UPPER BOUND OF SECOND HANKEL DETERMINANT FOR k-BI-SUBORDINATE FUNCTIONS

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In this work, we determine an upper bound of the functional $H_2(2) = a_2a_4 - a_3^2$ for functions belonging to a subclass of analytic bi-univalent functions which is defined by subordination conditions in the open unit disk $\mathbb{D}$. In addition, we get a smaller upper bound and more accurate estimation than the previous results and we correct their mistake.

Keywords: Univalent function, k-bi-subordinate functions, second Hankel determinant, subordination.

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1. **Introduction**

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{1}$$

A function $f : \mathbb{D} \to \mathbb{C}$ is called univalent on $\mathbb{D}$ if $f(z_1) \neq f(z_2)$ all $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$. Let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$.

A function $f \in \mathcal{A}$ is said to be *starlike*, if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{D}). \tag{2}$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike by $\mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be *convex*, if it satisfies the inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{D}). \tag{3}$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are convex by $\mathcal{C}$.

For two functions f and g which are analytic in $\mathbb{D}$, we say that the function f is *subordinate* to g, and write $f(z) \prec g(z)$, if there exists a *Schwarz function* w, that is a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$.
In particular, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$, [7].

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By means of the subordination, the conditions (2) and (3) are, respectively, equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \qquad \text{and} \qquad 1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}.$$

Ma and Minda [11] gave a unified presentation of various subclasses of starlike and convex functions by replacing the subordinate function $\frac{1+z}{1-z}$ by a more general analytic function φ with positive real part in the unit disk $\mathbb{D}$, symmetric with respect to the real axis and starlike with respect to $\varphi(0)=1$, and $\varphi'(0)>0$.

One of the important tools in the theory of univalent functions are the *Hankel determinants* which are used, for example, in showing that a function of bounded characteristic in $\mathbb{D}$, that is, a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [5].

In 1976, Noonan and Thomas [13] defined the *q-th Hankel determinant* for integers $n \geq 1$ and $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1)\,.$$

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \qquad \text{and} \qquad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix},$$

where the Hankel determinants $H_2(1)=a_3-a_2^2$ and $H_2(2)=a_2a_4-a_3^2$ are well-known as *Fekete-Szegö* and *second Hankel determinant* functionals, respectively. Further, Fekete and Szegö [8] introduced the generalized functional $a_3-\lambda a_2^2$, where λ is some real number. Problems in this field has also been argued by several authors (see for example [1, 4, 6, 9, 14, 15, 16, 20]).

In 1983, Sălăgean [17] introduced differential operator $D^k : \mathcal{A} \to \mathcal{A}$ defined by

$$D^0 f(z) = f(z), \qquad D^1 f(z) = Df(z) = zf'(z),$$

and in general

$$D^k f(z) = D(D^{k-1} f(z)), \qquad k \in \mathbb{N} = \{1, 2, \ldots\}\,.$$

We easily find that

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \qquad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,,$$

with $D^k f(0) = 0$.

The Koebe one-quarter theorem [7] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus every function $f \in \mathcal{S}$ has an inverse f^{-1}, such that

$$f^{-1}(f(z)) = z \ \ (z \in \mathbb{D})\,, \quad \text{and} \quad f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where the inverse f^{-1} has the power series expansion (see [10])

$$g(w) := f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \ldots. \tag{4}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$. For a brief history of functions in the class Σ and also different other characteristics of these functions see [2, 10, 18, 19, 21] and the references therein.

In this work, we assume that the function φ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, such that $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function has the power series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \; z \in \mathbb{D} \quad (B_1 > 0). \tag{5}$$

By means of the subordination, Bulut [3] defined the class $\mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$ of analytic bi-univalent functions as follows:

Definition 1.1. Let $m, k \in \mathbb{N}_0 \,; m > k$ and $\gamma \in \mathbb{C}\setminus\{0\}$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{D^m f(z)}{D^k f(z)} - 1\right) \prec \varphi(z) \tag{6}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^k g(w)} - 1\right) \prec \varphi(w), \tag{7}$$

where $z, w \in \mathbb{D}$ and the function $g = f^{-1}$ is defined by (4).

Remark 1.1. For $m = k+1$, we get the class $\mathcal{B}_{\Sigma}^{k+1,k}(\gamma;\varphi) = \mathcal{B}_{\Sigma,k}(\gamma;\varphi)$ of *k-bi-subordinate functions of complex order* $\gamma \in \mathbb{C}\setminus\{0\}$.

