SOME HARMONIC PROBLEMS ON THE TANGENT BUNDLE WITH A BERGER-TYPE DEFORMED SASAKI METRIC

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Let (M_{2k}, φ, g) be an almost anti-paraHermitian manifold and (TM, g_{BS}) be its tangent bundle with a Berger type deformed Sasaki metric g_{BS}. In this paper, we deal with the harmonicity of the canonical projection $\pi : TM \to M$ and a vector field ξ which is considered as a map $\xi : M \to TM$.

1. Introduction

Let M be an $2k-$dimensional Riemannian manifold with a Riemannian metric g. Throughout the paper, manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞.

An almost paracomplex manifold is an almost product manifold (M_{2k}, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y].$$

A paracomplex structure is an integrable almost paracomplex structure.

Let (M_{2k}, φ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \chi(M_{2k})$. If (M_{2k}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M_{2k}, φ, g) is said to be an almost anti-paraHermitian manifold. Moreover, (M_{2k}, φ, g) is said to be an anti-paraKähler if φ is parallel with respect to the Levi-Civita connection ∇ of g. As is well known, the anti-paraKähler condition ($\nabla \varphi = 0$) is equivalent to the paraholomorphicity of the anti-paraHermitian metric g, that is, $\Phi_\varphi g = 0$, where Φ_φ is the Tachibana operator [7].

In [1], the authors defined a new metric, which is called a Berger type deformed Sasaki metric, on the tangent bundle over an anti-paraKähler manifold. They studied the

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geodesics and curvature properties of the tangent bundle with the Berger type deformed Sasaki metric and gave the conditions for some almost anti-paraHermitian structures to be anti-paraKähler and quasi-anti-paraKähler on this setting. Motivated by the results presented in [6], we think up the paper. Clearly, in the present paper, we again consider the tangent bundle with the Berger type deformed Sasaki metric. First, we deal with the natural projection $\pi : TM \to M$ and find the conditions under which π is a totally geodesic or a harmonic map. Second, we consider a vector field on M as a map $\xi : M \to TM$ and obtain the conditions under which ξ is an isometric immersion, a totally geodesic or a harmonic map.

2. The Berger type deformed Sasaki metric on the tangent bundle

Let M be an $n-$dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle denoted by $\pi : TM \to M$. The tangent bundle TM is a $2n-$dimensional manifold. A system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i \right)$, $\bar{i} = n + i = n + 1, ..., 2n$, where (u^i) is the cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{ \frac{\partial}{\partial x^i} |_P \right\}$, P being an arbitrary point in U whose coordinates are (x^i). Summation over repeated indices is always implied.

Denote by ∇ the Levi-Civita connection of g. The local frame in horizontal distribution HTM, which is determined by ∇, is given by

$$E_i = \frac{\partial}{\partial x^i} - \Gamma_{i0}^h \frac{\partial}{\partial u^h}; \ i = 1, ..., n,$$

where $\Gamma_{i0}^h = u^j \Gamma_{ij}^h$ are the Christoffel symbols of ∇. The local frame in vertical distribution VTM which is defined by $\ker \pi_*$ is as

$$E_{\bar{i}} = \frac{\partial}{\partial u^i}; \ \bar{i} = n + 1, ..., 2n.$$

The local frame $\{E_\beta\} = (E_i, E_{\bar{i}})$ in TM is called an adapted frame. With respect to the adapted frame, the horizontal lift and vertical lift of a vector field $X = \xi^i \frac{\partial}{\partial x^i} \in \chi(M)$ are respectively as follows [10]:

$$^H X = \xi^i E_i \in \chi(\pi^{-1}(U)),$$
$$^V X = \xi^i E_{\bar{i}} \in \chi(\pi^{-1}(U)).$$

The system of local 1-forms $(dx^i, \delta u^i)$ in TM defines the local dual frame of the adapted frame $\{E_\beta\}$, where

$$\delta u^i =^H (dx^i) = du^i + \Gamma_{h0}^i dx^h.$$

Via natural lifts of a Riemannian metric g, new (pseudo-)Riemannian metrics can be induced on the tangent bundle TM over a Riemannian manifold (M, g). The well-known example of such metrics is the Sasaki metric. This metric on the tangent bundle TM was constructed by Sasaki in [8] and later studied its interesting geometric properties by some authors. Also, the various deformations of the Sasaki metric are considered and studied. One of them is the Berger type deformed Sasaki metric.

