CHARACTERIZATIONS, ADJOINTS AND PRODUCTS OF NUCLEAR PSEUDO-DIFFERENTIAL OPERATORS ON COMPACT AND HAUSDORFF GROUPS

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Characterizations of nuclear pseudo-differential operators from L^{p_1} into L^{p_2} on compact and Hausdorff groups are given for $1 \leq p_1, p_2 < \infty$. Explicit formulas for the adjoints from $L^{p_2'}$ into $L^{p_1'}$ and products of nuclear pseudo-differential operators from L^p into L^p, $1 \leq p < \infty$, on compact and Hausdorff groups are given.

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1. Introduction

Let G be a compact and Hausdorff group on which the left (and right) Haar measure is denoted by μ. Let ξ be an irreducible and unitary representation of G on a complex and separable Hilbert space X_ξ. Since G is compact, it is well known that X_ξ is finite-dimensional. We let d_ξ be the dimension of X_ξ. The number d_ξ is also known as the degree of the representation ξ of G on X_ξ. Let $\hat{G}$ be the set of all (equivalent classes) of irreducible and unitary representations of G, which is usually referred to as the *dual group* of G.

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Let $f \in L^p(G)$, $p \geq 1$. Then we define the Fourier transform $\hat{f}$, also denoted by $\mathcal{F}_G f$, of f by

$$\hat{f}(\xi) = \int_G f(x)\xi(x)^* dx, \quad \xi \in \hat{G}.$$

It is also well known that the Fourier inversion formula states that for a good class of functions in $L^p(G)$, $p \geq 1$,

$$f(x) = \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\hat{f}(\xi)), \quad x \in G.$$

The Fourier inversion formula can be looked at as a formula for the identity operator on $L^p(G)$, $p \geq 1$, and as such, is a perfect symmetry that gives us the identity operator on a suitable class of functions on G.

Good referenced for abstract harmonic analysis abound. See, for instance, [12], [10] and [4] for abstract harmonic analysis in general and group representations, dual group and the Fourier inversion formula in particular.

In order to obtain more interesting operators than the identity operator, we need to break the symmetry using *symbols* σ defined on the *phase space* $G \times \hat{G}$. To wit, let σ be a *suitable* function defined on $G \times \hat{G}$. Then for every point $(x, \xi) \in G \times \hat{G}$, $\sigma(x, \xi)$ is a $d_\xi \times d_\xi$ matrix. We define the *pseudo-differential operator* T_σ on G with symbol σ by

$$(T_\sigma f)(x) = \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x, \xi)\hat{f}(\xi)), \quad x \in G.$$

The aim of this paper is to give characterizations of nuclear pseudo-differential operators on compact and Hausdorff groups with applications to products and adjoints of nuclear pseudo-differential operators on these topological groups, thus extending the results in [6] from the unit circle at the origin to compact and Hausdorff groups.

In Section 2 of the paper, we recall the definitions and properties of nuclear operators on Banach spaces as generalizations of trace class operators on Hilbert spaces [3, 8, 9]. In Section 2 we give characterizations of nuclear pseudo-differential operators on compact and Hausdorff groups. Adjoints and products of nuclear pseudo-differential operators on compact and Hausdorff groups are given in, respectively, Section 3 and Section 4. Of particular note is that we can give explicit formulas for the adjoints in terms of the symbols, thus improving the corresponding results for $\mathbb{S}^1$ in [6].

While the focus of this paper is on exact formulas, we end this section by mentioning a related paper [5] on compact Lie groups giving sufficient conditions on the symbols to insure the desired mapping properties of the corresponding pseudo-differential operators. A point worth emphasizing is that the results in this paper are true for compact and Hausdorff groups instead of compact Lie groups. A recent paper [11] is devoted to the study of trace class pseudo-differential operators on compact and Hausdorff groups.

