ANALYSIS OF NATURAL FREQUENCIES OF A CRACKED VISCOELASTIC EULER-BERNOULLI BEAM BASED ON EQUIVALENT VISCOELASTIC SPRING MODELS

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In order to investigate effects of crack location or crack depth on the natural frequencies, the corresponding equations based on the finite element method and the approximate analytical method are presented by utilizing the principle of virtual work and compatibility conditions at the crack location, respectively. By numerical examples, the effectiveness and applicability of the two different methods are compared with those of the exact analytical method (EAM).

Keywords: viscoelastic, crack effect, natural frequency.

1. Introduction

In recent years, the research of crack effect and crack identification has attracted considerable attention due to the influences of the external loads, environment and self-defects [1]. Herein, to study the effects of cracks or defects on the vibration properties of the cracked viscoelastic beam structures, some papers are cited. By the Galerkin method and multiple scales method, Younesian et al. [2] analyzed the frequency responses of a cracked beam rested on a nonlinear viscoelastic foundation. Utilizing Fourier transform and regarding the crack as a massless rotation spring, Sarvestan et al. [3] presented a spectral finite element model for vibration analysis of a cracked viscoelastic beam. With the standard linear solid constitutive equation, Fu and Yang [4] presented the exact analytical method (EAM) to investigate the vibration properties of a viscoelastic Euler-Bernoulli cracked beam. However, to investigate effects of size or location of cracks on the natural frequencies, a transcendental equation must be solved numerically.

Based on the principle of virtual work and the compatibility conditions at the crack location, this paper extends the finite element method (FEM) and approximate analytical method (AAM) to overcome the weakness of solving eigenvalue problem, respectively. Then, by numerical examples, the accuracy and applicability are compared with those of the exact analytical method (EAM).

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2. **Theoretical model**

According to the constitutive equation of standard linear solid model[5], the relaxation modulus $Y(t)$ defined in time domain is given as

$$Y(t)=q_0+\left(\frac{q_1}{p_1}-q_0\right)\mathrm{e}^{-\frac{t}{p_1}},\quad p_1=\frac{\eta_2}{E_1+E_2},\quad q_0=\frac{E_1E_2}{E_1+E_2},\quad q_1=\frac{E_1\eta_2}{E_1+E_2}. \tag{1}$$

Where E_1 and E_2 are the elastic modulus of the elastic elements, η_2 is the viscous coefficient of a viscous element, ν is the Poisson's ratio, and *t* is the time.

There is a viscoelastic rectangular beam of length *L*, height *h* and width *b* in the coordinate axis *x*, *y*, and *z*, respectively. Let us consider that $w(x,t)$ and $\varphi(x,t)$ are the transverse deflection of the axial line and rotation angle of the beam cross section *A*. According to the hypothesis of the Euler-Bernoulli beam theory, the axial normal strain, rotation angle, and normal stress of the cross section are given as

$$\varepsilon(x,t)=-y\frac{\partial\varphi(x,t)}{\partial x},\quad \varphi(x,t)=\partial w(x,t)/\partial x,\quad \sigma(x,t)=Y(0)\varepsilon(x,t)+\dot{Y}(t)*\varepsilon(x,t). \tag{2}$$

Where $\dot{Y}(t)$ is the 1st derivative of $Y(t)$ with respect to the time *t*, the asterisk * denotes the convolution, i.e. $f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$.

And the bending moment $M(x,t)$ of the intact beam is

$$M(x,t)=-I\left[Y(0)\frac{\partial\varphi(x,t)}{\partial x}+\dot{Y}(t)*\frac{\partial\varphi(x,t)}{\partial x}\right]. \tag{3}$$

Where *I* is the moment of inertia of the neutral axis, and $I=\iint_A y^2\mathrm{d}y\mathrm{d}z$.