Definition 2.1. *Let (M_{2k}, φ, g) be an almost anti-paraHermitian manifold and TM be its tangent bundle. The Berger type deformed Sasaki metric on TM is defined by*

$$
\begin{aligned}
g_{BS}(^H X, {}^H Y) &= g(X, Y), & (1)\\
g_{BS}(^V X, {}^H Y) &= g_{BS}(^H X, {}^V Y) = 0,\\
g_{BS}(^V X, {}^V Y) &= g(X, Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)
\end{aligned}
$$

for all $X, Y \in \chi(M_{2k})$, where δ is some constant (if we consider the structure φ as almost complex, we get the metric in [9]).

The matrix of the Berger type deformed Sasaki metric with respect to the adapted frame $\{E_\beta\}$ is as follows:

$$g_{BS} = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} + \delta^2 g_{m0} g_{n0} \varphi_i^m \varphi_j^n \end{pmatrix} \qquad (2)$$

and its inverse

$$g_{BS}^{-1} = \begin{pmatrix} g^{ij} & 0 \\ 0 & g^{ij} - \frac{\delta^2}{1+\delta^2 g_{00}} \varphi_0^i \varphi_0^j \end{pmatrix}, \qquad (3)$$

where $g_{m0} = g_{mk} u^k$, $g_{00} = g_{mk} u^m u^k$, $\varphi_0^i = \varphi_m^j u^m$. For the Levi-Civita connection $^{BS}\nabla$ of the Berger type deformed Sasaki metric g_{BS}, we give the following proposition.

Proposition 2.1. [1] *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and TM its tangent bundle. The Levi-Civita connection $^{BS}\nabla$ of the Berger type deformed Sasaki metric g_{BS} on TM is locally given by*

$$\begin{cases} ^{BS}\nabla_{E_i} E_j = \Gamma_{ij}^h E_h - \frac{1}{2} R_{ij0}{}^h E_{\bar{h}}, \\ ^{BS}\nabla_{E_{\bar{i}}} E_j = -\frac{1}{2} R_{.j0i}^h E_h, \\ ^{BS}\nabla_{E_i} E_{\bar{j}} = -\frac{1}{2} R_{.i0j}^h E_h + \Gamma_{ij}^h E_{\bar{h}}, \\ ^{BS}\nabla_{E_{\bar{i}}} E_{\bar{j}} = \frac{\delta^2}{1+\delta^2} \varphi_j^k \varphi_0^h g_{ik} E_{\bar{h}}, \end{cases} \qquad (4)$$

where $R_{ijk}{}^h$ are the local components of the Riemannian curvature tensor field R of ∇ and $R_{.j0i}^h = R_{lj0}{}^s g^{lh} g_{si}$.

3. Main results

Let M and N be two Riemannian manifolds, $U \subset M$ be a domain with coordinates $(x^1, ..., x^m)$ and $V \subset N$ be a domain with coordinates $(u^1, ..., u^n)$. Also, let $f : U \to V$ be a map in which $u^\alpha = f^\alpha(x^1, ..., x^m)$. The second fundamental form of f denoted by $\beta(f)$ is

$$\beta(f) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)^\gamma = \left(\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \right) \frac{\partial}{\partial u^\gamma} \qquad (5)$$

and that of the tension field $\tau(f)$ of f is

$$\tau(f) = g^{ij} \left(\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \right) \frac{\partial}{\partial u^\gamma} \quad \text{(see [3])}. \qquad (6)$$

Remark that $\beta(f)$ and $\tau(f)$ may also be defined when N is endowed with a torsion free linear connection or M is a pseudo-Riemannian manifold.