Remark 1.2. If we set

$$m = k+1, \quad \gamma = 1 \quad \text{and} \quad \varphi(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \le \beta < 1)$$

in Definition 1.1, then the class $\mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$ reduces to the class $\mathcal{S}_{\Sigma,k}(\beta)$ of *k-bi-starlike functions*. In other words, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma,k}(\beta)$, if the following conditions are satisfied (see [15]):

$$\operatorname{Re}\left(\frac{D^{k+1} f(z)}{D^k f(z)}\right) > \beta \quad \textit{and} \quad \operatorname{Re}\left(\frac{D^{k+1} g(z)}{D^k g(z)}\right) > \beta.$$

For $k = 0$ and $k = 1$, we get the classes

$$\mathcal{S}_{\Sigma,0}(\beta) = \mathcal{S}_{\Sigma}^{*}(\beta) \quad \text{and} \quad \mathcal{S}_{\Sigma,1}(\beta) = \mathcal{K}_{\Sigma}(\beta),$$

which are the class of *bi-starlike functions* of order β and *bi-convex functions* of order β, respectively. In particular, we have the classes

$$\mathcal{S}_{\Sigma,0}(0) = \mathcal{S}_{\Sigma}^{*} \quad \text{and} \quad \mathcal{S}_{\Sigma,1}(0) = \mathcal{K}_{\Sigma},$$

which are the class of *bi-starlike functions* and *bi-convex functions*, respectively.

Example 1.1. *If we set* $f(z) = \dfrac{z}{1-z}$ *and* $\varphi(z) = \dfrac{1+z}{1-z}$ *where* $z \in \mathbb{D}$*, then both* $f(z)$ *and* $g(w) = f^{-1}(w) = \dfrac{w}{1+w}$ *are univalent in* $\mathbb{D}$ *and so* $f \in \Sigma$*. On other the hand, conditions* (6) *and* (7) *hold for* $k = 1$*,* $m = 2$ *and* $\gamma = 1$*, that is,*

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+z}{1-z} \prec \frac{1+z}{1-z}, \quad \text{this is equvalent with} \quad \operatorname{Re}\left(\frac{1+z}{1-z}\right) > 0$$

and

$$1 + \frac{wg''(w)}{g'(w)} = \frac{1-w}{1+w} \prec \frac{1+w}{1-w}, \quad \text{this is equvalent with} \quad \operatorname{Re}\left(\frac{1-w}{1+w}\right) > 0.$$

Therefore $f \in \mathcal{B}_{\Sigma}^{2,1}\left(1; \dfrac{1+z}{1-z}\right)$*, in other words* f *is* 1*-bi-convex function (bi-convex function). Since every convex function is a starlike function, so also* f *is* 1*-bi-starlike function (bi-starlike function).*

Theorem 1.1. [6, Theorem 2.1] *Let the function f given by* (1) *be in the class* $\mathcal{S}^{*}_{\Sigma}(\beta)$ $(0 \le \beta < 1)$. *Then*

$$\left|a_2a_4 - a_3^2\right| \le \begin{cases} \frac{4(1-\beta)^2}{3}\left(4\beta^2 - 8\beta + 5\right) & , \quad \beta \in \left[0, \frac{29-\sqrt{137}}{32}\right] \\ \\ (1-\beta)^2\left(\frac{13\beta^2-14\beta-7}{16\beta^2-26\beta+5}\right) & , \quad \beta \in \left(\frac{29-\sqrt{137}}{32}, 1\right). \end{cases}$$

Corollary 1.1. [6, Corollary 2.2] *Let the function f given by* (1) *be in the class* $\mathcal{S}^{*}_{\Sigma}$. *Then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{20}{3}.$$

Theorem 1.2. [6, Theorem 2.3] *Let the function f given by* (1) *be in the class* $\mathcal{K}_{\Sigma}(\beta)$ $(0 \le \beta < 1)$. *Then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{(1-\beta)^2}{24}\left(\frac{5\beta^2 + 8\beta - 32}{3\beta^2 - 3\beta - 4}\right).$$

Corollary 1.2. [6, Corollary 2.4] *Let the function f given by* (1) *be in the class* $\mathcal{K}_{\Sigma}$. *Then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{3}.$$