Supposing that the transverse crack $j\ (j=1,2,\cdots,N)$ is always open, which means the crack can be equivalent as a massless viscoelastic torsion spring [1]. Let us denote the bending moment and equivalent viscoelastic torsion spring of the crack *j* with crack depth d_j at $x=x_j$ by $M_j(t)$ and $k_j(t)$, respectively, the rotation angle $\Delta_j(t)$ of the equivalent torsion spring can be expressed as

$$M_j(t)=-\left[k_j(0)\Delta_j(t)+\dot{k}_j(t)*\Delta_j(t)\right]. \tag{4}$$

Based on the equation of the rotation angle $\phi(x,t)$ for a cracked beam [4], the bending moment in time domain and Laplace domain are given as, respectively.

$$M(x,t)=-\left[(EI)_{\mathrm{e}}(x,0)\frac{\partial\phi(x,t)}{\partial x}+\dot{(EI)}_{\mathrm{e}}(x,t)*\frac{\partial\phi(x,t)}{\partial x}\right],\quad \bar{M}(x,s)=-s(\overline{EI})_{\mathrm{e}}(x,s)\frac{\partial\bar{\phi}(x,s)}{\partial x}. \tag{5}$$

Where the superscript "—" denotes the Laplace transform of the function with respect to the time *t*, and *s* is the Laplace transform parameter.

Utilizing the equation of the rotation angle and Dirac delta function $\delta(x)$ [4], the equivalent bending stiffness of the viscoelastic cracked beam in Laplace domain is written as

$$\frac{1}{(\overline{EI})_{\mathrm{e}}(x,s)}=\frac{1}{\overline{Y}(s)I}+\sum_{j=1}^{N}\frac{1}{\overline{k}_j(s)}\delta(x-x_j). \tag{6}$$

3. Methods for vibration of a viscoelastic cracked beam

3.1. Exact analytical method

Based on the separation of variables method, an exact analytical method is presented to analyze the viscoelastic cracked beam with open cracks in reference [4]. Below is a brief progress as follows.

The equivalent stiffness of the crack j ($j=1,\cdots,N$) in time domain and Laplace domain are given as, respectively,

$$k_j(t)=\mu_j IY(t),\quad \overline{k}_j(s)=\mu_j I\overline{Y}(s). \tag{7}$$

Where the parameter $\mu_j=(0.9/h)\left[(d_j/h)-1\right]^2\Big/\left\{(d_j/h)\left[2-(d_j/h)\right]\right\}$.

The free vibration equation of the Euler-Bernoulli beam is

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2}-\frac{\partial^2 M(x,t)}{\partial x^2}=0. \tag{8}$$

Where ρ is the density of the beam.

Introduce the following dimensionless variables and parameters

$$w^*=\frac{w}{L},\; m^*=\frac{M}{E_1L^3},\; \xi=\frac{x}{L},\; \xi_j=\frac{x_j}{L},\; \mu_j^*=\mu_jL,\; I^*=\frac{I}{L^4},\; A^*=\frac{A}{L^2},\; \rho^*=\frac{\rho L^2}{E_1T^2},\; t^*=\frac{t}{T}. \tag{9}$$

Based on the separation of variables method [6], the vibration solutions can be assumed as

$$w^*(\xi,t^*)=W^*(\xi)\mathrm{e}^{\mathrm{i}\omega t^*},\; m^*(\xi,t^*)=M^*(\xi)\mathrm{e}^{\mathrm{i}\omega t^*}. \tag{10}$$

Where $W^*(\xi)$ and $M^*(\xi)$ are the dimensionless mode functions of the transverse displacement and bending moment for the cracked beam, $\mathrm{i}=\sqrt{-1}$, ω is the complex eigenfrequency, and the real part and imaginary part of ω are the natural frequency and decrement coefficient [7,8], respectively.

Then, the dimensionless mode functions of the bending moment and shearing force can be derived. With the corresponding boundary conditions, the set of linear equations is given as

$$[\boldsymbol{A}]\{\boldsymbol{C}\}=\boldsymbol{0}. \tag{11}$$

Where $[\boldsymbol{A}]$ is a 4×4 coefficient vector, and $\{\boldsymbol{C}\}=\{C_1,C_2,C_3,C_4\}^{\mathrm{T}}$.

If there exists a nonzero solution of $\{\boldsymbol{C}\}$, the necessary and sufficient condition is stated that the determinant of the coefficients vector is zero, which is a transcendental equation. Here is the basic process of the present exact analytical method (EAM) [4]. By solving the equation, the complex eigenfrequency ω can be obtained with the different boundary conditions.