We note that the canonical projection $\pi : (TM, g_{BS}) \to (M_{2k}, \varphi, g)$ is a Riemannian submersion. Direct computations give

$$\begin{aligned} \beta(\pi)(E_i, E_j) &= \beta(\pi)(E_{\bar{i}}, E_{\bar{j}}) = 0, \\ \beta(\pi)_v(E_{\bar{i}}, E_j) &= \frac{1}{2} R_{.j0i}^h (\pi(v)) \frac{\partial}{\partial x^h}. \end{aligned} \qquad (7)$$

From (7), we obtain the following theorem.

Theorem 3.1. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. The Riemannian submersion $\pi : (TM, g_{BS}) \to (M_{2k}, \varphi, g)$ is totally geodesic if and only if ∇ is locally flat. Moreover π is a harmonic map.*

Let h be another anti-paraHermitian metric on M with respect to an almost paracomplex structure φ_1. We take in consideration the projection $\pi : (TM, g_{BS}) \rightarrow (M_{2k}, \varphi_1, h)$. Then, we have

$$
\begin{aligned}
\beta\left(\pi\right)\left(E_{\bar{i}}, E_{\bar{j}}\right) &= 0, & (8)\\
\beta\left(\pi\right)_v\left(E_{\bar{i}}, E_j\right) &= \frac{1}{2} R^h_{.j0i}\left(\pi(v)\right) \frac{\partial}{\partial x^h}, \\
\beta\left(\pi\right)\left(E_i, E_j\right) &= \left({}^h\Gamma^h_{ij} - \Gamma^h_{ij}\right) \frac{\partial}{\partial x^h},
\end{aligned}
$$

where ${}^h\Gamma^h_{ij}$ are the Christoffel symbols of the metric h. Hence we get the proposition below.

Proposition 3.1. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. $\pi : (TM, g_{BS}) \rightarrow (M_{2k}, \varphi_1, h)$ is totally geodesic if and only if (M_{2k}, φ, g) is locally flat and $I : (M_{2k}, \varphi, g) \rightarrow (M_{2k}, \varphi_1, h)$ is totally geodesic.*

Note that if $\pi : (TM, g_{BS}) \rightarrow (M_{2k}, \varphi_1, h)$ is totally geodesic, then (M_{2k}, φ, g) and (M_{2k}, φ_1, h) are locally flat.

Let g and h be two anti-paraHermitian metrics on M. It is said that h is harmonic with respect to g if

$$
g^{ij}\left({}^h\Gamma^h_{ij} - \Gamma^h_{ij}\right) = 0 \text{ (see [2]).} \qquad (9)
$$

The equations in (8) give the following proposition.

Proposition 3.2. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. $\pi : (TM, g_{BS}) \rightarrow (M_{2k}, \varphi_1, h)$ is harmonic if and only if h is harmonic with respect to g.*

Now, we consider a vector field ξ as $\xi : (M_{2k}, \varphi, g) \rightarrow (TM, g_{BS})$. First, we should prove the proposition below.

Proposition 3.3. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. $\xi : (M_{2k}, \varphi, g) \rightarrow (TM, g_{BS})$ is an isometric immersion if and only if $\nabla \xi = 0$.*

Proof. The following equation holds

$$
\xi_{*,p} X = \left({}^H X + {}^V(\nabla_X \xi)\right)_{\xi(p)}, \quad \forall p \in M. \qquad (10)
$$

If we suppose

$$
\begin{aligned}
\overset{1}{g}_p(X, Y) &= g_{BS_{\xi(p)}}(\xi_{*,p} X, \xi_{*,p} Y) = g_p(X, Y) + g_p(\nabla_X \xi, \nabla_Y \xi) \\
&\quad + \delta^2 g_p(\nabla_X \xi, \varphi \xi) g_p(\nabla_Y \xi, \varphi \xi),
\end{aligned}
$$

then ξ is an isometric immersion if and only if $\overset{1}{g} = g$. It is obvious that $\overset{1}{g} = g$ if and only if $\nabla_X \xi = 0$ for all $X \in \chi(M)$, that is, $\nabla \xi = 0$. $\qquad \square$

By using the equations (4) and (10), we get the proposition below.