2. Nuclear Pseudo-Differential Operators on $L^p(G)$

We first recall the basic notions of nuclear operators on Banach spaces. Let T be a bounded linear operator from a complex Banach space X into another complex Banach space Y such that there exist sequences $\{x'_n\}_{n=1}^\infty$ in the dual space X' of X and $\{y_n\}_{n=1}^\infty$ in Y such that

$$\sum_{n=1}^\infty \|x'_n\|_{X'}\|y_n\|_Y < \infty$$

and

$$Tx = \sum_{n=1}^\infty x'_n(x)y_n, \quad x \in X.$$

Then we call $T : X \to Y$ a *nuclear operator* and if $X = Y$, then its *trace* $\operatorname{tr}(T)$ is defined by

$$\operatorname{tr}(T) = \sum_{n=1}^\infty x'_n(y_n).$$

It can be proved that the definition of a nuclear operator and the definition of the trace of a nuclear operator are independent of the choices of the sequences $\{x'_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$. A good reference is the book [7].

The main tool that we use is the following result in [1, 2, 3].

Theorem 2.1. *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces. Then a bounded linear operator $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$, $1 \le p_1, p_2 < \infty$, is nuclear if and only if there exist sequences $\{g_n\}_{n=1}^\infty$ in $L^{p'_1}(X_1, \mu_1)$ and $\{h_n\}_{n=1}^\infty$ in $L^{p_2}(X_2, \mu_2)$ such that for all $f \in L^{p_1}(X_1, \mu_1)$,*

$$(Tf)(x) = \int_{X_1} K(x, y)f(y)\, d\mu_1(y), \quad x \in X_2,$$

where

$$K(x, y) = \sum_{n=1}^\infty h_n(x)g_n(y), \quad x \in X_2, y \in X_1,$$

and

$$\sum_{n=1}^\infty \|g_n\|_{L^{p'_1}(X_1,\mu_1)}\|h_n\|_{L^{p_2}(X_2,\mu_2)} < \infty.$$

The function K on $X_2 \times X_1$ in Theorem 2.1 is called the *kernel* of the nuclear operator $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$.

Let (X, μ) be a σ-finite measure space. Let $T : L^p(X, \mu) \to L^p(X, \mu)$, $1 \le p < \infty$, be a nuclear operator. Then by Theorem 2.1, we can find sequences $\{g_n\}_{n=1}^\infty$ in $L^{p'}(X, \mu)$ and $\{h_n\}_{n=1}^\infty$ in $L^p(X, \mu)$ such that

$$\sum_{n=1}^\infty \|g_n\|_{L^{p'}(X,\mu)}\|h_n\|_{L^p(X,\mu)} < \infty$$

and for all $f \in L^p(X, \mu)$,

$$(Tf)(x) = \int_X K(x,y) f(y)\, d\mu(y), \quad x \in X,$$

where

$$K(x,y) = \sum_{n=1}^{\infty} h_n(x) g_n(y), \quad x, y \in X.$$

The trace $\mathrm{tr}(T)$ of $T : L^p(X,\mu) \to L^p(X,\mu)$ is given by

$$\mathrm{tr}(T) = \int_X K(x,x)\, d\mu(x). \tag{1}$$

We can now give a characterization of nuclear pseudo-differential operators from $L^{p_1}(G)$ into $L^{p_2}(G)$, where G is a compact and Hausdorff group.

Theorem 2.2. *Let σ be a function on $G \times \hat{G}$, where G is a compact and Hausdorff group. Then the pseudo-differential operator $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$, $1 \leq p_1, p_2 < \infty$, is nuclear if and only if there exist sequences $\{g_k\}_{k=1}^{\infty} \in L^{p_1'}(G)$ and $\{h_k\}_{k=1}^{\infty} \in L^{p_2}(G)$ such that*

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p_1'}(G)} \|h_k\|_{L^{p_2}(G)} < \infty$$

and

$$\sigma(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x) \widehat{\overline{g_k}}(\xi)^*, \quad (x,\xi) \in G \times \hat{G}.$$