3.2. Finite element method

To analyze free vibration of the viscoelastic beam, Hamilton's principle [8] and Newton's second law [9] had been employed. On this basis, by regarding the crack as a massless rotation spring and considering the additional virtual work of the bending moment at the crack location [10], the principle of virtual work for the free vibration of the viscoelastic cracked beam is presented as

$$\int_0^L \left\{ \rho A \frac{\partial^2 w(x,t)}{\partial t^2} \Theta w(x,t) - M(x,t) \Theta \left[\frac{\partial^2 w(x,t)}{\partial x^2} \right] \right\} \mathrm{d}x - \sum_{j=1}^{N} M_j \Theta \Delta_j = 0. \tag{12}$$

Where Θ is the variational operator.

The vibration solutions [8] can be expressed as

$$w(x,t) = W(x)\mathrm{e}^{\mathrm{i}\omega t}, \quad \varphi(x,t) = \frac{\mathrm{d}w(x,t)}{\mathrm{d}x} = \Phi(x)\mathrm{e}^{\mathrm{i}\omega t}, \quad M(x,t) = M(x)\mathrm{e}^{\mathrm{i}\omega t}. \tag{13}$$

Where $W(x)$, $\Phi(x)$, and $M(x)$ are the mode functions of deflection, rotation angle and bending moment, respectively.

By combining Eqs. (3) , (13) and the 1st equation of Eq.(1), we have

$$M(x) = -I \frac{q_0 + \mathrm{i}\omega q_1}{1 + \mathrm{i}\omega p_1} \frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2}, \quad \Phi(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{14}$$

When $E_1 \to \infty$, the parameter $p_1 = \eta_2/(E_1 + E_2)$ is reduced to zero, the 1st equation of Eq. (14) is degenerated into the mode function of bending moment for the Kelvin-Voigt beam [11] as follows

$$M(x) = -I\left(q_0 + \mathrm{i}\omega q_1\right) \frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2}. \tag{15}$$

Suppose that the crack j is located at the end point of beam element, i.e. $x = x_j$ in fig. 1, the relative rotation angle of the equivalent rotation spring model at the crack location by using 2nd equation of Eq. (13) is given as

$$\Delta(x_j, t) = \varphi_{j\mathrm{R}}(t) - \varphi_{j\mathrm{L}}(t) = \Delta(x_j)\mathrm{e}^{\mathrm{i}\omega t}. \tag{16}$$

Where $\varphi_{j\mathrm{L}}(t)$ and $\varphi_{j\mathrm{R}}(t)$ are the rotation angles of two adjacent beam elements at the both sides of the crack location $x = x_j$, see fig. 1.

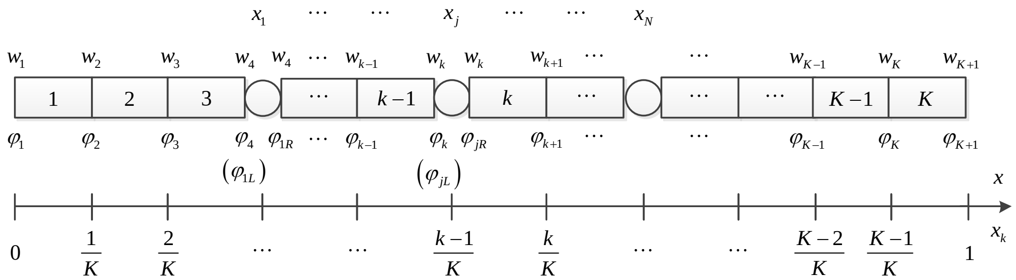


Fig. 1. Finite element mesh of the viscoelastic cracked beam

Utilizing the 2nd equation of Eq. (13) and Eq. (16), we have

$$\Delta(x_j) = \Phi_{j\mathrm{R}}(x) - \Phi_{j\mathrm{L}}(x). \tag{17}$$

By combining Eqs. (1), (7) (16) and the Laplace transform of Eq. (3), the bending moment at the crack location $x = x_j$ is given as

$$M(x_j, t) = -\mu_j I \Delta(x_j)\left[Y(0)\mathrm{e}^{\mathrm{i}\omega t} + \dot{Y}(t) * \mathrm{e}^{\mathrm{i}\omega t}\right] = -\mu_j I \frac{q_0 + \mathrm{i}\omega q_1}{1 + \mathrm{i}\omega p_1}\Delta(x_j)\mathrm{e}^{\mathrm{i}\omega t}. \tag{18}$$

