# RICCI SOLITON ON MANIFOLDS WITH COSYMPLECTIC METRIC

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*The m-quasi-Einstein metric emerged from m-Bakry-Emery Ricci tensor $S_m^f$ is the generalization of Einstein metric as well as of Ricci soliton. The quasi-Einstein metric is useful to construct the warped product Einstein metrics. Using geometric techniques of warped product many examples of generalized Sasakian-space-forms (GSSF) with non-constant functions were obtained in* [1]. *Physicists are interested in studying quasi-Einstein metrics to understand string theory which is mathematically similar to the study of Ricci soliton. The solution of the Ricci flow equation on a Riemannian manifold can be seen as quasi-Einstein metrics or Ricci solitons which are extremely important for both mathematicians and physicists.*

*In this paper, we study GSSF $M^{2n+1}(f_1, f_2, f_3)$ with cosymplectic metric admitting Ricci soliton and $*$-Ricci soliton and give examples.*

**Keywords:** Ricci soliton, $*$-Ricci soliton, generalized Sasakian-space-form, cosymplectic manifold.
**MSC2020**: 53C 15, 53C 40, 53C 50, 53D 15.

## 1. **Introduction**

Hamilton examined manifolds with positive curvature with the help of an efficient technique of Ricci flow [5]. The Ricci soliton is one of the solutions of this flow. The Ricci soliton known as quasi-Einstein metrics in physics has its applications in mathematical physics. Also, there are many problems in sciences which can be modeled as differential equations having its solutions as quasi-Einstein metrics.

On a Riemannian manifold, a Ricci soliton is a triplet $(g, V, \lambda)$ where $g$ is a Riemannian metric, $V$ a potential field and $\lambda$ a real scalar such that the Ricci tensor i.e., $Ric$ satisfies the following equation [4]

$$Ric + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1}$$

where $\mathcal{L}_V$ is the Lie-derivative. The Ricci soliton is said to be shrinking, steady or expanding according as $\lambda$ is $> 0$, $= 0$ or $< 0$, respectively.

The Ricci curvature $S$ is defined as [15]

$$S(U, Y) = \sum_{i=1}^{2n+1} g(R(e_i, U)Y, e_i), \tag{2}$$

$\forall\ U, Y \in TM$, where $e_i$ are local orthonormal vector fields on $M^{2n+1}$.

Tachibana [12] introduced $*$-Ricci tensor on an almost Hermitian manifold. Hamada [7] studied the $*$-Ricci tensor on real hypersurfaces in non-flat complex space forms and

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defined the $*$-Ricci tensor on an almost contact metric manifold $M$ as follows :

$$S^*(U,Y) = \frac{1}{2}\text{trace}(Z \mapsto R(U,\phi Y)\phi Z), \text{ for any } U,Y,Z \in TM, \tag{3}$$

where $\phi$ is a $(1,1)$-tensor field and $R$ is a Riemann curvature tensor.

Kaimakamis and Panagiotidou [9] defined $*$-Ricci soliton on a Riemannian manifold $(M,g)$ as

$$\frac{1}{2}\mathcal{L}_V g + Ric^* = \lambda g, \tag{4}$$

where $g$ is a Riemannian metric, $\lambda$ a real constant, and $V$ a smooth vector field. $*$-Ricci solitons of real hypersurfaces in non-flat complex space forms with potential vector field being the structure vector field $\xi$ was studied by them in this paper.

Almost contact geometry and topics closely related to it grab the attention and interest of many geometers because of its pure geometrical point of view and its application in many areas of physics. In the last few decades, Ricci solitons and $*$-Ricci solitons on an almost contact metric manifold were studied extensively by many geometers.

In differential geometry, the curvature tensor $R$ is very significant to understand the manifold. The Ricci tensor $S$, scalar curvature $r$ can be obtained on the manifold using $R$. It is also known that the $R$ depends on sectional curvatures entirely.

A Riemannian manifold $M$ is called real-space-form if its curvature tensor satisfies

$$R(U,Y)Z = c\big(g(Y,Z)U - g(U,Z)Y\big), \tag{5}$$

where $c$ is constant sectional curvature, $g$ a Riemannian metric and $U,Y,Z \in TM$. The examples are the Euclidean spaces, the hyperbolic spaces and the spheres with $c = 0$, $c < 0$ and $c > 0$, respectively.