Proof. Suppose that $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is nuclear, where $1 \leq p_1, p_2 < \infty$. Then by Theorem 2.1, there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G)$ such that

$$\sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}(G)} \|g_k\|_{L^{p_1'}(G)} < \infty$$

and for all $f \in L^{p_1}(G)$,

$$\begin{aligned}
(T_\sigma f)(x) &= \sum_{\eta \in \hat{G}} \sum_{i,j=1}^{d_\eta} d_\eta (\eta(x)\sigma(x,\eta))_{ij} \hat{f}(\eta)_{ji} \\
&= \int_G \sum_{\eta \in \hat{G}} \sum_{i,j=1}^{d_\eta} d_\eta ((\eta(x)\sigma(x,\eta))_{ij} \overline{\eta(y)_{ij}} f(y)\, d\mu(y) \\
&= \int_G \left(\sum_{k=1}^{\infty} h_k(x) g_k(y) \right) f(y)\, d\mu(y) \qquad (2)
\end{aligned}$$

for all $x \in G$. Let ξ be a fixed but arbitrary element in $\hat{G}$. Then for $1 \leq m, n \leq d_\xi$, we define the function f on G by

$$f(y) = \xi(y)_{nm}, \quad y \in G.$$

Since

$$\int_G \xi(y)_{nm}\overline{\eta(y)_{ij}}d\mu(y) = \frac{1}{d_\xi}$$

if and only if $\xi = \eta$ and $n = i$ and $m = j$, and is zero otherwise, it follows from (2) that

$$(\xi(x)\sigma(x,\xi))_{nm} = \sum_{k=1}^{\infty} h_k(x)\overline{(\widehat{\overline{g_k}}(\xi))_{mn}}, \quad (x,\xi) \in G \times \hat{G}.$$

Thus,

$$\sigma(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x)\widehat{\overline{g_k}}(\xi)^*, \quad (x,\xi) \in G \times \hat{G}.$$

Conversely, suppose that there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G)$ such that

$$\sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}(G)}\|g_k\|_{L^{p'_1}(G)} < \infty$$

and

$$\sigma(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x)\widehat{\overline{g_k}}(\xi)^*, \quad (x,\xi) \in G \times \hat{G}.$$

Then for all $f \in L^{p_1}(G)$,

$$\begin{aligned}(T_\sigma f)(x) &= \sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\hat{f}(\xi)) \\ &= \sum_{\xi\in\hat{G}} \sum_{n,m=1}^{d_\xi} d_\xi \sum_{k=1}^{\infty} h_k(x)\widehat{\overline{g_k}}(\xi)^*_{nm}\hat{f}(\xi)_{mn} \\ &= \sum_{\xi\in\hat{G}} \sum_{n,m=1}^{d_\xi} d_\xi \sum_{k=1}^{\infty} h_k(x)\overline{\widehat{\overline{g_k}}(\xi)_{mn}}\hat{f}(\xi)_{mn}\end{aligned}$$

for all $x \in G$. Using the Fourier transform of f, we obtain for all $f \in L^{p_1}(G)$,

$$\begin{aligned}(T_\sigma f)(x) &= \int_G \sum_{\xi\in\hat{G}} \sum_{n,m=1}^{d_\xi} d_\xi \xi(y)_{nm} \hat{f}(\xi)_{mn} \sum_{k=1}^{\infty} h_k(x) g_k(y)\, d\mu(y) \\ &= \int_G \sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(y)\hat{f}(\xi)) \sum_{k=1}^{\infty} h_k(x) g_k(y)\, d\mu(y) \\ &= \int_G \left(\sum_{k=1}^{\infty} h_k(x) g_k(y)\right) f(y)\, d\mu(y)\end{aligned}$$

for all x in G. Therefore by Theorem 2.1, $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is nuclear. □

The following theorem is another characterization of nuclear pseudo-differential operators from $L^{p_1}(G)$ into $L^{p_2}(G)$, where G is a compact and Hausdorff group. In the proof given below, we use the fact that a compact and Hausdorff group is unimodular and hence the left Haar measure and the right Haar measure coincide. See [13].