The class $\mathcal{S}_{\Sigma,k}(\beta)$ of k-bi-starlike functions is defined by Orhan et al. [15] and they obtained an upper bound for the second Hankel determinant of functions $f \in \mathcal{S}_{\Sigma,k}(\beta)$ (see [15, Theorem 2.1]). They got for $\eta, \mu \le 1$

$$\left|a_2a_4 - a_3^2\right| \le T_1 + (\eta+\mu)T_2 + (\eta^2+\mu^2)T_3 + (\eta+\mu)^2T_4 = G(\eta,\mu),$$

where

$$T_1 = T_1(p) = \frac{(1-\beta)^2}{3(2^{3k})}\left[\left((1-\beta)^2\frac{(3(2^k) + 2^{2k} - \text{ï?‘}\tfrac{1}{2}3^{k+1})}{2^{2k}} + \frac{1}{4}\right)p^4 - \frac{p^3}{2} + 2p\right] \ge 0$$

$$T_2 = T_2(p) = \frac{(1-\beta)^2p^2(4-p^2)}{2^{2k+2}}\left[\frac{1}{3^{2k}} + \frac{(1-\beta)}{4(3^k)}\right] \ge 0$$

$$T_3 = T_3(p) = \frac{(1-\beta)^2p(4-p^2)(p-2)}{24(2^{3k})} \le 0$$

$$T_4 = T_4(p) = \frac{(1-\beta)^2}{16(9^k)}\frac{(4-p^2)^2}{4} \ge 0.$$

They claimed that

$$T_3 + 2T_4 > 0 \qquad \text{for} \qquad p \in [0,2),$$

to maximize the function $G(\eta,\mu)$ on the closed square $[0,1]\times[0,1]$. But there is a mistake in their proof. Now we give a *counterexample* that this inequality is not true:

If we choose

$$\beta = 0, \qquad k = 10 \qquad \text{and} \qquad p = 0,9,$$

then we have

$$T_3 + 2T_4 = -3,134800373 \times 10^{-11} < 0.$$

The main purpose of this paper is that, by using a different method from the one in [15], to determine the functional $H_2(2) = a_2a_4 - a_3^2$ for functions belonging to the subclass of analytic bi-univalent functions $\mathcal{B}^{m,k}_{\Sigma}(\gamma;\varphi)$ which is defined by subordination principle in the open unit disk $\mathbb{D}$. In addition, we get more accurate estimation than the previous results and we give the correction of [15, Theorem 2.1].

In order to prove our main results, we need the following lemmas.

Lemma 1.1. [7, p. 190] *Let u be analytic function in the unit disk $\mathbb{D}$, with $u(0) = 0$, and $|u(z)| < 1$ for all $z \in \mathbb{D}$, with the power series expansion*

$$u(z) = \sum_{n=1}^{\infty} c_n z^n.$$

Then, $|c_n| \leq 1$ for all $n \in \mathbb{N}$. Furthermore, $|c_n| = 1$ for some $n \in \mathbb{N}$ if and only if $u(z) = e^{i\theta} z^n$, $\theta \in \mathbb{R}$.

Lemma 1.2. [9] *If $\psi(z) = \sum\limits_{n=1}^{\infty} \psi_n z^n$, $z \in \mathbb{D}$, is a Schwarz function, then*

$$\psi_2 = x\left(1 - \psi_1^2\right),$$
$$\psi_3 = \left(1 - \psi_1^2\right)\left(1 - |x|^2\right)s - \psi_1\left(1 - \psi_1^2\right)x^2,$$

for some x, s, with $|x| \leq 1$ and $|s| \leq 1$.

2. **Main Results**

Whilst Lemma 1.1 holds for complex-valued c_n $(n \in \mathbb{N})$, in this paper we restrict our attention to the case of real valued c_1.

Theorem 2.1. *Let the function f given by* (1) *be in the class $\mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq B_1|\gamma|^2 \times \begin{cases} R & \text{if} \quad Q \leq 0, P \leq -Q \\ P + Q + R & \text{if} \quad (Q \geq 0,\ P \geq -\frac{Q}{2}),\ or,\ \ (Q \leq 0, P \geq -Q) \\ \frac{4PR - Q^2}{4P} & \text{if} \quad Q > 0, P \leq -\frac{Q}{2}, \end{cases}$$

where

$$P = \left|\frac{-\left[(2^m - 2^k)(2^{2k} - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)\right]\gamma^2 B_1^3}{(4^m - 4^k)(2^m - 2^k)^4} + \frac{B_3}{(4^m - 4^k)(2^m - 2^k)}\right|$$
$$- 2\left(\frac{|\gamma| B_1^2}{4(2^m - 2^k)^2(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)}\right) - \frac{B_1}{(4^m - 4^k)(2^m - 2^k)} + \frac{B_1}{(3^m - 3^k)^2},$$

$$Q = 2\left(\frac{|\gamma| B_1^2}{4(2^m - 2^k)^2(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)}\right) + \frac{B_1}{(4^m - 4^k)(2^m - 2^k)} - \frac{2B_1}{(3^m - 3^k)^2},$$