In contact geometry the notion of GSSF was studied by replacing constant quantities $(c+3)/4$ and $(c-1)/4$ with smooth functions. An almost contact metric manifold is called a GSSF if [1]

$$\begin{aligned} R(U,Y)Z = {} & f_1\big(g(Y,Z)U - g(U,Z)Y\big) + f_2\big(g(U,\varphi Z)\varphi Y \\ & -g(Y,\varphi Z)\varphi U + 2g(U,\varphi Y)\varphi Z\big) + f_3\big(g(U,Z)\eta(Y)\xi \\ & -g(Y,Z)\eta(U)\xi + \eta(U)\eta(Z)Y - \eta(Y)\eta(Z)U\big), \end{aligned} \tag{6}$$

for smooth functions $f_1$, $f_2$, $f_3$ on $M$ and $U,Y,Z \in TM$. Many interesting examples of GSSF with non-constant functions were constructed using Riemannian submersions, warped products, product manifolds, conformal transformations, D-conformal deformations, and D-homothetic deformations [1]. Several authors have studied manifolds with GSSF and their submanifolds.

From a geometrical point of view and because of applications in many areas of physics almost contact geometry is widely studied. Cosymplectic manifolds give an important class of almost contact metric manifolds. The significance of the study of cosymplectic manifolds for the geometric illustration of time-dependent mechanics is extensively recognized.

Cho and Kimura [4] studied Ricci soliton on real hypersurfaces in a complex space form. Cho [3] showed that constant sectional curvature is 0 of a cosymplectic 3-manifold admitting a Ricci soliton with a transversal vector field or the Reeb potential vector field. Singh and Lalmalsawma [11] studied Ricci solitons on $\alpha$-cosymplectic manifolds. Hui et al. [8] studied Ricci solitons on 3-dimensional generalized Sasakian-space-form.

Recently, Wang [13] studied Ricci soliton on a cosymplectic manifold $M^3$. The Ricci soliton on GSSF $M^3$ with quasi Sasakian metric was studied by Sarkar and Biswas [10].

In most of the existing literature, the nature of the Ricci soliton constant is determined by comparing $(\mathcal{L}_V R)(U,\xi)\xi$, obtained by using two methods: first by Lie-differentiating Ricci soliton equation and using commutation formula in it and other by taking Lie-derivative of curvature tensor $R(U,\xi)\xi$. However, in the case of cosymplectic manifold, we get $(\mathcal{L}_V R)(U,\xi)\xi = 0$ in both ways so no conclusion can be obtained using this.

In view of this, we study GSSF with cosymplectic metric admitting Ricci soliton or $*$-Ricci soliton and we obtain $(\mathcal{L}_V Ric)(U,Y)$ using commutation formula and Ricci soliton equation and on the other hand by taking Lie-derivative of $Ric(U,Y)$. Then comparing $(\mathcal{L}_V Ric)(U,Y)$ obtained from both ways to get the value of $\lambda$. The paper is organized as follows: In section 2, we collect formulas which are useful in subsequent sections. In section 3, we study Ricci soliton or $*$-Ricci soliton on GSSF with cosymplectic metric and provide examples.

## 2. **Preliminaries**

An almost contact metric manifold $M^{2n+1}$ is a smooth manifold such that its structure tensors $(\phi,\xi,\eta,g)$ satisfies the following conditions [2]

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1, \tag{7}$$

$$\phi\xi = 0,\ \eta\ o\ \phi = 0,\ \eta(U) = g(U,\xi), \tag{8}$$

$$g(\phi U,\phi Y) = g(U,Y) - \eta(U)\eta(Y), \tag{9}$$

where $\phi$ is a $(1,1)$-tensor field, $\xi$ a structure vector field, $\eta$ a 1-form and $g$ a Riemannian metric and $U$, $Y \in TM$.

Following Goldberg and Yano [6], $M(\phi,\xi,\eta,g)$ is called an almost cosymplectic manifold if $d\eta = 0$ and $d\Phi = 0$, where $\Phi$ is a fundamental 2-form such that $\Phi(X,Y) = g(X,\phi Y)$. Further, if the associated almost contact structure on an almost cosymplectic manifold is normal, then it is called a cosymplectic manifold. On a cosymplectic manifold, we have

$$(\nabla_U\phi)(Y) = 0, \tag{10}$$

$$\nabla_U\xi = 0, \tag{11}$$

$$(\nabla_U\eta)(Y) = g(\nabla_U\xi, Y) = 0, \tag{12}$$

$$R(U,Y)\xi = 0, \tag{13}$$

$\forall\ U,Y \in TM$.