Theorem 2.3. *Let σ be a function on $G \times \hat{G}$, where G is a compact and Hausdorff group. Then the pseudo-differential operator $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is a nuclear operator for $1 \le p_1, p_2 < \infty$ if and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G)$ such that*

$$\sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}(G)} \|g_k\|_{L^{p_1'}(G)} < \infty$$

and for all x and y in G,

$$\sum_{\xi\in G} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\xi(y)^*) = \sum_{k=1}^{\infty} h_k(x) g_k(y).$$

Proof. Suppose that $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is a nuclear operator. Then by Theorem 2.2, there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G)$ such that

$$\sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}(G)} \|g_k\|_{L^{p_1'}(G)} < \infty$$

and

$$(\xi(x)\sigma(x,\xi))_{nm} = \sum_{k=1}^{\infty} h_k(x) \overline{(\widehat{\overline{g_k}}(\xi))_{mn}}, \quad (x,\xi) \in G \times \hat{G},$$

for all n and m with $1 \le n, m \le d_\xi$. Let $y \in G$. Then

$$(\xi(x)\sigma(x,\xi))_{nm} \overline{\xi(y)_{nm}} = \int_G \xi(z)_{nm} \overline{\xi(y)_{nm}} \sum_{k=1}^{\infty} h_k(x) g_k(z)\, d\mu(z)$$

and hence

$$\begin{aligned}
&\sum_{n,m=1}^{d_\xi} d_\xi (\xi(x)\sigma(x,\xi))_{nm}\overline{\xi(y)_{nm}} \\
&= \int_G \sum_{n,m=1}^{d_\xi} d_\xi \xi(z)_{nm}\overline{\xi(y)_{nm}} \sum_{k=1}^\infty h_k(x)g_k(z)\,d\mu(z), \quad (x,\xi)\in G\times\hat{G}.
\end{aligned}$$

So, for all x and y in G,

$$\sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\xi(y)^*) = \int_G \sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(z)\xi(y)^*) \sum_{k=1}^\infty h_k(x)g_k(z)\,d\mu(z).$$

Since

$$\begin{aligned}
\sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(z)\xi(y)^*) &= \sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(z\cdot y^{-1}) \\
&= \delta(z\cdot y^{-1}), \quad z,y\in G,
\end{aligned}$$

where δ and $\cdot$ are, respectively, the Dirac delta and the binary operation on the group G, it follows that

$$\begin{aligned}
\sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\xi(y)^*) &= \int_G \delta(z\cdot y^{-1}) \sum_{k=1}^\infty h_k(x)g_k(z)\,d\mu(z) \\
&= \int_G \delta(w) \sum_{k=1}^\infty h_k(x)g_k(w\cdot y)\,d\mu(w) \\
&= \sum_{k=1}^\infty h_k(x)g_k(y)
\end{aligned}$$

for all x and y in G. Conversely, let $\{g_k\}_{k=1}^\infty$ and $\{h_k\}_{k=1}^\infty$ be sequences in, respectively, $L^{p_1'}(G)$ and $L^{p_2}(G)$ such that

$$\sum_{k=1}^\infty \|h_k\|_{L^{p_2}(G)}\|g_k\|_{L^{p_1'}(G)} < \infty$$

and for all x and y in G,

$$\sum_{\xi\in\hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\xi(y)^*) = \sum_{k=1}^\infty h_k(x)g_k(y).$$

Then for all $f \in L^{p_1}(G)$,

$$\begin{aligned}
(T_\sigma f)(x) &= \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\hat{f}(\xi)) \\
&= \sum_{\xi \in \hat{G}} \sum_{m,n=1}^{d_\xi} d_\xi (\xi(x)\sigma(x,\xi))_{mn} \hat{f}(\xi)_{nm} \\
&= \int_G \sum_{\xi \in \hat{G}} \sum_{m,n=1}^{d_\xi} d_\xi (\xi(x)\sigma(x,\xi)_{mn} \overline{\xi(y)_{mn}} f(y)\, d\mu(y) \\
&= \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\xi(y)^*) f(y)\, d\mu(y) \\
&= \int_G \left(\sum_{k=1}^{\infty} h_k(x) g_k(y) \right) f(y)\, d\mu(y)
\end{aligned}$$

for all $x \in G$. This completes the proof. □

Theorem 2.3 gives the trace of a nuclear pseudo-differential operator from $L^p(G)$ for $1 \leq p < \infty$. Indeed, we have the following well-known fact.