$$R = \frac{B_1}{(3^m - 3^k)^2}.$$

Proof. Let $f \in \mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$. Then by definition of subordination and Lemma 1.1, there exist two Schwarz functions u and v, of the form $u(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ and $v(z) = \sum\limits_{n=1}^{\infty} d_n z^n$, $z \in \mathbb{D}$ such that

$$1 + \frac{1}{\gamma}\left(\frac{D^m f(z)}{D^k f(z)} - 1\right) = \varphi(u(z)) \tag{8}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^k g(w)} - 1\right) = \varphi(v(w)), \tag{9}$$

where

$$\varphi(u(z)) = 1 + B_1c_1z + (B_1c_2 + B_2c_1^2)z^2 + (B_1c_3 + 2B_2c_1c_2 + B_3c_1^3)z^3 + \cdots \tag{10}$$

and

$$\varphi(v(w)) = 1 + B_1 d_1 w + (B_1 d_2 + B_2 d_1^2) w^2 + (B_1 d_3 + 2B_2 d_1 d_2 + B_3 d_1^3) w^3 + \cdots . \tag{11}$$

From (8), (10) and (9), (11), we have

$$(2^m - 2^k) a_2 = \gamma B_1 c_1 \tag{12}$$

$$(3^m - 3^k) a_3 - 2^k (2^m - 2^k) a_2^2 = \gamma (B_1 c_2 + B_2 c_1^2) \tag{13}$$

$$(4^m - 4^k) a_4 - \left[3^k (2^m - 2^k) + 2^k (3^m - 3^k)\right] a_2 a_3 + 2^{2k} (2^m - 2^k) a_2^3$$
$$= \gamma (B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_1^3) \tag{14}$$

and

$$-(2^m - 2^k) a_2 = \gamma B_1 d_1 \tag{15}$$

$$(3^m - 3^k)(2a_2^2 - a_3) - 2^k (2^m - 2^k) a_2^2 = \gamma (B_1 d_2 + B_2 d_1^2) \tag{16}$$

$$-(4^m - 4^k)\left(5a_2^3 - 5a_2 a_3 + a_4\right) + \left[3^k (2^m - 2^k) + 2^k (3^m - 3^k)\right] a_2 \left(2a_2^2 - a_3\right)$$
$$-2^{2k} (2^m - 2^k) a_2^3 = \gamma (B_1 d_3 + 2B_2 d_1 d_2 + B_3 d_1^3), \tag{17}$$

respectively. From (12) and (15), we get that

$$c_1 = -d_1 \tag{18}$$

and

$$a_2 = \frac{\gamma B_1 c_1}{2^m - 2^k}. \tag{19}$$

Nevertheless, from (13) and (16), we get

$$a_3 = \frac{\gamma^2 B_1^2 c_1^2}{(2^m - 2^k)^2} + \frac{\gamma B_1 (c_2 - d_2)}{2(3^m - 3^k)}. \tag{20}$$

Furthermore, from (14) and (17), we obtain

$$a_4 = \frac{\left[(2^m - 2^k)(3^k - 2^{2k}) + 2^k (3^m - 3^k)\right] \gamma^3 B_1^3 c_1^3}{(4^m - 4^k)(2^m - 2^k)^3} + \frac{5\gamma^2 B_1^2 c_1 (c_2 - d_2)}{4(2^m - 2^k)(3^m - 3^k)}$$
$$+ \frac{\gamma B_1 (c_3 - d_3)}{2(4^m - 4^k)} + \frac{\gamma B_2 c_1 (c_2 + d_2)}{(4^m - 4^k)} + \frac{\gamma B_3 c_1^3}{(4^m - 4^k)}. \tag{21}$$

Therefore, after calculations we have

$$\left|a_2 a_4 - a_3^2\right| = \left| \frac{-\left[(2^m - 2^k)(2^{2k} - 3^k) - 2^k (3^m - 3^k) + (4^m - 4^k)\right] \gamma^4 B_1^4 c_1^4}{(4^m - 4^k)(2^m - 2^k)^4} \right.$$
$$+ \frac{\gamma^3 B_1^3 c_1^2 (c_2 - d_2)}{4(2^m - 2^k)^2 (3^m - 3^k)} + \frac{\gamma^2 B_1 B_2 c_1^2 (c_2 + d_2)}{(4^m - 4^k)(2^m - 2^k)}$$
$$+ \frac{\gamma^2 B_3 B_1 c_1^4}{(4^m - 4^k)(2^m - 2^k)} + \frac{\gamma^2 B_1^2 c_1 (c_3 - d_3)}{2(4^m - 4^k)(2^m - 2^k)}$$
$$\left. - \frac{\gamma^2 B_1^2 (c_2 - d_2)^2}{4(3^m - 3^k)^2} \right|. \tag{22}$$