The following commutation formulae will be useful to obtain our results. On a Riemannian manifold $M$ [14], we have

$$(\nabla_Z\mathcal{L}_V g)(U,Y) = g((\mathcal{L}_V\nabla)(Z,U),Y) + g((\mathcal{L}_V\nabla)(Z,Y),U), \tag{14}$$

and

$$(\mathcal{L}_V R)(U,Y)Z = (\nabla_U\mathcal{L}_V\nabla)(Y,Z) - (\nabla_Y\mathcal{L}_V\nabla)(U,Z), \tag{15}$$

$\forall\ U,\ Y,\ Z \in TM$.

## 3. **Solitons on GSSF admitting Cosymplectic metric**

In this section we study solitons on GSSF with cosymplectic metric.

**Lemma 3.1.** [6] *Let $M(\phi,\eta,g)$ be a cosymplectic manifold. Then for any $U \in TM$, $K(U,\xi) = 0$.*

**Theorem 3.1.** *Let $M^{2n+1}(f_1,f_2,f_3)$ be a GSSF with cosymplectic metric, then $f_1 = f_3$.*

*Proof.* From (6), we find that $R(\xi,U)\xi = (f_3 - f_1)U$ and hence $g(R(\xi,U)U,\xi) = f_1 - f_3$, for any unit vector field $U$ orthogonal to $\xi$ and using Lemma 3.1, we get $f_1 = f_3$. □

Next, we obtain

**Theorem 3.2.** *Let $M^{2n+1}(f_1,f_2,f_3)$ be a GSSF with cosymplectic metric. Then, $f_1 = f_2 = f_3$.*

*Proof.* The Ricci identity is given by

$$\nabla_U\nabla_Y\phi - \nabla_Y\nabla_U\phi - \nabla_{[U,Y]}\phi = R(U,Y)\phi - \phi R(U,Y), \tag{16}$$

$\forall\ U,\ Y \in TM$. Using (10) in (16), we obtain

$$R(U,Y)\phi = \phi R(U,Y). \tag{17}$$

Also, on cosymplectic manifold [6], we have

$$R(\phi U, \phi Y) = R(U,Y). \tag{18}$$

From (18), we have

$$R(Z,\phi W)Y = R(\phi Z, \phi^2 W)Y. \tag{19}$$

Taking inner product of (19) with a vector field $U$ on $M$, we obtain

$$g(R(Z,\phi W)Y,U) = -g(R(Y,U)\phi Z, W) + \eta(W)g(R(Y,U)\phi Z,\xi). \tag{20}$$

Again from (18), we obtain

$$R(\phi Z, W)Y = R(\phi^2 Z, \phi W)Y. \tag{21}$$

Now, taking inner product of (21) with a vector field $U$ on $M$, we find

$$g(R(\phi Z,W)Y,U) = -g(R(Y,U)Z,\phi W) - \eta(Z)g(R(Y,U)\phi W,\xi). \tag{22}$$

Using (17)$\sim$(22), we find

$$g(R(\phi Z,W)Y,U) + g(R(Z,\phi W)Y,U) \quad = \quad 0. \tag{23}$$

Also, from (6) we have

$$\begin{aligned} g(R(Z,\phi W)Y,U) &= f_1\{g(\phi W,Y)g(Z,U) - g(Z,Y)g(\phi W,U)\} \\ &+ f_2\{g(Z,\phi Y)g(\phi^2 W,U) - g(\phi W,\phi Y)g(\phi Z,U) + 2g(Z,\phi^2 W)g(\phi Y,U)\}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} g(R(\phi Z,W)Y,U) &= f_1\{g(W,Y)g(\phi Z,U) - g(\phi Z,Y)g(W,U)\} \\ &+ f_2\{g(\phi Z,\phi Y)g(\phi W,U) - g(W,\phi Y)g(\phi^2 Z,U) + 2g(\phi Z,\phi W)g(\phi Y,U)\}, \end{aligned} \tag{25}$$

for vector fields $U$, $Y$, $Z$ and $W$ orthogonal to $\xi$. Adding (24) and (25), we get

$$\begin{aligned} g(R(\phi Z,W)Y,U) + g(R(Z,\phi W)Y,U) &= (f_2 - f_1)\{g(U,Z)g(\phi Y,W) \\ &- g(Y,Z)g(\phi U,W) - g(U,\phi Z)g(Y,W) + g(Y,\phi Z)g(U,W)\}. \end{aligned} \tag{26}$$

Then, from (23) and (26), we obtain

$$f_1 = f_2. \tag{27}$$

Using Theorem 3.1 together with (27) completes the proof of the theorem. □

Next, we have

**Theorem 3.3.** *Let $M^{2n+1}(f_1,f_2,f_3)$ be a GSSF with cosymplectic metric. Then,*

$$S^*(U,Y) = S(U,Y), \tag{28}$$

$\forall\ U,\ Y \in TM$.