Corollary 2.4. Let $T_\sigma : L^p(G) \to L^p(G)$ be a nuclear operator for $1 \leq p < \infty$. Then the trace $\mathrm{tr}(T_\sigma)$ of T_σ is given by

$$\mathrm{tr}(T_\sigma) = \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\sigma(x,\xi))\, d\mu(x).$$

Proof. By the trace formula (1) and Theorem 2.3,

$$\begin{aligned}
\mathrm{tr}(T_\sigma) &= \int_G \sum_{k=1}^{\infty} h_k(x) g_k(x)\, d\mu(x) \\
&= \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x,\xi)\xi(x)^*)\, d\mu(x)
\end{aligned}$$

So,

$$\begin{aligned}
\mathrm{tr}(T_\sigma) &= \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\xi(x)^{-1}\sigma(x,\xi))\, d\mu(x) \\
&= \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\sigma(x,\xi))\, d\mu(x).
\end{aligned}$$

□

3. **Adjoints**

We give in this section a formula for the symbols of the adjoints of nuclear pseudo-differential operators from $L^{p_1}(G)$ into $L^{p_2}(G)$ for $1 \leq p_1, p_2 < \infty$, where G is a compact and Hausdorff group.

Theorem 3.1. *Let σ be a function on $G \times \hat{G}$ such that $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is a nuclear operator for $1 \leq p_1, p_2 < \infty$. Then the adjoint T_σ^* of T_σ is a nuclear operator from $L^{p_2'}(G)$ into $L^{p_1'}(G)$ with symbol τ given by*

$$\tau(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \overline{g_k}(x), \quad (x,\xi) \in G \times \hat{G}.$$

Proof. Let $f \in L^{p_1}(G)$ and $g \in L^{p_2'}(G)$. Then

$$\int_G (T_\sigma f)(x)\overline{g(x)}\, d\mu(x) = \int_G f(x)\overline{(T_\sigma^* g)(x)} d\mu(x).$$

So,

$$\int_G \left(\int_G \sum_{\xi \in \hat{G}} \sum_{i,j=1}^{d_\xi} d_\xi (\xi(x)\sigma(x,\xi))_{ij} \overline{\xi(y)_{ij}} f(y)\, d\mu(y) \right) \overline{g(x)}\, d\mu(x)$$

$$= \int_G f(x) \overline{\left(\int_G \sum_{\xi \in \hat{G}} \sum_{i,j=1}^{d_\xi} d_\xi (\xi(x)\tau(x,\xi))_{ij} \overline{\xi(y)_{ij}} g(y)\, d\mu(y) \right)} d\mu(x). \tag{1}$$

Now, let γ and η be elements in $\hat{G}$. Then for $1 \leq t, m \leq d_\gamma$ and $1 \leq n, l \leq d_\eta$, we let f and g be functions on G be defined by

$$f(x) = \gamma(x)_{tm}, \quad x \in G,$$

and

$$g(x) = \eta(x)_{nl}, \quad x \in G.$$

Therefore by (1),

$$\int_G (\gamma(x)\sigma(x,\gamma))_{tm} \overline{\eta(x)_{nl}} d\mu(x) = \int_G \gamma(x)_{tm} \overline{(\eta(x)\tau(x,\eta))_{nl}} d\mu(x)$$

and we get

$$\overline{\int_G (\gamma(x)\sigma(x,\gamma))_{tm} \overline{\eta(x)_{nl}} d\mu(x)} = \int_G (\eta(x)\tau(x,\eta))_{nl} \overline{\gamma(x)_{tm}} d\mu(x).$$