Proposition 3.4. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. The second fundamental form $\beta(\xi)$ and the tension field $\tau(\xi)$ of the map $\xi : (M_{2k}, \varphi, g) \rightarrow (TM, g_{BS})$ are given by*

$$
\begin{aligned}
\beta\left(\xi\right)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= -\frac{1}{2}\left(\left(\nabla_j \xi^k\right) R^h_{.ikm} \xi^m + (\nabla_i \xi^l) R^h_{.jlm} \xi^m\right) E_h & (11)\\
&\quad + \left(-\frac{1}{2} R^h_{ijm} \xi^m + \nabla_i \nabla_j \xi^h + (\nabla_i \xi^m)(\nabla_j \xi^n) A^h_{mn}\right) E_{\bar{h}},
\end{aligned}
$$

$$\tau(\xi) = \left(g^{ij}(\nabla_j \xi^k) R^h_{.ikm} \xi^m \right) E_h \tag{12}$$
$$+ \left(g^{ij}(\nabla_i \nabla_j \xi^h) + g^{ij}(\nabla_i \xi^m)(\nabla_j \xi^n) A^h_{mn} \right) E_{\bar{h}},$$

where $A^h_{mn} = \frac{\delta^2}{1+\delta^2} \varphi^k_n \varphi^h_0 g_{mk}$.

The equation (12) gives the following theorem.

Theorem 3.2. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. The map $\xi : (M_{2k}, \varphi, g) \to (TM, g_{BS})$ is harmonic if and only if*

$$g^{ij}(\nabla_j \xi^k) R^h_{.ikm} \xi^m = 0, \qquad g^{ij}(\nabla_i \nabla_j \xi^h) + g^{ij}(\nabla_i \xi^m)(\nabla_j \xi^n) A^h_{mn} = 0. \tag{13}$$

As a direct consequence of Proposition 3.3 and (11), we obtain the theorem below.

Theorem 3.3. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. If $\xi : (M_{2k}, \varphi, g) \to (TM, g_{BS})$ is isometric immersion, then ξ is totally geodesic. Furthermore, ξ is harmonic.*

Remark 3.1. *ξ is parallel if and only if ξ is a minimal isometric immersion.*

Now, we investigate the harmonicity of the Berger type deformed Sasaki metric g_{BS} and the Sasaki metric g_S with respect to each other. By using the Christoffel symbols of these metrics, we state the following two propositions (for the Christoffel symbols of the Sasaki metric see [10]).

Proposition 3.5. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. Suppose that $I : (TM, g_{BS}) \to (TM, g_S)$ is the identity map. Then, the following holds:*

1) I cannot be totally geodesic;

2) The tension field $\tau_{g_{BS}}(I)$ of I is given by

$$\tau_{g_{BS}}(I_{TM}) = \left(-\frac{\delta^2 n}{1+\delta^2} \varphi^h_0 + \frac{\delta^4 g_{00} u^h}{(1+\delta^2)(1+\delta^2 g_{00})} \right) E_{\bar{h}}. \tag{14}$$

So, g_S cannot be harmonic with respect to g_{BS}.

Proposition 3.6. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. Suppose that $I : (TM, g_S) \to (TM, g_{BS})$ is the identity map. Then, the tension field $\tau_{g_S}(I)$ of I is given by*

$$\tau_{g_{BS}}(I_{TM}) = \left(\frac{\delta^2 n}{1+\delta^2} \varphi^h_0 \right) E_{\bar{h}}. \tag{15}$$

Finally, we study the conditions for the vanishing of the second fundamental form of the identity map between (TM, g_{BS}) and $(TM, {}^{BS}\overset{m}{\tilde{\nabla}})$, where ${}^{BS}\overset{m}{\tilde{\nabla}}$ is the mean connection of the Schouten-Van Kampen connection associated with the Levi-Civita connection of the Berger type deformed Sasaki metric. The Schouten-Van Kampen connection ${}^{BS}\tilde{\nabla}$ associated with ${}^{BS}\nabla$ is defined by