According to Lemma 1.2 and (18), we find that

$$c_2 - d_2 = \left(1 - c_1^2\right)(x - y) \qquad \text{and} \qquad c_2 + d_2 = \left(1 - c_1^2\right)(x + y) \tag{23}$$

and

$$c_3 = \left(1 - c_1^2\right)\left(1 - |x|^2\right) s - c_1 \left(1 - c_1^2\right) x^2 \quad \text{and}$$
$$d_3 = \left(1 - d_1^2\right)\left(1 - |y|^2\right) t - d_1 \left(1 - d_1^2\right) y^2,$$

where

$$c_3 - d_3 = (1 - c_1^2)\left[(1 - |x|^2) s - (1 - |y|^2) t\right] - c_1 (1 - c_1^2)(x^2 + y^2), \tag{24}$$

for some $x,\ y,\ s,\ t$ with $|x|\leq 1,\ |y|\leq 1,\ |s|\leq 1$ and $|t|\leq 1$. Applying (23) and (24) in (22), it follows that

$$\begin{aligned}
&\left|a_2a_4-a_3^2\right|\\
=\ & B_1|\gamma|^2\Bigg|\left[\frac{-\left[(2^m-2^k)(2^{2k}-3^k)-2^k(3^m-3^k)+(4^m-4^k)\right]\gamma^2B_1^3}{(4^m-4^k)(2^m-2^k)^4}+\frac{B_3}{(4^m-4^k)(2^m-2^k)}\right]c_1^4\\
&+\left[\frac{\gamma B_1^2(x-y)}{4(2^m-2^k)^2(3^m-3^k)}+\frac{B_2(x+y)}{(4^m-4^k)(2^m-2^k)}\right]c_1^2\left(1-c_1^2\right)-\frac{B_1c_1^2\left(1-c_1^2\right)}{2(4^m-4^k)(2^m-2^k)}\left(x^2+y^2\right)\\
&-\frac{B_1\left(1-c_1^2\right)^2}{4(3^m-3^k)^2}(x-y)^2+\frac{B_1c_1\left(1-c_1^2\right)}{2(4^m-4^k)(2^m-2^k)}\left[\left(1-|x|^2\right)s-\left(1-|y|^2\right)t\right]\Bigg|.
\end{aligned}$$

Since $|c_1|\leq 1$, we assume that $c_1=c\in[0,1]$. So we have

$$\begin{aligned}
&\left|a_2a_4-a_3^2\right|\\
\leq\ & B_1|\gamma|^2\Bigg\{\left|\frac{-\left[(2^m-2^k)(2^{2k}-3^k)-2^k(3^m-3^k)+(4^m-4^k)\right]\gamma^2B_1^3}{(4^m-4^k)(2^m-2^k)^4}+\frac{B_3}{(4^m-4^k)(2^m-2^k)}\right|c^4\\
&+\left[\frac{|\gamma|B_1^2}{4(2^m-2^k)^2(3^m-3^k)}+\frac{|B_2|}{(4^m-4^k)(2^m-2^k)}\right]c^2\left(1-c^2\right)(|x|+|y|)\\
&+\frac{B_1c^2\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}\left(|x|^2+|y|^2\right)+\frac{B_1\left(1-c^2\right)^2}{4(3^m-3^k)^2}(|x|+|y|)^2\\
&+\frac{B_1c\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}\left[\left(1-|x|^2\right)|s|+\left(1-|y|^2\right)|t|\right]\Bigg\}\\
\leq\ & B_1|\gamma|^2\Bigg\{\left|\frac{-\left[(2^m-2^k)(2^{2k}-3^k)-2^k(3^m-3^k)+(4^m-4^k)\right]\gamma^2B_1^3}{(4^m-4^k)(2^m-2^k)^4}+\frac{B_3}{(4^m-4^k)(2^m-2^k)}\right|c^4\\
&+\left[\frac{|\gamma|B_1^2}{4(2^m-2^k)^2(3^m-3^k)}+\frac{|B_2|}{(4^m-4^k)(2^m-2^k)}\right]c^2\left(1-c^2\right)(|x|+|y|)\\
&+\frac{B_1c^2\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}\left(|x|^2+|y|^2\right)+\frac{B_1\left(1-c^2\right)^2}{4(3^m-3^k)^2}(|x|+|y|)^2\\
&+\frac{B_1c\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}\left[\left(1-|x|^2\right)+\left(1-|y|^2\right)\right]\Bigg\}\\
=\ & B_1|\gamma|^2\Bigg\{\left|\frac{-\left[(2^m-2^k)(2^{2k}-3^k)-2^k(3^m-3^k)+(4^m-4^k)\right]\gamma^2B_1^3}{(4^m-4^k)(2^m-2^k)^4}+\frac{B_3}{(4^m-4^k)(2^m-2^k)}\right|c^4\\
&+\frac{2B_1c\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}+\left[\frac{|\gamma|B_1^2}{4(2^m-2^k)^2(3^m-3^k)}+\frac{|B_2|}{(4^m-4^k)(2^m-2^k)}\right]c^2\left(1-c^2\right)(|x|+|y|)\\
&+\left[\frac{B_1c^2\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}-\frac{B_1c\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)}\right]\left(|x|^2+|y|^2\right)+\frac{B_1\left(1-c^2\right)^2}{4(3^m-3^k)^2}(|x|+|y|)^2\Bigg\}.
\end{aligned}$$