Thus,

$$\overline{((\gamma(\cdot)\sigma(\cdot,\gamma))_{tm})^\wedge(\eta)_{ln}} = ((\eta(\cdot)\tau(\cdot,\eta))_{nl})^\wedge(\gamma)_{mt} \tag{2}$$

for $1 \leq t, m \leq d_\gamma$, $1 \leq n, l \leq d_\eta$ and all γ and η in $\hat{G}$. Since T_σ is a nuclear operator, it follows that there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p_1'}(G)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G)$ such that

$$\sum_{k=1}^\infty \|h_k\|_{L^{p_2}(G)} \|g_k\|_{L^{p_1'}(G)} < \infty$$

and for all (y, γ) in $G \times \hat{G}$,

$$(\gamma(y)\sigma(y,\gamma))_{tm} = \sum_{k=1}^\infty h_k(y)\overline{(\widehat{g_k}(\gamma))_{mt}}, \quad 1 \leq m, t \leq d_\gamma.$$

So, for all $(x, \eta) \in G \times \hat{G}$,

$$\begin{aligned} ((\eta(x)\tau(x,\eta))_{nl} &= \sum_{\gamma \in \hat{G}} d_\gamma \mathrm{tr}[\gamma(x)((\eta(\cdot)\tau(\cdot,\eta))_{nl})^\wedge(\gamma)] \\ &= \sum_{\gamma \in \hat{G}} \sum_{t,m=1}^{d_\gamma} d_\gamma \gamma(x)_{tm}((((\eta(\cdot)\tau(\cdot,\eta))_{nl})^\wedge(\gamma))_{mt} \end{aligned}$$

Hence for all $(x, \eta) \in G \times \hat{G}$, we get by (2)

$$\begin{aligned} ((\eta(x)\tau(x,\eta)))_{nl} &= \sum_{\gamma \in \hat{G}} \sum_{t,m=1}^{d_\gamma} d_\gamma \gamma(x)_{tm} \overline{(((\gamma(\cdot)\sigma(\cdot,\gamma))_{tm})^\wedge(\eta))_{ln}}. \\ &= \sum_{\gamma \in \hat{G}} \sum_{m,t=1}^{d_\gamma} d_\gamma \gamma(x)_{tm} \int_G \overline{((\gamma(y)\sigma(y,\gamma))_{tm})} \eta(y)_{nl} d\mu(y) \\ &= \sum_{\gamma \in \hat{G}} \sum_{m,t=1}^{d_\gamma} d_\gamma \gamma(x)_{tm} \int_G \sum_{k=1}^\infty (\overline{h_k(y)}(\widehat{g_k}(\gamma))_{mt} \eta(y)_{nl} d\mu(y) \\ &= \sum_{k=1}^\infty \overline{\widehat{h_k}(\eta)_{ln}} \sum_{\gamma \in \hat{G}} d_\gamma \mathrm{tr}(\gamma(x)\widehat{g_k}(\gamma)) \\ &= \sum_{k=1}^\infty \overline{\widehat{h_k}(\eta)_{ln}} \overline{g_k}(x) \\ &= \sum_{k=1}^\infty \widehat{h_k}(\eta)^*_{nl} \overline{g_k}(x) \end{aligned}$$

for $1 \leq n, l \leq d_\eta$. Thus, for all $(x, \eta) \in G \times \hat{G}$, we get

$$\eta(x)\tau(x,\eta) = \sum_{k=1}^\infty \widehat{h_k}(\eta)^* \overline{g_k}(x)$$

and hence

$$\tau(x,\eta) = \eta(x)^* \sum_{k=1}^{\infty} \widehat{h_k}(\eta)^* \overline{g_k}(x).$$

□

As an application of Theorem 2.2 and Theorem 3.1, we give a criterion for the self-adjointness of nuclear pseudo-differential operators.

Corollary 3.2. Let σ be a function on $G \times \hat{G}$ such that $T_\sigma : L^2(G) \to L^2(G)$ is nuclear. Then $T_\sigma : L^2(G) \to L^2(G)$ is self-adjoint if and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ in $L^2(G)$ such that

$$\sum_{k=1}^{\infty} \|h_k\|_{L^2(G)} \|g_k\|_{L^2(G)} < \infty,$$

$$\sum_{k=1}^{\infty} h_k(x) \widehat{\overline{g_k}}(\xi)^* = \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \overline{g_k}(x), \quad (x,\xi) \in G \times \hat{G},$$

and

$$\sigma(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x) \widehat{\overline{g_k}}(\xi)^*, \quad (x,\xi) \in G \times \hat{G}.$$

We can give another formula for the adjoints of nuclear operators and another criterion for the self-adjointness of nuclear operators in terms of symbols.