$${}^{BS}\tilde{\nabla}_{\tilde{X}}\tilde{Y} = V^{BS}\nabla_{\tilde{X}} V\tilde{Y} + H^{BS}\nabla_{\tilde{X}} H\tilde{Y}, \tag{16}$$

where $\tilde{X}$ and $\tilde{Y} \in \chi(TM)$, and V and H are the vertical and horizontal projections. The local components of ${}^{BS}\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ are as follows:

$$\begin{cases} {}^{BS}\tilde{\nabla}_{E_i} E_j = \Gamma^h_{ij} E_h, \\ {}^{BS}\tilde{\nabla}_{E_{\bar{i}}} E_j = {}^{BS}\nabla_{E_{\bar{i}}} E_j, \\ {}^{BS}\tilde{\nabla}_{E_i} E_{\bar{j}} = \Gamma^h_{ij} E_{\bar{h}}, \\ {}^{BS}\tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = {}^{BS}\nabla_{E_{\bar{i}}} E_{\bar{j}}. \end{cases} \tag{17}$$

The mean connection of $^{BS}\tilde{\nabla}$ is $^{BS}\overset{m}{\tilde{\nabla}} = {}^{BS}\tilde{\nabla} - \frac{1}{2}T$, where T is the torsion tensor field of $^{BS}\tilde{\nabla}$. For the local components of $^{BS}\overset{m}{\tilde{\nabla}}$, we find

$$\begin{cases} {}^{BS}\overset{m}{\tilde{\nabla}}_{E_i}E_j = {}^{BS}\tilde{\nabla}_{E_i}E_j, \\ {}^{BS}\overset{m}{\tilde{\nabla}}_{E_{\bar{i}}}E_j = \frac{1}{2}{}^{BS}\nabla_{E_i}E_j, \\ {}^{BS}\overset{m}{\tilde{\nabla}}_{E_i}E_{\bar{j}} = {}^{BS}\nabla_{E_i}E_{\bar{j}} - \frac{1}{4}R^h_{.i0j}E_h, \\ {}^{BS}\overset{m}{\tilde{\nabla}}_{E_{\bar{i}}}E_{\bar{j}} = {}^{BS}\nabla_{E_{\bar{i}}}E_{\bar{j}}. \end{cases} \tag{18}$$

It is clear that $^{BS}\overset{m}{\tilde{\nabla}}$ is a torsion-free connection. Moreover, note that $^{BS}\overset{m}{\tilde{\nabla}} = {}^{BS}\tilde{\nabla}$ if and only if $R = 0$. The following proposition ends the paper.

Proposition 3.7. *Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric. The second fundamental form of the map $I : (TM, g_{BS}) \rightarrow (TM, {}^{BS}\overset{m}{\tilde{\nabla}})$ satisfies $\beta(I)(E_i, E_j) = \beta(I)(E_{\bar{i}}, E_{\bar{j}}) = 0$, $\beta(I)(E_{\bar{i}}, E_j) = \frac{1}{4}R^h_{.j0i}E_h$. Hence $\tau(I) = \text{trace}\beta(I) = 0$. Moreover if $R \neq 0$ then $\beta(I) \neq 0$.*

4. Conclusions

To investigate the harmonicity of maps between Riemannian manifolds that are naturally constructed from one to another is particularly interesting. An example is the tangent bundle TM on a Riemannian manifold (M, g), equipped with the Sasaki metric g_S. In [5], it was proved that the only vector fields V defining harmonic maps from a compact Riemannian manifold (M, g) to (TM, g_S) are the parallel vector fields. Ishihara [4] independently obtained the same result and moreover gave an explicit expression of the tension field associated to a vector field. Also note that Oniciuc [6] proved the same result when TM is equipped with the Cheeger-Gromoll metric. Our paper contains some results about the harmonicity of the tangent bundle projection $\pi : TM \rightarrow M$ and, on the other direction, of a vector field $\xi : M \rightarrow TM$, both considered as maps between two distinguished Riemannian manifolds: a given almost anti-paraHermitian manifold (M_{2k}, φ, g) and its tangent bundle (TM, g_{BS}), endowed with a Berger type deformed Sasaki metric. The last results are concerning the Schouten-Van Kampen connection.

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