Substitution of Eqs. (13), (14), (16) and (18) into Eq. (12), one obtain

$$\int_0^L \left[(\mathrm{i}\omega)^2 \frac{1+\mathrm{i}\omega p_1}{q_0 + \mathrm{i}\omega q_1} E_1 \rho A W(x) \Theta W(x) + E_1 I \frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2} \Theta \frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2}\right]\mathrm{d}x + \sum_{j=1}^{N} E_1 I \mu_j \Delta(x_j) \Theta\left[\Delta(x_j)\right] = 0. \tag{19}$$

The beam is divided into K units of the finite element along the axis (see fig. 1), there have $L_e = L/K$ and $x_k = (k-1)/K$ ($k = 1, \cdots, K+1$). With the interpolation function, the mode function of the k-th beam element is given as

$$W(x) = S p_{\mathrm{e}}^{(k)}. \tag{20}$$

Where S is the shape function vector composed with a series of two-node Hermite's interpolation function [12], $p_{\mathrm{e}}^{(k)}$ is the nodal displacement vector of the k-th beam element, and

$$S = \left[S_1 \;\; S_2 \;\; S_3 \;\; S_3\right], \quad p_{\mathrm{e}}^{(k)} = \left[W_k \;\; \Phi_k \;\; W_{k+1} \;\; \Phi_{k+1}\right]^{\mathrm{T}}. \tag{21}$$

$$\begin{cases} S_1(x) = 1 - 3\beta^2 + 2\beta^3, \quad S_2(x) = \left(\beta - 2\beta^2 + \beta^3\right)L_{\mathrm{e}}, \\ S_3(x) = 3\beta^2 - 2\beta^3, \qquad S_4(x) = \left(\beta^3 - \beta^2\right)L_{\mathrm{e}}, \qquad \beta = x/L_{\mathrm{e}}. \end{cases} \tag{22}$$

Utilizing the 2nd derivative of Eq. (20) with respect to the variable x, one obtain

$$\frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2} = B p_{\mathrm{e}}^{(k)}, \quad B = \frac{\mathrm{d}^2 S}{\mathrm{d}x^2}. \tag{23}$$

Suppose that the crack is regarded as a massless torsion spring, and the length of crack element is zero [10]. Then, Eq. (17) can be rewritten as

$$\Delta(x_j) = S_{\mathrm{c}} c_{\mathrm{e}}^{(j)}, \;\; S_{\mathrm{c}} = [-1 \;\; 1], \;\; c_{\mathrm{e}}^{(j)} = \left[\Phi_{j\mathrm{L}} \;\; \Phi_{j\mathrm{R}}\right]^{\mathrm{T}}. \tag{24}$$

Where $c_{\mathrm{e}}^{(j)}$ is the nodal displacement vector with two degrees of freedom, S_{c} is the matrix of shape function for the j-th crack element.

By substituting Eqs. (20), (23) and (24) into Eq. (19), we have

$$\sum_{k=1}^{K} \Theta\left(p_{\mathrm{e}}^{(k)\mathrm{T}}\right)\left[(\mathrm{i}\omega)^2 \frac{1+\mathrm{i}\omega p_1}{q_0 + \mathrm{i}\omega q_1} E_1 M_{\mathrm{e}}^{(k)} + K_{\mathrm{e}}^{(k)}\right] p_{\mathrm{e}}^{(k)} + \sum_{j=1}^{N} \Theta\left(c_{\mathrm{e}}^{(j)\mathrm{T}}\right) K_{\mathrm{c,e}}^{(j)} c_{\mathrm{e}}^{(j)} = 0. \tag{25}$$

Where the superscripts k and j denote the k-th beam element and j-th crack element, respectively, and

$$M_{\mathrm{e}} = \int_0^{L_{\mathrm{e}}} \rho A S^{\mathrm{T}} S \mathrm{d}x, \quad K_{\mathrm{e}} = \int_0^{L_{\mathrm{e}}} E_1 I B^{\mathrm{T}} B \mathrm{d}x, \quad K_{\mathrm{c,e}} = E_1 I \mu_j S_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}}. \tag{26}$$

Where $M_{\mathrm{e}}^{(k)}$ $(k=1,\cdots,K)$ and $K_{\mathrm{e}}^{(k)}$ are the mass matrix and stiffness matrix of the *k*-th beam element, respectively. $K_{\mathrm{c,e}}^{(j)}(j=1,\cdots,N)$ is the stiffness matrix of the *j*-th crack element.