Now, for $\lambda=|x|\leq 1$ and $\mu=|y|\leq 1$, we obtain

$$\left|a_2a_4-a_3^2\right|\leq B_1|\gamma|^2\left[T_1+(\lambda+\mu)T_2+(\lambda^2+\mu^2)T_3+(\lambda+\mu)^2T_4\right]=B_1|\gamma|^2F(\lambda,\mu),$$

where

$$T_1 = T_1(c) = \left| \frac{-\left[(2^m-2^k)(2^{2k}-3^k)-2^k(3^m-3^k)+(4^m-4^k)\right]\gamma^2 B_1^3}{(4^m-4^k)(2^m-2^k)^4} + \frac{B_3}{(4^m-4^k)(2^m-2^k)} \right| c^4$$

$$+\frac{2B_1 c\left(1-c^2\right)}{2(4^m-4^k)(2^m-2^k)} \geq 0$$

$$T_2 = T_2(c) = \left[\frac{|\gamma| B_1^2}{4(2^m-2^k)^2(3^m-3^k)} + \frac{|B_2|}{(4^m-4^k)(2^m-2^k)}\right] c^2\left(1-c^2\right) \geq 0$$

$$T_3 = T_3(c) = \frac{B_1 c(c-1)(1-c^2)}{2(4^m-4^k)(2^m-2^k)} \leq 0$$

$$T_4 = T_4(c) = \frac{B_1\left(1-c^2\right)^2}{4(3^m-3^k)^2} \geq 0.$$

We now need to maximize the function $F(\lambda,\mu)$ on the closed square $[0,1]\times[0,1]$ for $c\in[0,1]$. With regards to $F(\lambda,\mu)=F(\mu,\lambda)$, it is sufficient that we investigate the maximum of

$$G(\lambda) = F(\lambda,\lambda) = T_1 + 2\lambda T_2 + 2\lambda^2(T_3+2T_4), \tag{25}$$

on $\lambda\in[0,1]$ according to $c\in(0,1)$, $c=0$ and $c=1$.

Firstly, if we let $c=1$, then we obtain

$$\max\{G(\lambda):\lambda\in[0,1]\} = \left| \frac{-\left[(2^m-2^k)(2^{2k}-3^k)-2^k(3^m-3^k)+(4^m-4^k)\right]\gamma^2 B_1^3}{(4^m-4^k)(2^m-2^k)^4} + \frac{B_3}{(4^m-4^k)(2^m-2^k)} \right|.$$

Secondly, letting $c=0$, so we get

$$G(\lambda) = \frac{4B_1}{4(3^m-3^k)^2}\lambda^2,$$

hence we can see that

$$\max\{G(\lambda):\lambda\in[0,1]\} = G(1) = \frac{B_1}{(3^m-3^k)^2}.$$

Finally, we let $c\in(0,1)$. Considering equation (25) for $0\leq\lambda\leq 1$ we get

(**i**) If $T_3+2T_4\geq 0$, it is clear that

$$G'(\lambda) = 4(T_3+2T_4)\lambda + 2T_2 > 0$$

for $0<\lambda<1$ and any fixed $c\in(0,1)$, that is $G(\lambda)$ is an increasing function. Hence

$$\max\{G(\lambda):\lambda\in[0,1]\} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

(**ii**) If $T_3+2T_4<0$, then we consider for critical point

$$\lambda_0 = \frac{-(T_2)}{2(T_3+2T_4)} = \frac{T_2}{2k}$$

for any fixed $c\in(0,1)$, where $k=-(T_3+2T_4)>0$, the following two cases:

Case 1. For $\lambda_0 = \dfrac{T_2}{2k} > 1$, it follows that $k < \dfrac{T_2}{2} \leq T_2$, and so $T_2+T_3+2T_4\geq 0$. Therefore,

$$G(0) = T_1 \leq T_1 + 2(T_2+T_3+2T_4) = G(1).$$

Case 2. For $\lambda_0 = \dfrac{T_2}{2k} \leq 1$, since $T_2\geq 0$, we get that $\frac{T_2^2}{2k}\leq T_2$. Therefore,

$$G(0) = T_1 \leq T_1 + \frac{T_2^2}{2k} = G(\lambda_0) \leq T_1 + T_2.$$

Considering the above cases for point of c, it follows that the function $G(\lambda)$ gets its maximum when $T_3 + 2T_4 \geq 0$, it means

$$\max\{G(\lambda) : \lambda \in [0,1]\} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Therefore, $\max F(\lambda, \mu) = F(1,1)$ on the boundary of the square.

Let $K : [0,1] \to \mathbb{R}$,

$$\begin{aligned} K(c) &= B_1|\gamma|^2 \max F(\lambda,\mu) = B_1|\gamma|^2 F(1,1) \\ &= B_1|\gamma|^2(T_1 + 2T_2 + 2T_3 + 4T_4). \end{aligned} \tag{26}$$

By replacing the values of $T_1,\ T_2,\ T_3$ and T_4 in the above function K, we have

$$\begin{aligned} K(c) =& B_1|\gamma|^2 \Bigg\{ \Bigg[\left| \frac{-\left[(2^m-2^k)(2^{2k}-3^k) - 2^k(3^m-3^k) + (4^m-4^k)\right]\gamma^2 B_1^3}{(4^m-4^k)(2^m-2^k)^4} + \frac{B_3}{(4^m-4^k)(2^m-2^k)} \right| \\ & - 2\left(\frac{|\gamma| B_1^2}{4(2^m-2^k)^2(3^m-3^k)} + \frac{|B_2|}{(4^m-4^k)(2^m-2^k)} \right) - \frac{B_1}{(4^m-4^k)(2^m-2^k)} + \frac{B_1}{(3^m-3^k)^2} \Bigg] c^4 \\ & + \Bigg[2\left(\frac{|\gamma| B_1^2}{4(2^m-2^k)^2(3^m-3^k)} + \frac{|B_2|}{(4^m-4^k)(2^m-2^k)} \right) + \frac{B_1}{(4^m-4^k)(2^m-2^k)} - \frac{2B_1}{(3^m-3^k)^2} \Bigg] c^2 \\ & + \frac{B_1}{(3^m-3^k)^2} \Bigg\}. \end{aligned}$$

Suppose $c^2 = u$ and for the simplicity, set

$$\begin{aligned} P =& \left| \frac{-\left[(2^m-2^k)(2^{2k}-3^k) - 2^k(3^m-3^k) + (4^m-4^k)\right]\gamma^2 B_1^3}{(4^m-4^k)(2^m-2^k)^4} + \frac{B_3}{(4^m-4^k)(2^m-2^k)} \right| \\ & - 2\left(\frac{|\gamma| B_1^2}{4(2^m-2^k)^2(3^m-3^k)} + \frac{|B_2|}{(4^m-4^k)(2^m-2^k)} \right) - \frac{B_1}{(4^m-4^k)(2^m-2^k)} + \frac{B_1}{(3^m-3^k)^2}, \\ Q =& 2\left(\frac{|\gamma| B_1^2}{4(2^m-2^k)^2(3^m-3^k)} + \frac{|B_2|}{(4^m-4^k)(2^m-2^k)} \right) + \frac{B_1}{(4^m-4^k)(2^m-2^k)} - \frac{2B_1}{(3^m-3^k)^2}, \\ R =& \frac{B_1}{(3^m-3^k)^2}. \end{aligned} \tag{27}$$