*Proof.* Taking inner product of (6) with a vector field $W$ on $M$, we find

$$\begin{aligned} g(R(U,Y)Z,W) &= f_1\{g(Y,Z)g(U,W) - g(U,Z)g(Y,W)\} \\ &+ f_2\{g(U,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi U,W) + 2g(U,\phi Y)g(\phi Z,W)\} \\ &+ f_3\{(g(U,Z)\eta(Y) - g(Y,Z)\eta(U))\eta(W) + (\eta(U)g(Y,W) - \eta(Y)g(U,W))\eta(Z)\}. \end{aligned} \tag{29}$$

Putting $U = e_i = W$ in (29) and using (2), we obtain

$$S(U,Y) = F_2 g(U,Y) - F_3\eta(U)\eta(Y), \tag{30}$$

$\forall\, U, Y \in TM$, where $F_2 = 2nf_1 + 3f_2 - f_3$ and $F_3 = 3f_2 + (2n-1)f_3$.

Now, replacing $Y$ by $\phi Y$, $Z$ by $\phi Z$ in (6) and using $\eta \circ \phi = 0$, we have

$$R(U,\phi Y)\phi Z = f_1\{g(\phi Y,\phi Z)U - g(U,\phi Z)\phi Y\} + f_2\{g(U,\phi^2 Z)\phi^2 Y - g(\phi Y,\phi^2 Z)\phi U + 2g(U,\phi^2 Y)\phi^2 Z\} + f_3\{-g(\phi Y,\phi Z)\eta(U)\xi\}. \tag{31}$$

Further, taking inner product of (31) with $Z$, we have

$$g(R(U,\phi Y)\phi Z, Z) = f_1\{g(\phi Y,\phi Z)g(U,Z) - g(U,\phi Z)g(\phi Y,Z)\} + f_2\{g(U,\phi^2 Z)g(\phi^2 Y,Z) - g(\phi Y,\phi^2 Z)g(\phi U,Z) + 2g(U,\phi^2 Y)g(\phi^2 Z,Z)\} + f_3\{-g(\phi Y,\phi Z)\eta(U)\eta(Z)\}. \tag{32}$$

Putting $Z = e_i$ in (32), tracing from $i = 1$ to $i = 2n+1$ and using (3), we obtain

$$S^*(U,Y) = F_1(g(U,Y) - \eta(U)\eta(Y)), \tag{33}$$

where $F_1 = f_1 + (1+2n)f_2$. Using Theorem 3.2 in (30) and (33), we get (28). □

**Theorem 3.4.** *Let $M^{2n+1}(f_1, f_2, f_3)$ be a GSSF with cosymplectic metric then, $\xi f_1 = 0$.*

*Proof.* Differentiating (13) along $Z \in TM$ and using (11), we obtain

$$(\nabla_Z R)(U,Y)\xi = 0, \tag{34}$$

$\forall\, U,\, Y \in TM$.

If GSSF admits cosymplectic metric, then

$$Q\xi = 0. \tag{35}$$

Differentiating (35) with respect to $U \in TM$ and using (11), we obtain

$$(\nabla_U Q)\xi = 0. \tag{36}$$

Also, if GSSF admits cosymplectic metric, then

$$QU = (2n+2)f_1(U - \eta(U)\xi). \tag{37}$$

Covariant derivative of (37) along $\xi$ gives

$$(\nabla_\xi Q)U = (2n+2)\xi f_1(U - \eta(U)\xi). \tag{38}$$

From second Bianchi identity, we have

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} R)(U,\xi)Y, e_i) = g((\nabla_U Q)\xi, Y) - g((\nabla_\xi Q)U, Y). \tag{39}$$

Taking inner product of (34) with $W$ and then putting $Z = U = e_i$ and tracing with respect to basis $\{e_i\}_{i=1}^{2n+1}$ on $M$, we find

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} R)(e_i,Y)\xi, W) = 0. \tag{40}$$

From (39) and (40), we obtain

$$g((\nabla_U Q)\xi, Y) - g((\nabla_\xi Q)U, Y) = 0. \tag{41}$$

Using (36) in (41), we find that

$$(\nabla_\xi Q)U = 0. \tag{42}$$

Using (38) in (42), we obtain the required result. □

**Remark 3.1.** *In view of Theorem 3.3, the study of the $*$-Ricci soliton is same as the study of the Ricci soliton.*

Next, we give examples of Ricci solitons and $*$-Ricci solitons on GSSFs with cosymplectic metric.