When the crack location $x=x_j$ $(j=1,2,\cdots,N)$ is equal to the element nodal coordinate $x_k=(k-1)/K$ $(k=2,\cdots,K)$, it means that the *j*-th equivalent torsion spring is connected by the *k*-th beam element and $(k-1)$-th beam element, and the nodal vector of the $(k-1)$-th beam element can be expressed as

$$p_{\mathrm{e}}^{(k-1)}=\left[W_{k-1}\ \ \Phi_{k-1}\ \ W_k\ \ \Phi_k\right]^{\mathrm{T}}. \tag{27}$$

In addition, the nodal vector of the *k*-th beam element Eq. (21) can be rewritten as

$$p_{\mathrm{e}}^{(k)}=\left[W_k\ \ \Phi_{j\mathrm{R}}\ \ W_{k+1}\ \ \Phi_{k+1}\right]^{\mathrm{T}}. \tag{28}$$

Where $\Phi_{j\mathrm{R}}$ is a new independent variable. Considering of the crack effect, two degrees of freedom for the *j*-th crack element in Eq. (24) are presented as $\varphi_{j\mathrm{L}}=\varphi_k$ and $\varphi_{j\mathrm{R}}\neq\varphi_k$, respectively.

Then, Eq. (25) is rewritten as

$$\left[(\mathrm{i}\omega)^2\frac{1+\mathrm{i}\omega p_1}{q_0+\mathrm{i}\omega q_1}E_1M_{\mathrm{p}}+K_{\mathrm{p}}\right]\{\boldsymbol{p}\}=\mathbf{0}. \tag{29}$$

Where

$$M_{\mathrm{p}}\{\boldsymbol{p}\}=\sum_{k=1}^{K}M_{\mathrm{e}}^{(k)}p_{\mathrm{e}}^{(k)},\ \ K_{\mathrm{p}}\{\boldsymbol{p}\}=\sum_{k=1}^{K}K_{\mathrm{e}}^{(k)}p_{\mathrm{e}}^{(k)}+\sum_{j=1}^{N}K_{\mathrm{c,e}}^{(j)}c_{\mathrm{e}}^{(j)}. \tag{30}$$

As the number of generalized node displacements is $2(K+1)+N$, the vector $\{\boldsymbol{p}\}$ is expressed as

$$\{\boldsymbol{p}\}=\left[W_1\ \ \Phi_1\ \ W_2\ \ \Phi_2\cdots W_{k-1}\ \ \Phi_{k-1}\ \ W_k\ \ \Phi_k\ \ \Phi_{j\mathrm{R}}\ \ W_{k+1}\ \ \Phi_{k+1}\cdots W_N\ \ \Phi_N\ \ W_{N+1}\ \ \Phi_{N+1}\right]^{\mathrm{T}}. \tag{31}$$

Then, the frequency equation is presented as follows

$$\det\left[(\mathrm{i}\omega)^2\frac{1+\mathrm{i}\omega p_1}{q_0+\mathrm{i}\omega q_1}E_1M_{\mathrm{p}}+K_{\mathrm{p}}\right]=0. \tag{32}$$

Applying the dimensionless form of Eq. (32) and utilizing Matlab programs, the dimensionless eigenfrequency of the viscoelastic cracked beam can be obtained with the different boundary conditions.

3.3 Approximate analytical method

For a simple-supported elastic beam with open cracks in reference [13], it was stated that the mode function of the elastic cracked beam was composed with the mode function of an undamaged beam and a polynomial function which showed the effect of cracks. By supposing that the maximum potential energy of the beam was equal to the maximum kinetic energy for an arbitrary *k*-th mode, the approximate analytical expression of the k-th natural frequency for a simple-supported elastic beam with cracks were derived by employing the compatibility conditions at the crack locations. In addition, Manevich and Kołakowski [9] claimed that for a simple-supported beam, the mode function of the viscoelastic beam was the same to that of the elastic beam. On this basis, the approximate analytical expression of the complex eigenfrequency for a simple-supported viscoelastic cracked beam is presented in this subsection.

