ideal of T $x,y,z\in T$. Then

$$(\lambda\circ\mu\circ\nu)(z)=\bigvee_{z=uvw}\{\lambda(u)\wedge\mu(v)\wedge\nu(w)\}$$

[If $z=uvw$, then $xyz=xy(uvw)=(xyu)vw$. Since λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left ideal of T, so $\lambda(xyu)\vee\frac{1-k}{2}\geq\lambda(u)$]. Thus

$$
\begin{aligned}
(\lambda\circ\mu\circ\nu)(z) &= \bigvee_{z=uvw}\{\lambda(u)\wedge\mu(v)\wedge\nu(w)\}\\
&\le \bigvee_{z=uvw}\left\{\left(\lambda(xyu)\vee\frac{1-k}{2}\right)\wedge\mu(v)\wedge\nu(w)\right\}\\
&\le \bigvee_{xyz=abc}\left\{(\lambda(a)\wedge\mu(b)\wedge\nu(c))\vee\frac{1-k}{2}\right\}
\end{aligned}
$$

$$
=\left(\bigvee_{xyz=abc}\{\lambda(a)\wedge\mu(b)\wedge\nu(c)\}\right)\vee\frac{1-k}{2}
$$

$$
=(\lambda\circ\mu\circ\nu)(xyz)\vee\frac{1-k}{2}.
$$

Thus $(\lambda\circ\mu\circ\nu)(xyz)\vee\frac{1-k}{2}\ge(\lambda\circ\mu\circ\nu)(z).$

If $(\lambda\circ\mu\circ\nu)(xyz)=0,$ then $(\lambda\circ\mu\circ\nu)(xyz)=0\le(\lambda\circ\mu\circ\nu)(xyz)\vee\frac{1-k}{2}.$ Hence $\lambda\circ\mu\circ\nu$ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left ideal of $T.$

Similarly, we can show that $\lambda\circ\mu\circ\nu$ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal of $T.$ Hence, $\lambda\circ\mu\circ\nu$ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy two sided ideal of $T.$

Lemma 3.1 *The intersection of any family of $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left (resp. right, lateral) ideals of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left (resp. right, lateral) ideal of $T.$*

Proof. Let $\{\lambda_i\}_{i\in I}$ be a family of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideals of T and $x,y,z\in T.$ Since each λ_i is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of $T,$ we have $\lambda_i(xyz)\vee\frac{1-k}{2}\ge\lambda_i(z)$ for all $i\in I.$ Thus

$$((\bigwedge_{i\in I}\lambda_i)(xyz))\vee\tfrac{1-k}{2}=(\bigwedge_{i\in I}(\lambda_i(xyz))\vee\tfrac{1-k}{2}$$

$$=\bigwedge_{i\in I}(\lambda_i(xyz)\vee\tfrac{1-k}{2})\ge\bigwedge_{i\in I}(\lambda_i(z))=(\bigwedge_{i\in I}\lambda_i)(z).$$

Hence $\bigwedge_{i\in I}\lambda_i$ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of $T.$

In a similar way we can prove the following lemma.

Lemma 3.2 *The union of any family of $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left (resp. right, lateral) ideals of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$ -fuzzy left (resp. right, lateral) ideal of $T.$*

Theorem 3.8 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (resp. right, lateral) ideal of T if and only if $\mathbf{T}\circ^k\mathbf{T}\circ^k\lambda\le\lambda^k$ (resp. $\lambda\circ^k\mathbf{T}\circ^k\mathbf{T}\le\lambda^k$, $\mathbf{T}\circ^k\lambda\circ^k\mathbf{T}\le\lambda^k$), where $\lambda^k=\lambda\vee\frac{1-k}{2}$ and* $\mathbf{T}$ is the fuzzy subset of T mapping every element of T on 1.