According to

$$\max_{0\leq u\leq 1}(Pu^2 + Qu + R) = \begin{cases} R & \text{if} \quad Q \leq 0, P \leq -Q \\ P+Q+R & \text{if} \quad (Q \geq 0,\ P \geq -\frac{Q}{2}),\ or,\ \ (Q \leq 0, P \geq -Q) \\ \frac{4PR-Q^2}{4P} & \text{if} \quad Q > 0, P \leq -\frac{Q}{2} \end{cases},$$

it follows that

$$\left|a_2a_4 - a_3^2\right| \leq B_1|\gamma|^2 \times \begin{cases} R & \text{if} \quad Q \leq 0, P \leq -Q \\ P+Q+R & \text{if} \quad (Q \geq 0,\ P \geq -\frac{Q}{2}),\ or,\ \ (Q \leq 0, P \geq -Q) \\ \frac{4PR-Q^2}{4P} & \text{if} \quad Q > 0, P \leq -\frac{Q}{2} \end{cases},$$

where $P,\ Q$ and R are given by (27). This completes the proof. □

For

$$m = k+1, \quad \gamma = 1 \qquad \text{and} \qquad \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \quad (0 \le \beta < 1)$$

in Theorem 2.1, we get the following correction of the estimates in [15, Theorem 2.1]:

Corollary 2.1. *Let the function f given by (1) be in the class $\mathcal{S}_{\Sigma,k}(\beta)$ $(0 \le \beta < 1)$. Then*

$$|a_2a_4 - a_3^2| \le 2(1-\beta) \times \begin{cases} R & \text{if} \quad Q \le 0, P \le -Q \\ P+Q+R & \text{if} \quad (Q \ge 0,\, P \ge -\frac{Q}{2}),\ or,\ (Q \le 0, P \ge -Q) \\ \frac{4PR-Q^2}{4P} & \text{if} \quad Q > 0, P \le -\frac{Q}{2}, \end{cases}$$

where

$$P = (1-\beta)\left\{ \left| \frac{-\left[2^{2k}+3(2^k)-3^{k+1}\right](1-\beta)^2}{3(2^{5k-3})} + \frac{1}{3(2^{3k-1})} \right| - \frac{1-\beta}{(2^{2k})(3^k)} - \frac{1}{2^{3k-1}} + \frac{1}{2(3^{2k})} \right\},$$

$$Q = (1-\beta)\left[\frac{1-\beta}{(2^{2k})(3^k)} + \frac{1}{2^{3k-1}} - \frac{1}{3^{2k}} \right],$$

$$R = \frac{1-\beta}{2(3^{2k})}.$$

For $k = 0$ in Corollary 2.1, we get the following result that is an improvement of the estimates which in Theorem 1.1.

Corollary 2.2. *Let the function f given by (1) be in the class $\mathcal{S}_{\Sigma}^{*}(\beta)$ $(0 \le \beta < 1)$. Then*

$$|a_2a_4 - a_3^2| \le 2(1-\beta)^2 \begin{cases} \frac{2}{3}\left(4\beta^2 - 8\beta + 3\right) & , \quad 0 \le \beta \le \frac{29-\sqrt{649}}{32} \\ \frac{13\beta^2-14\beta-15}{32\beta^2-52\beta-6} & , \quad \frac{29-\sqrt{649}}{32} \le \beta \le \frac{1}{2} \\ \frac{19\beta^2-50\beta+39}{32\beta^2-76\beta+54} & , \quad \frac{1}{2} \le \beta < 1 \end{cases}.$$

For $\beta = 0$, Corollary 2.2 yields the following coefficient estimates for bi-starlike functions. This result is an improvement of the estimates obtained in Corollary 1.1.

Corollary 2.3. *Let the function f given by (1) be in the class $\mathcal{S}_{\Sigma}^{*}$. Then*

$$|a_2a_4 - a_3^2| \le 4.$$

For $k = 1$ in Corollary 2.1, we get the following result that is an improvement of the estimates in Theorem 1.2.

Corollary 2.4. *Let the function f given by (1) be in the class $\mathcal{K}_{\Sigma}(\beta)$ $(0 \le \beta < 1)$. Then*

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{24} \cdot \frac{11\beta^2 - 40\beta + 48}{3\beta^2 - 9\beta + 10}.$$

For $\beta = 0$, Corollary 2.4 yields the following coefficient estimates for bi-convex functions. This result is an improvement of the estimates obtained in Corollary 1.2.

Corollary 2.5. *Let the function f given by (1) be in the class $\mathcal{K}_{\Sigma}$. Then*

$$|a_2a_4 - a_3^2| \le \frac{1}{5}.$$

3. Conclusion

In the final section, we found improved upper bounds for the functional $|H_2(2)|$ for functions in the class $\mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$. The technique of proof for Theorem 2.1 can be extended to other classes of functions similar to $\mathcal{B}_{\Sigma}^{m,k}(\gamma;\varphi)$ as for example $M_{\Sigma}(\varphi,\beta)$ introduced in Definition 1.1 of [12], in order to improve previous estimates by their Theorem 2.1. Sharp estimates for $|H_2(2)|$ are for now open problems.

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