Theorem 3.2. *Let σ be a function on $G \times \hat{G}$ such that $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is nuclear for $1 \le p_1, p_2 < \infty$. Then the symbol τ of the adjoint $T_\sigma^* : L^{p_2'}(G) \to L^{p_1'}(G)$ is given by*

$$\tau(x,\xi) = \xi(x)^* \sum_{\eta \in \hat{G}} d_\eta \int_G \mathrm{tr}[(\eta(y)\sigma(y,\eta))^* \eta(x)] \xi(y)\, d\mu(y),$$

which is the same as

$$\tau(x,\xi) = \xi(x)^* \sum_{\eta \in \hat{G}} d_\eta \overline{\overline{\mathrm{tr}(\sigma(\cdot,\eta)^* \eta(\cdot)^* \eta(x))}^{\wedge}}(\xi)$$

for all $(x,\xi) \in G \times \hat{G}$.

Proof. Let $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ be a nuclear operator for $1 \le p_1, p_2 < \infty$. Then there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G)$ such that

$$\sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}(G)} \|g_k\|_{L^{p_1'}(G)} < \infty$$

and for all $(y,\eta) \in G \times \hat{G}$,

$$\eta(y)\sigma(y,\eta) = \sum_{k=1}^{\infty} h_k(y) \widehat{\overline{g_k}}(\eta)^*$$

or equivalently

$$(\eta(y)\sigma(y,\eta))^* = \sum_{k=1}^{\infty} \overline{h_k}(y)\widehat{\overline{g_k}}(\eta).$$

Let $(x,\xi) \in G \times \hat{G}$. Then

$$\mathrm{tr}[(\eta(y)\sigma(y,\eta))^*\eta(x)]\xi(y) = \sum_{k=1}^{\infty} \overline{h_k}(y)\xi(y)\mathrm{tr}[\widehat{\overline{g_k}}(\eta)\eta(x)].$$

So, for all $(x,\xi) \in G \times \hat{G}$,

$$\begin{aligned}\int_G \mathrm{tr}[(\eta(y)\sigma(y,\eta))^*\eta(x)]\xi(y)\,d\mu(y) &= \sum_{k=1}^{\infty} \mathrm{tr}[\widehat{\overline{g_k}}(\eta)\eta(x)] \int_G \overline{h_k}(y)\xi(y)\,d\mu(y) \\ &= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^*\mathrm{tr}[\widehat{\overline{g_k}}(\eta)\eta(x)].\end{aligned}$$

Therefore by Theorem 3.1,

$$\begin{aligned}&\sum_{\eta\in\hat{G}} d_\eta \int_G \mathrm{tr}[(\eta(y)\sigma(y,\eta))^*\eta(x)]\xi(y)\,d\mu(y) \\ =\ & \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \sum_{\eta\in\hat{G}} d_\eta \mathrm{tr}[\widehat{\overline{g_k}}(\eta)\eta(x)] \\ =\ & \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^*\overline{g_k}(x) \\ =\ & \xi(x)\tau(x,\xi)\end{aligned}$$

for all $(x,\xi) \in G \times \hat{G}$. □

We can also give another criterion for the self-adjointness of nuclear pseudo-differential operators on compact and Hausdorff groups.

Corollary 3.4. Let σ be a function on $G\times\hat{G}$ be such that $T_\sigma : L^2(G) \to L^2(G)$ is a nuclear pseudo-differential operator. Then $T_\sigma : L^2(G) \to L^2(G)$ is self-adjoint if and only if

$$\sigma(x,\xi) = \xi(x)^* \sum_{\eta\in\hat{G}} d_\eta \overline{((\mathrm{tr}(\sigma(\cdot,\eta)^*\eta(\cdot)^*\eta(x)))^{\wedge}(\xi)}$$

for all $(x,\xi) \in G \times \hat{G}$.