Proof. Suppose λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of a ternary semigroup $T.$ Let $x\in T.$ If $(\mathbf{T}\circ^k\mathbf{T}\circ^k\lambda)(x)=0,$ then $(\mathbf{T}\circ^k\mathbf{T}\circ^k\lambda)(x)\le\lambda(x).$ Otherwise, there exist elements $p,q,r\in T$ such that $x=pqr.$ Then we have

$$\left(\mathbf{T}\circ^k \mathbf{T}\circ^k \lambda\right)(x) = (\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\vee\frac{1-k}{2}$$
$$= \left(\bigvee_{x=pqr}\left\{\mathbf{T}(p)\wedge\mathbf{T}(q)\wedge\lambda(r)\right\}\right)\vee\frac{1-k}{2}$$
$$= \bigvee_{x=pqr}\lambda(r)\vee\frac{1-k}{2}$$

$$\leq\left(\bigvee_{x=pqr}\lambda(pqr)\vee\frac{1-k}{2}\right)\vee\frac{1-k}{2}$$
$$=\lambda(x)\vee\frac{1-k}{2}=\lambda^k(x).$$

This implies that $\mathbf{T}\circ^k \mathbf{T}\circ^k \lambda \leq \lambda^k$.

Conversely, assume that λ satisfies the given condition. Let $x,y,z\in T.$ Then

$$\lambda^k(xyz)\geq\left(\mathbf{T}\circ^k \mathbf{T}\circ^k \lambda\right)(xyz)$$
$$=(\mathbf{T}\circ\mathbf{T}\circ\lambda)(xyz)\vee\frac{1-k}{2}$$
$$=\left(\bigvee_{xyz=pqr}\left\{\mathbf{T}(p)\wedge\mathbf{T}(q)\wedge\lambda(r)\right\}\right)\vee\frac{1-k}{2}$$
$$\geq\mathbf{T}(x)\wedge\mathbf{T}(y)\wedge\lambda(z)\vee\frac{1-k}{2}=\lambda(z)\vee\frac{1-k}{2}\geq\lambda(z).$$

So $\lambda(xyz)\vee\frac{1-k}{2}\geq\lambda(z).$ This shows that λ satisfies the condition $(T2b).$ So λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of $T.$

Similarly, we can prove cases for λ be $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right and lateral ideal of $T.$

Definition 3.3 *A fuzzy subset λ of a ternary semigroup T is called an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T if it satisfies:*

$(T5a)$ $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}$ and

$(T5a')$ $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}$

for all $x\in T.$

Theorem 3.9 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T if and only if $U(\lambda;t)(\neq\phi)$ is a quasi-ideal of T for all $t\in(\frac{1-k}{2},1].$*

Proof. Let λ be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of $T.$ Let $x\in TTU(\lambda;t)\cap TU(\lambda;t)T\cap U(\lambda;t)TT.$ Then there exist $t_1,t_2,t_3,t_4,t_5,t_6\in T$ and $s_1,s_2,s_3\in U(\lambda;t)$ such that $x=t_1t_2s_1$, $x=t_3s_2t_4$ and $x=s_3t_5t_6.$ Now,

$$(\lambda\circ\mathbf{T}\circ\mathbf{T})(x)=\bigvee_{x=pqr}\{\lambda(p)\wedge\mathbf{T}(q)\wedge\mathbf{T}(r)\}\geq\lambda(s_3)\mathbf{T}(t_5)\wedge\mathbf{T}(t_6))=\lambda(s_3)\geq t.$$

Also, $(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)=\bigvee_{x=uvw}\left\{\mathbf{T}(u)\wedge\mathbf{T}(v)\wedge\lambda(w)\right\}\geq\mathbf{T}(t_1)\wedge\mathbf{T}(t_2)\wedge\lambda(s_1)=\lambda(s_1)\geq t,$

And $(\mathbf{T}\circ\lambda\circ\mathbf{T})(x)=\bigvee_{x=lmn}\{\mathbf{T}(l)\wedge\lambda(m)\wedge\mathbf{T}(n)\}\geq\mathbf{T}(t_3)\wedge\lambda(s_2)\wedge\mathbf{T}(t_4)=\lambda(s_2)\geq t.$

Thus by hypothesis

$$\max\{\lambda(x),\tfrac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}\geq t>\tfrac{1-k}{2}.$$

This implies that $\lambda(x)\geq t,$ that is $x\in U(\lambda;t).$ This shows that $TTU(\lambda;t)\cap TU(\lambda;t)T\cap U(\lambda;t)TT\subseteq U(\lambda;t).$

Similarly $TTU(\lambda;t)\cap TTU(\lambda;t)TT\cap U(\lambda;t)TT\subseteq U(\lambda;t).$ Thus $U(\lambda;t)$ is a quasi-ideal of T.