**Example 3.1.** *Consider the manifold $M^{2n+1}$ with structures $\{\phi, \xi, \eta, g\}$*

$$\begin{cases} \phi(e_l) = e_{l+1},\ \phi(e_{l+1}) = -e_l,\ \text{for } l = 1, 3, \dots, 2n-3, 2n-1, \\ \phi(e_{2n+1}) = 0, e_{2n+1} = \xi = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \cdots + \frac{\partial}{\partial x^{2n+1}}, \\ \eta = \frac{1}{2n-1}(dx^1 + dx^2 + \cdots + dx^{2n} - dx^{2n+1}), \end{cases} \tag{43}$$

$$g_{lp} = \begin{cases} \frac{(2n-2)^2+2n}{(2n-1)^2},\ l = p = 1, \dots, 2n \\ \frac{-2n+3}{(2n-1)^2},\ l \neq p, 1 \leq l, p \leq 2n \\ \frac{-2}{(2n-1)^2},\ l = 1, \dots, 2n, p = 2n+1 \\ \frac{2n+1}{(2n-1)^2},\ l = p = 2n+1. \end{cases} \tag{44}$$

*Then, a local orthonormal frame of $TM$ at each point of $M$ is given by*

$$e_l = \frac{\partial}{\partial x^l} + \frac{\partial}{\partial x^{2n+1}}, l = 1, \cdots, 2n \quad \text{and} \quad e_{2n+1} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \cdots + \frac{\partial}{\partial x^{2n+1}},$$

*Also, we have*

$$[e_l, e_p] = 0,\ \nabla_{e_l} e_p = 0,\ l, p = 1, \cdots, 2n+1, \tag{45}$$

$$R(e_l, e_p)e_q = 0,\ S^*(e_l, e_p) = 0,\ l, p, q = 1, 2, 3. \tag{46}$$

*Further, $V$ on $M$ is given by*

$$V = \lambda(x_1 e_1 + x_2 e_2 + \cdots + x_{2n+1} e_{2n+1}).$$

*Then, we have*

$[V, e_l] = -\lambda e_l - \lambda e_{2n+1}$ *for* $l = 1, \cdots, 2n$, $[V, e_{2n+1}] = -\lambda e_1 - \lambda e_2 - \cdots - \lambda e_{2n+1}$.

*Now, we can see that*

$$(\mathcal{L}_V g)(e_l, e_p) + 2S^*(e_l, e_p) = 2\lambda g(e_l, e_p),$$

*for arbitrary soliton constant $\lambda$ and $l, p = 1, \cdots, 2n+1$.*

*Hence, $M^{2n+1}(0, 0, 0)$ is the GSSF with cosymplectic metric admitting $*$-Ricci soliton with shrinking, steady or expanding $*$-Ricci soliton according as $\lambda > 0$, $= 0$ or $< 0$, respectively.*

**Example 3.2.** *Consider the manifold $M^3$ with structures $\{\phi, \xi, \eta, g\}$*

$$\begin{cases} \phi(X^1, Y^1, Z) = (-Y^1, X^1, 0), \quad \xi = -\frac{\partial}{\partial z}, \\ g = \frac{1}{x^2}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta, \quad x \neq 0, \quad \eta = -dz. \end{cases} \tag{47}$$

*Then, a local orthonormal frame of $TM$ is given by*

$$e_1 = x\frac{\partial}{\partial x}, e_2 = x\frac{\partial}{\partial y}, e_3 = -\frac{\partial}{\partial z}.$$

*Moreover, we have*

$$[e_1, e_3] = 0,\ [e_1, e_2] = e_2,\ [e_2, e_3] = 0, \tag{48}$$

$$\begin{cases} \nabla_{e_1} e_1 = 0,\ \nabla_{e_2} e_1 = -e_2,\ \nabla_{e_3} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_2} e_2 = e_1, \\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_3} e_3 = 0, \end{cases} \tag{49}$$

$$\begin{cases} S(e_l, e_l) = -1, \quad l = 1, 2, \; S(e_3, e_3) = 0, \\ S(e_l, e_p) = 0, \qquad l \neq p, l, p = 1, 2, 3. \end{cases} \tag{50}$$