At first, we take the viscoelastic beam with a single crack for an example. Suppose that there is a transverse open crack with depth d_1 at $x = x_1$, the cracked beam is considered as a massless viscoelastic spring connected by two intact viscoelastic sub-beams. Let $w^{(k)}(x,t)$ is the transverse deflection of the k-th sub-beam, and here $0 \le x \le x_1$ and $x_1 \le x \le L$ correspond to $k=1$ and $k=2$, respectively.

Substituting Eqs. (6), (7) and the Laplace equation of Eq. (1) into the 2nd equation of Eq. (5), using the inverse Laplace transform and ignoring the effect of the crack, the equations of motion of the k-th intact sub-beam are

$$\left(1+p_1\frac{\partial}{\partial t}\right)M^{(k)}(x,t) = -I\left(q_0+q_1\frac{\partial}{\partial t}\right)\frac{\partial^2 w^{(k)}(x,t)}{\partial x^2}. \quad (k=1,2) \tag{33}$$

$$\left(1+p_1\frac{\partial}{\partial t}\right)\rho A\frac{\partial^2 w^{(k)}(x,t)}{\partial t^2} = -I\left(q_0+q_1\frac{\partial}{\partial t}\right)\frac{\partial^4 w^{(k)}(x,t)}{\partial x^4}. \quad (k=1,2) \tag{34}$$

The boundary conditions of the simple-supported beam are

$$w^{(1)}(0,t)=0,\ \ M^{(1)}(0,t)=0,\ \ w^{(2)}(L,t)=0,\ \ M^{(2)}(L,t)=0. \tag{35}$$

And the compatibility conditions [14] at $x = x_1$ are given as follows

$$\begin{cases} w^{(1)}(x,t)=w^{(2)}(x,t),\ \ M^{(1)}(x,t)=M^{(2)}(x,t), \\ \dfrac{\partial M^{(1)}(x,t)}{\partial x}=\dfrac{\partial M^{(2)}(x,t)}{\partial x},\ \ \dfrac{\partial w^{(2)}(x,t)}{\partial x}-\dfrac{\partial w^{(1)}(x,t)}{\partial x}=\Delta(x,t). \end{cases} \tag{36}$$

Where $\Delta(x,t)$ is the relative rotation angle due to the crack effect and the Laplace transform of $\Delta(x,t)$ is defined by Eq. (4).

Combining Eqs. (1), (4), (7) and the 4th equation of Eq. (36) with the inverse Laplace transform, and then, combining Eq. (34), one obtain

$$\left(q_0+q_1\frac{\partial}{\partial t}\right)\left[\frac{\partial w^{(2)}(x_1,t)}{\partial x}-\frac{\partial w^{(1)}(x_1,t)}{\partial x}-\frac{1}{\mu_1}\frac{\partial^2 w^{(2)}(x_1,t)}{\partial x^2}\right]=0. \tag{37}$$

Similar to Eq. (13), the vibration solutions can be expressed as

$$w^{(k)}(x,t)=W^{(k)}(x)\mathrm{e}^{\mathrm{i}\omega t},\quad M^{(k)}(x,t)=m^{(k)}(x)\mathrm{e}^{\mathrm{i}\omega t}. \quad (k=1,2) \tag{38}$$

Substituting Eq. (38) into Eqs. (33) and (37), respectively

$$m^{(k)}(x)=-I\frac{q_0+\mathrm{i}\omega q_1}{1+\mathrm{i}\omega p_1}\frac{\mathrm{d}^2 W^{(k)}(x)}{\mathrm{d}x^2}. \quad (k=1,2) \tag{39}$$

$$\frac{\mathrm{d}W^{(2)}(x_1)}{\mathrm{d}x}-\frac{\mathrm{d}W^{(1)}(x_1)}{\mathrm{d}x}=\frac{1}{\mu_1}\frac{\mathrm{d}^2W^{(2)}(x_1)}{\mathrm{d}x^2}. \tag{40}$$