Conversely, assume that $(\neq\phi)U(\lambda;t)$ is a quasi-ideal of T for all $t\in(\tfrac{1-k}{2},1].$ Let $x\in T$ be such that

$$\max\{\lambda(x),\tfrac{1-k}{2}\}<\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}.$$

Choose $t\in(\tfrac{1-k}{2},1]$ such that

$$\max\{\lambda(x),\tfrac{1-k}{2}\}<t\leq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}.$$

This implies that $\lambda(x)<t$ and $(\lambda\circ\mathbf{T}\circ\mathbf{T})(x)\geq t,(\mathbf{T}\circ\lambda\circ\mathbf{T})(x)\geq t$ and $(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\geq t.$

If $(\lambda\circ\mathbf{T}\circ\mathbf{T})(x)\geq t,$ then

$(\lambda\circ\mathbf{T}\circ\mathbf{T})(x)=\bigvee_{x=uvw}\{\lambda(u)\wedge\mathbf{T}(v)\wedge\mathbf{T}(w)\}=\bigvee_{x=uvw}\lambda(u)\geq t.$ This implies that there exists $z\in U(\lambda;t)$ such that $x=abz$ for some $a,b\in T.$ Similarly $(\mathbf{T}\circ\lambda\circ\mathbf{T})(x)\geq t$ implies $m\in U(\lambda;t)$ such that $x=lmn$ for some $l,n\in T,$ and $(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\geq t$ implies $r\in U(\lambda;t)$ such that $x=pqr$ for some $p,q\in T.$ Thus $x\in TTU(\lambda;t)\cap TU(\lambda;t)T\cap U(\lambda;t)TT\subseteq U(\lambda;t).$ This implies that $\lambda(x)\geq t,$ which is a contradiction, because $\lambda(x)<t.$ Hence $\max\{\lambda(x),\tfrac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}.$ Similarly, $\max\{\lambda(x),\tfrac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}.$

This shows that λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.

Theorem 3.10 *Let A be a non-empty subset of a ternary semigroup T. Define a fuzzy subset λ of T by:*

$$\lambda(x)=\begin{cases}1 & if\ x\in A\\ \leq\dfrac{1-k}{2} & otherwise.\end{cases}$$

Then A is a quasi-ideal of T if and only if λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.

Proof. Let A be a quasi-ideal of T and $x\in T.$ If $x\in A$, then $\lambda(x)=1.$ This implies that $\max\{\lambda(x),\tfrac{1-k}{2}\}=1\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}.$ If $x\notin A,$ then $x\notin TTA\cap TAT\cap ATT$ and $x\notin TTA\cap TTATT\cap ATT.$ So $\lambda(x)\leq\tfrac{1-k}{2}.$ Since $x\notin TTA\cap TAT\cap ATT,$

therefore, $\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}\neq 1.$ This implies that

$\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}\leq\tfrac{1-k}{2}=\max\{\lambda(x),\tfrac{1-k}{2}\}.$ Similarly, since $x\notin TTA\cap TTATT\cap ATT.$

Therefore, $\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}\neq 1.$

Hence, in any case $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}$, and $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}$. So λ is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.

Conversely, assume that λ is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T. Let $x\in TTA\cap TAT\cap ATT$. Then there exist $a,b,c\in A$ and $t_1,t_2,t_3,t_4,t_5,t_6\in T$ such that $x=at_1t_2$, $x=t_3bt_4$, and $x=t_5t_6c$. Now,

$(\lambda\circ\mathbf{T}\circ\mathbf{T})(x)=\bigvee_{x=pqr}\{\lambda(p)\wedge\mathbf{T}(q)\wedge\mathbf{T}(r)\}\geq\lambda(a)\wedge\mathbf{T}(t_1)\wedge\mathbf{T}(t_2)=1.$ This implies that $(\lambda\circ\mathbf{T}\circ\mathbf{T})(x)=1$. Similarly, $(\mathbf{T}\circ\lambda\circ\mathbf{T})(x)=1$ and $(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)=1$. Thus by hypothesis, $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}=1.$

This implies that $\lambda(x)=1$, that is $x\in A$. This shows that $TTA\cap TAT\cap ATT\subseteq A$. .Similarly, $TTA\cap TTATT\cap ATT\subseteq A$. Hence A is a quasi-ideal of T.