*The potential vector field $V$ on $M$ is given by*

$$V = e_1 + \frac{y}{x} e_2 + z e_3.$$

*Then, we have*

$$[V, e_l] = 0, \quad l = 1, 2 \quad \text{and} \quad [V, e_3] = e_3.$$

*Now, we can see that*

$$(\mathcal{L}_V g)(e_l, e_p) + 2S(e_l, e_p) = 2\lambda g(e_l, e_p),$$

*for $\lambda = -1$ and $l, p = 1, 2, 3$.*

*Hence, $M^3(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$ is the GSSF with cosymplectic metric admitting expanding Ricci soliton.*

**Theorem 3.5.** *Let $M^{2n+1}(f_1, f_2, f_3)$ be a GSSF with cosymplectic metric admitting Ricci soliton or $*$-Ricci soliton, then*

$$(\mathcal{L}_V R)(U, \xi)\xi = 0. \tag{51}$$

*Proof.* Using (33) in (4), we get

$$(\mathcal{L}_V g)(U, Y) = 2\lambda g(U, Y) - 2F_1 g(\phi U, \phi Y). \tag{52}$$

Covariant differentiation of (52) along $Z \in TM$, gives

$$\begin{aligned} (\nabla_Z \mathcal{L}_V g)(U, Y) &= -2F_1 g((\nabla_Z \phi)U, \phi Y) - 2F_1 g((\nabla_Z \phi)Y, \phi U) \\ &- 2(ZF_1) g(\phi U, \phi Y). \end{aligned} \tag{53}$$

Using (14) in (53), we obtain

$$\begin{aligned} g((\mathcal{L}_V \nabla)(Z, U), Y) + g((\mathcal{L}_V \nabla)(Z, Y), U) &= -2F_1 g((\nabla_Z \phi)U, \phi Y) \\ &-2F_1 g((\nabla_Z \phi)Y, \phi U) - 2(ZF_1) g(\phi U, \phi Y). \end{aligned} \tag{54}$$

Similarly, we have

$$\begin{aligned} g((\mathcal{L}_V \nabla)(U, Y), Z) + g((\mathcal{L}_V \nabla)(U, Z), Y) &= -2F_1 g((\nabla_U \phi)Y, \phi Z) \\ &-2F_1 g((\nabla_U \phi)Z, \phi Y) - 2(UF_1) g(\phi Y, \phi Z). \end{aligned} \tag{55}$$

$$\begin{aligned} g((\mathcal{L}_V \nabla)(Y, Z), U) + g((\mathcal{L}_V \nabla)(Y, U), Z) &= -2F_1 g((\nabla_Y \phi)Z, \phi U) \\ &-2F_1 g((\nabla_Y \phi)U, \phi Z) - 2(YF_1) g(\phi Z, \phi U). \end{aligned} \tag{56}$$

Adding (55) and (56), then subtracting (54) from it, we find

$$\begin{aligned} & g((\mathcal{L}_V \nabla)(U, Y), Z) = F_1 g((\nabla_Z \phi)U, \phi Y) + F_1 g((\nabla_Z \phi)Y, \phi U) \\ & +(ZF_1) g(\phi U, \phi Y) - F_1 g((\nabla_Y \phi)Z, \phi U) - F_1 g((\nabla_Y \phi)U, \phi Z) - (YF_1) g(\phi Z, \phi U) \\ & -F_1 g((\nabla_U \phi)Y, \phi Z) - F_1 g((\nabla_U \phi)Z, \phi Y) - (UF_1) g(\phi Y, \phi Z), \end{aligned} \tag{57}$$

$\forall\, U, Y, Z \in TM$. Taking $Y = \xi$ in (57) and using (10), Theorem 3.2 and Theorem 3.4, we obtain