4. **Products**

The following theorem shows that the product of two nuclear pseudo-differential operators on $L^p(G)$ is a nuclear pseudo-differential operator on $L^p(G)$ for $1 \le p < \infty$.

Theorem 4.1. *Let $T_\sigma : L^p(G) \to L^p(G)$ be a nuclear pseudo-differential operator, i.e., there exist by Theorem 2.2 sequences $\{g_k\}_{k=1}^\infty$ in $L^{p'}(G)$ and $\{h_k\}_{k=1}^\infty$ in $L^p(G)$ such that*

$$\sum_{k=1}^{\infty} \|h_k\|_{L^p(G)} \|g_k\|_{L^{p'}(G)} < \infty$$

and

$$\sigma(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x) \widehat{\overline{g_k}}(\xi)^*, \quad (x,\xi) \in G \times \hat{G}.$$

Let $T_\tau : L^p(G) \to L^p(G)$ be a bounded linear operator. Then $T_\tau T_\sigma : L^p(G) \to L^p(G)$ is a nuclear pseudo-differential operator $T_\lambda : L^p(G) \to L^p(G)$, where

$$\lambda(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h'_k(x) \widehat{\overline{g_k}}(\xi)^*$$

for all $(x,\xi) \in G \times \hat{G}$, where

$$h'_k(x) = \sum_{\eta \in \hat{G}} \mathrm{tr}[\eta(x)\tau(x,\eta)\widehat{h_k}(\eta)], \quad x \in G,$$

for all positive integers k.

Proof. Let $f \in L^p(G)$. Then

$$\begin{aligned}
&(T_\tau T_\sigma f)(x) \\
= &\sum_{\eta \in \hat{G}} d_\eta \mathrm{tr}[\eta(x)\tau(x,\eta)\widehat{T_\sigma f}(\eta)] \\
= &\sum_{\eta \in \hat{G}} d_\eta \mathrm{tr}\left[\eta(x)\tau(x,\eta) \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(y)\sigma(y,\xi)\hat{f}(\xi)\eta(y)^*\, d\mu(y)\right]
\end{aligned}$$

for all $x \in G$. Therefore using the nuclearity of T_σ,

$$\begin{aligned}
&(T_\tau T_\sigma f)(x) \\
= &\sum_{\eta \in \hat{G}} d_\eta \mathrm{tr}\left[\eta(x)\tau(x,\eta) \int_G \sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}\left(\sum_{k=1}^{\infty} h_k(y) \widehat{\overline{g_k}}(\xi)^* \hat{f}(\xi)\right) \eta(y)^*\, d\mu(y)\right] \\
= &\sum_{\eta \in \hat{G}} d_\eta \mathrm{tr}\left[\eta(x)\tau(x,\eta) \sum_{\xi \in \hat{G}} \sum_{k=1}^{\infty} \widehat{h_k}(\eta) d_\xi \mathrm{tr}(\widehat{\overline{g_k}}(\xi)^* \hat{f}(\xi))\right] \\
= &\sum_{\xi \in \hat{G}} \sum_{k=1}^{\infty} \sum_{\eta \in \hat{G}} d_\xi d_\eta \mathrm{tr}[\eta(x)\tau(x,\eta)\widehat{h_k}(\eta)] \mathrm{tr}(\widehat{\overline{g_k}}(\xi)^* \hat{f}(\xi)) \\
= &\sum_{\xi \in \hat{G}} d_\xi \mathrm{tr}(\xi(x)\lambda(x,\xi)\hat{f}(\xi)), \quad x \in G,
\end{aligned}$$

where

$$\begin{aligned}\lambda(x,\xi) &= \xi(x)^* \sum_{k=1}^{\infty} \sum_{\eta\in\hat{G}} d_\eta \mathrm{tr}[\eta(x)\tau(x,\eta)\widehat{h_k}(\eta)]\widehat{\overline{g_k}}(\xi)^* \\ &= \xi(x)^* \sum_{k=1}^{\infty} h'_k(x)\widehat{\overline{g_k}}(\xi)^*\end{aligned}$$

for all $(x,\xi) \in G \times \hat{G}$. This completes the proof. □

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