Corollary 3.7 *A non-empty subset A of a ternary semigroup T is a quasi-ideal of T if and only if the characteristic function C_A , of A , is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.*

Lemma 3.3 *The intersection of any family of $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideals of a ternary semigroup T is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.*

Proof. The proof is straightforward.

Theorem 3.11 *Every $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy left (resp. right, lateral) ideal of a ternary semigroup T is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.*

Proof. Suppose λ is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy left ideal of T and $x\in T$. Consider,

$$(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)=\bigvee_{x=abc}\{\mathbf{T}(a)\wedge\mathbf{T}(b)\wedge\lambda(c)\}=\bigvee_{x=abc}\lambda(c)\leq\bigvee_{x=abc}\lambda(abc)\vee\tfrac{1-k}{2}$$

$=\lambda(x)\vee\frac{1-k}{2}$. So $(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\leq\lambda(x)\vee\frac{1-k}{2}$. Hence $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\lambda\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x).$ Also $\max\{\lambda(x),\frac{1-k}{2}\}\geq\min\{(\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ\lambda)(x)\}.$

Hence λ is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of T.

The converse of the above theorem is not true in general.

Example 4. *Let* $T=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

be the ternary semigroup under matrix multiplication. Then

$$Q=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$$

be the quasi-ideal of T, which is neither a left ideal, nor a right ideal, nor a lateral ideal of T. Then by Corollary 3.7, the characteristic function C_Q of Q is an $(\bar{\in},\bar{\in}\vee\overline{q_k})$ -fuzzy quasi-ideal of T, but neither $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy left ideal, nor $(\bar{\in}\vee\overline{q_k})$ -fuzzy right ideal, nor $(\bar{\in},\bar{\in}\vee\overline{q_k})$-fuzzy lateral ideal of T.

Definition 3.4 *A fuzzy subset* λ *of a ternary semigroup* T *is called an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy bi-ideal of* T *if it satisfies* $(T1a)$ *and*

$(T6a)$ $(xuyvz)_{\min\{t,r,s\}}\overline{\in}\lambda$ implies $x_t\overline{\in}\vee\overline{q_k}\lambda$ or $y_r\overline{\in}\vee\overline{q_k}\lambda$ or $z_s\overline{\in}\vee\overline{q_k}\lambda$ for all $x,y,z,u,v\in T$ and $t,r,s\in(0,1]$.

Theorem 3.12 *Let* λ *be a fuzzy subset of a ternary semigroup* T*. Then the following conditions are equivalent:*

$(T6a)$ $(xuyvz)_{\min\{t,r,s\}}\overline{\in}\lambda$ implies $x_t\overline{\in}\vee\overline{q_k}\lambda$ or $y_r\overline{\in}\vee\overline{q_k}\lambda$ or $z_s\overline{\in}\vee\overline{q_k}\lambda$;

$(T6b)$ $\max\{\lambda(xuyvz),\frac{1-k}{2}\}\geq\min\{\lambda(x),\lambda(y),\lambda(z)\}$, for all $x,y,z,u,v\in T$ and $t,r,s\in(0,1]$.

Proof. The proof is similar to the proof of Theorem 3.4.

Corollary 3.8 *A fuzzy subset* λ *of a ternary semigroup* T *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy bi- ideal of* T *if it satisfies* $(T1b)$ *and* $(T6b)$.

Theorem 3.13 *A fuzzy subset* λ *of a ternary semigroup* T *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy bi-ideal of* T *if and only if* $U(\lambda;t)(\neq\phi)$ *is a bi-ideal of* T *for all* $t\in(\frac{1-k}{2},1]$.

Proof. The proof is similar to the proof of Theorem 3.5.

Theorem 3.14 *Let* A *be a non-empty subset of a ternary semigroup* T*. Define a fuzzy subset* λ *of* T *by:*

$$\lambda(x)=\begin{cases}1 & if\ x\in A\\ \leq\dfrac{1-k}{2} & otherwise.\end{cases}$$

Then A is a bi-ideal of T if and only if λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal of T.