$$g((\mathcal{L}_V \nabla)(U, \xi), Z) = 0.$$

Hence,

$$(\mathcal{L}_V \nabla)(U, \xi) = 0. \tag{58}$$

Further, differentiating (58) along an arbitrary $Y \in TM$, we get

$$(\nabla_Y \mathcal{L}_V \nabla)(U, \xi) = 0. \tag{59}$$

Using (59) in (15), we obtain

$$(\mathcal{L}_V R)(U,Y)\xi = 0. \tag{60}$$

Putting $Y=\xi$ in (60), we find

$$(\mathcal{L}_V R)(U,\xi)\xi = 0. \tag{61}$$

Putting $Y=\xi, Z=\xi$ in (6), we get

$$R(U,\xi)\xi = (f_1-f_3)(U-\eta(U)\xi). \tag{62}$$

Lie-differentiating (62) along $V$, we have

$$\begin{aligned}(\mathcal{L}_V R)(U,\xi)\xi &= V(f_1-f_3)(U-\eta(U)\xi) \\ &+ (f_1-f_3)(g(\mathcal{L}_V\xi,U)\xi - 2\eta(\mathcal{L}_V\xi)U - (\mathcal{L}_V\eta)(U)\xi).\end{aligned} \tag{63}$$

Putting $Y=\xi$ in (33), we obtain

$$S^*(U,\xi) = 0. \tag{64}$$

Putting $Y=\xi$ in (4) and using (64), we get

$$(\mathcal{L}_V g)(U,\xi) = 2\lambda\eta(U). \tag{65}$$

Lie-differentiating $\eta(U)=g(U,\xi)$ along $V$ and using (65), we find

$$(\mathcal{L}_V\eta)(U) - g(\mathcal{L}_V\xi,U) - 2\lambda\eta(U) = 0. \tag{66}$$

Taking Lie-derivative of $g(\xi,\xi)=1$ along $V$ and using (65), we have

$$\eta(\mathcal{L}_V\xi) = -\lambda. \tag{67}$$

Using (66) and (67) in (63), we obtain

$$(\mathcal{L}_V R)(U,\xi)\xi = (V(f_1-f_3)+2\lambda(f_1-f_3))(U-\eta(U)\xi), \tag{68}$$

for any $U\in TM$. Putting $f_3=f_1$ for cosymplectic metric in (68), we get

$$(\mathcal{L}_V R)(U,\xi)\xi = 0. \tag{69}$$

From (61) and (69), we get the required result. □

Now, we have following

**Theorem 3.6.** *Let $M^{2n+1}(f_1,f_2,f_3)$ be a GSSF with cosymplectic metric admitting Ricci solion or $*$-Ricci soliton, then*

$$(2n-1)\sum_{i=1}^{2n+1}(e_ie_i-\nabla_{e_i}e_i)f_1 = 2n(Vf_1+2\lambda f_1-4(n+1)f_1^2). \tag{70}$$

*Proof.* Using (10) in (57), we get

$$\begin{aligned}g((\mathcal{L}_V\nabla)(U,Y),Z) &= (ZF_1)g(\phi U,\phi Y)-(YF_1)g(\phi U,\phi Z) \\ &- (UF_1)g(\phi Y,\phi Z),\end{aligned} \tag{71}$$

$\forall\ U,\ Y,\ Z\in TM$.

Which gives

$$(\mathcal{L}_V\nabla)(U,Y) = (\nabla F_1)g(\phi U,\phi Y)+(YF_1)\phi^2U+(UF_1)\phi^2Y. \tag{72}$$

Differentiating (72) along $Z$ and using (10), (11) and (12), we obtain

$$\begin{aligned}(\nabla_Z\mathcal{L}_V\nabla)(U,Y) &= (\nabla_Z\nabla F_1)g(\phi U,\phi Y)+(ZYF_1-(\nabla_ZY)F_1)\phi^2U \\ &+ (ZUF_1-(\nabla_ZU)F_1)\phi^2Y.\end{aligned} \tag{73}$$

Using (73) in (15), we find

$$(\mathcal{L}_V R)(Z,U)Y = (\nabla_Z \nabla F_1)g(\phi U, \phi Y) - (\nabla_U \nabla F_1)g(\phi Z, \phi Y) \\ +(ZYF_1 - (\nabla_Z Y)F_1)\phi^2 U - (UYF_1 - (\nabla_U Y)F_1)\phi^2 Z. \tag{74}$$

Contracting (74) over $Z$, we get

$$\begin{aligned}(\mathcal{L}_V S)(U,Y) &= \sum_{i=1}^{2n+1} g(\nabla_{e_i}\nabla F_1, e_i)g(\phi U, \phi Y) + g(\nabla_U \nabla F_1, \phi^2 Y) \\ &+ \sum_{i=1}^{2n+1}(e_i Y F_1 - (\nabla_{e_i} Y)F_1)g(-U + \eta(U)\xi, e_i) \\ &+ 2n(UYF_1 - (\nabla_U Y)F_1).\end{aligned} \tag{75}$$