Proof. The proof is similar to the proof of Theorem 3.3.

Corollary 3.9 *A non-empty subset* A *of a ternary semigroup* T *is a bi-ideal of* T *if and only if the characteristic function* C_A *, of* A *, is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy bi-ideal of* T.

Lemma 3.3 *The intersection of any family of* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy bi-ideals of a ternary semigroup* T *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy bi-ideal of* T.

Proof. The proof is similar to the proof of Lemma 3.1.

Lemma 3.4 *Let* λ,μ *and* ν *be* $(\overline{\in},\overline{\in}\vee\overline{q_k})$ *-fuzzy bi-ideals of a ternary semigroup* T*. Then* $\lambda\circ\mu\circ\nu$ *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy bi-ideal of* T.

Proof. The proof is straightforward.

Definition 3.5 *A fuzzy subset* λ *of a ternary semigroup* T *is called an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy generalized bi-ideal of* T *if it satisfies* $(T6a)$ *or* $(T6b)$.

Theorem 3.15 *A fuzzy subset* λ *of a ternary semigroup* T *is an* $(\overline{\in},\overline{\in}\vee\overline{q_k})$*-fuzzy generalized bi-ideal of* T *if and only if* $U(\lambda;t)(\neq\phi)$ *is a generalized bi-ideal of* T *for all* $t\in(\frac{1-k}{2},1]$.

Proof. The proof is similar to the proof of Theorem 3.5.

Theorem 3.16 *Let* A *be a non-empty subset of a ternary semigroup* T*. Define a fuzzy subset* λ *of* T *by:*

$$\lambda(x)=\begin{cases}1 & if\ x\in A\\ \leq \frac{1-k}{2} & otherwise.\end{cases}$$

Then A is a generalized bi-ideal of T if and only if λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy generalized bi-ideal of T.

Proof. The proof is similar to the proof of Theorem 3.3.

Corollary 3.10 *A non-empty subset A of a ternary semigroup T is a generalized bi-ideal of T if and only if the characteristic function C_A, of A, is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy generalized bi-ideal of T.*

Definition 3.6 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T if it satisfies*

$(T7a)$ $(uvxyz)_t\overline{\in}\lambda$ implies $x_t\overline{\in}\vee\overline{q_k}\lambda$ for all $x,y,z,u,v\in T$ and $t\in(0,1]$.

Theorem 3.17 *Let λ be a fuzzy subset of a ternary semigroup T, $x,y,z,u,v\in T$ and $t\in(0,1]$. Then the following conditions are equivalent:*

$(T7a)$ $(uvxyz)_t\overline{\in}\lambda$ implies $x_t\overline{\in}\vee\overline{q_k}\lambda$;

$(T7b)$ $\max\{\lambda(uvxyz),\frac{1-k}{2}\}\geq\lambda(x)$.

Proof. The proof is similar to the proof of Theorem 3.4.

Corollary 3.10 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T if it satisfies $(T7b)$.*

Theorem 3.18 *A fuzzy subset λ of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T if and only if $U(\lambda;t)(\neq\phi)$ is an interior-ideal of T for all $t\in(\frac{1-k}{2},1]$.*

Proof. The proof is similar to the proof of Theorem 3.5.

Theorem 3.19 *Let A be a non-empty subset of a ternary semigroup T. Define a fuzzy subset λ of T by:*

$$\lambda(x)=\begin{cases}1 & if\ x\in A\\ \leq \frac{1-k}{2} & otherwise.\end{cases}$$

Then A is an interior-ideal of T if and only if λ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T.

Proof. The proof is similar to the proof of Theorem 3.3.

Corollary 3.11 *A non-empty subset A of a ternary semigroup T is an interior-ideal of T if and only if the characteristic function C_A, of A, is an an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T.*

Lemma 3.5 *The intersection of any family of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideals of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T.*

Proof: The proof is straightforward.

Lemma 3.6 *Every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy lateral ideal of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T.*

Proof. The proof is straightforward.

Corollary 3.12 *Every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of a ternary semigroup T is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of T.*

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