Lie-differentiating (30) along $V$, we obtain

$$\begin{aligned}(\mathcal{L}_V S)(U,Y) &= (VF_2)g(U,Y) + F_2(\mathcal{L}_V g)(U,Y) - (VF_3)\eta(U)\eta(Y) \\ &- F_3((\mathcal{L}_V \eta)(U)\eta(Y) + (\mathcal{L}_V \eta)(Y)\eta(U)).\end{aligned} \tag{76}$$

Using (52) and (66) in (76), we get

$$(\mathcal{L}_V S)(U,Y) = (VF_2 + 2F_2\lambda - 2F_2F_1)g(U,Y) + (2F_2F_1 - VF_3 \\ -4\lambda F_3)\eta(U)\eta(Y) - F_3(g(\mathcal{L}_V\xi, U)\eta(Y) + g(\mathcal{L}_V\xi, Y)\eta(U)), \tag{77}$$

$\forall\ U,\ Y \in TM$. Using Theorem 3.2 in (77), we obtain

$$\begin{aligned}(\mathcal{L}_V S)(U,Y) &= (VF_1 - 2F_1^2 + 2\lambda F_1)g(\phi U, \phi Y) - 2\lambda F_1\eta(U)\eta(Y) \\ &- F_1(g(\mathcal{L}_V\xi, U)\eta(Y) + g(\mathcal{L}_V\xi, Y)\eta(U)).\end{aligned} \tag{78}$$

From (75) and (78), we get

$$\sum_{i=1}^{2n+1} g(\nabla_{e_i}\nabla F_1, e_i)g(\phi U, \phi Y) + g(\nabla_U \nabla F_1, \phi^2 Y) + 2n(UYF_1 - (\nabla_U Y)F_1) \\ + \sum_{i=1}^{2n+1}(e_i Y F_1 - (\nabla_{e_i} Y)F_1)g(-U + \eta(U)\xi, e_i) = (VF_1 - 2F_1^2 + 2\lambda F_1)g(\phi U, \phi Y) \\ -2\lambda F_1\eta(U)\eta(Y) - F_1(g(\mathcal{L}_V\xi, U)\eta(Y) + g(\mathcal{L}_V\xi, Y)\eta(U)). \tag{79}$$

Putting $U = Y = e_j$ in (79), tracing over $j = 1$ to $j = 2n+1$ and using (67), we get

$$2n\sum_{i=1}^{2n+1}(e_i e_i - \nabla_{e_i} e_i)F_1 - \sum_{j=1}^{2n+1}(e_j e_j - \nabla_{e_j} e_j)F_1 + g(\nabla_\xi \nabla F_1, \xi) = 2n(VF_1 - 2F_1^2 + 2\lambda F_1),$$

wherein using Theorem 3.4, we find

$$(2n-1)\sum_{i=1}^{2n+1}(e_i e_i - \nabla_{e_i} e_i)F_1 = 2n(VF_1 - 2F_1^2 + 2\lambda F_1). \tag{80}$$

Using Theorem 3.2 in (80), we get (70). □

**Corollary 3.1.** *Under the assumption of Theorem 3.6 if $f_1$ is constant, then*
*(i) $M$ is flat and the soliton vector field is homothetic if $f_1 = 0$,*
*(ii) $\lambda = (2n+2)f_1$ and hence the soliton is expanding or shrinking according as $f_1 < 0$ or $f_1 > 0$, respectively,*
*(iii) the soliton is steady if $f_1 = 0$ and $\lambda = (2n+2)f_1$.*

*Proof.* If $f_1$ is constant, then (70) gives

$$f_1(\lambda - (2n+2)f_1) = 0. \quad (81)$$

Hence, we have

**Case (i)**: Consider $f_1 = 0$. Using this in Theorem 3.2, we obtain $f_1 = f_2 = f_3 = 0$. Hence, from (6) we find $M$ is flat. Using this in (1), we obtain

$$\mathcal{L}_V g = 2\lambda g,$$

which gives $V$ is homothetic.

**Case (ii)**: Let $f_1 \neq 0$. Then, from (81) we get $\lambda = (2n+2)f_1$. Hence $M$ is shrinking or expanding accordingly as $f_1 > 0$ or $f_1 < 0$, respectively.

**Case (iii)**: Suppose $f_1 = 0$ and $\lambda = (2n+2)f_1$, then $\lambda = 0$. Hence, the soliton is steady.

This completes the proof. □

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