By extending the method of an elastic beam used by Bakhtiari-Nejad et al. [13], the n-th mode functions of the viscoelastic beam with a single crack are given as

$$\begin{cases} W_n^{(1)}(x)=\Omega_n\left[W_{\mathrm{nc},n}(x)+A_{1,n}+B_{1,n}x+C_{1,n}x^2+D_{1,n}x^3\right], & (0\le x\le x_1) \\ W_n^{(2)}(x)=\Omega_n\left[W_{\mathrm{nc},n}(x)+A_{2,n}+B_{2,n}x+C_{2,n}x^2+D_{2,n}x^3\right]. & (x_1\le x\le L) \end{cases} \tag{41}$$

Where $W_{\mathrm{nc},n}(x)$ is the n-th mode function of the intact beam, for a simple-supported beam there has $W_{\mathrm{nc},n}(x)=\sin(n\pi x/L)$ [9,13]. Ω_n is the relevant constant. $A_{k,n}$, $B_{k,n}$, $C_{k,n}$ and $D_{k,n}$ （$k=1,2$）are the undetermined functions.

Substitution of Eqs. (38) and (41) into Eq. (35) , the first three equations of Eq. (36), and Eq.(40), respectively, one obtain

$$\begin{cases} A_{1,n}=0,\ A_{2,n}=x_1\dfrac{1}{\mu_1}\left(\dfrac{n\pi}{L}\right)^2\sin\left(\dfrac{n\pi x_1}{L}\right),\ C_{1,n}=0,\ C_{2,n}=0,\ D_{1,n}=D_{2,n}=0, \\ B_{2,n}=-\dfrac{x_1}{L}\dfrac{1}{\mu_1}\left(\dfrac{n\pi}{L}\right)^2\sin\left(\dfrac{n\pi x_1}{L}\right),\ B_{1,n}=\dfrac{L-x_1}{L}\dfrac{1}{\mu_1}\left(\dfrac{n\pi}{L}\right)^2\sin\left(\dfrac{n\pi x_1}{L}\right). \end{cases} \tag{42}$$

Substitution of Eq. (42) into Eq. (41), the approximate expressions of mode functions for the viscoelastic beam with a single crack are presented as

$$\begin{cases} W_n^{(1)}(x)=\Omega_n\left[\sin\left(\dfrac{n\pi x}{L}\right)+\dfrac{(L-x_1)x}{L}\dfrac{1}{\mu_1}\left(\dfrac{n\pi}{L}\right)^2\sin\left(\dfrac{n\pi x_1}{L}\right)\right], & (0\le x\le x_1) \\ W_n^{(2)}(x)=\Omega_n\left[\sin\left(\dfrac{n\pi x}{L}\right)+\dfrac{(L-x)x_1}{L}\dfrac{1}{\mu_1}\left(\dfrac{n\pi}{L}\right)^2\sin\left(\dfrac{n\pi x_1}{L}\right)\right]. & (x_1\le x\le L) \end{cases} \tag{43}$$

Utilizing the principle of virtual work for a simple-supported viscoelastic beam with an open crack, Eqs. (12), (38), (39) and (43) are combined as follows

$$\begin{aligned} &\rho A(\mathrm{i}\omega)^2(1+\mathrm{i}\omega p_1)\left\{\frac{L}{2}+\frac{x_1^2(L-x_1)^2}{3L}\left[\frac{1}{\mu_1}\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x_1}{L}\right)\right]^2+\frac{2}{\mu_1}\sin^2\left(\frac{n\pi x_1}{L}\right)\right\}+ \\ &I(q_0+\mathrm{i}\omega q_1)\left[\left(\frac{n\pi}{L}\right)^4\frac{L}{2}+\frac{1}{\mu_1}\left(\frac{n\pi}{L}\right)^4\sin^2\left(\frac{n\pi x_1}{L}\right)\right]=0. \end{aligned} \tag{44}$$

Therefore, utilizing Eq. (9), the approximate analytical value of the dimensionless eigenfrequency for the viscoelastic beam with an open crack based on the standard linear solid model can be obtained.

When $d_1\to 0$ or $\mu_1\to\infty$, Eq. (44) is degenerated into the expression for a viscoelastic intact beam with standard linear solid equation as follows

$$\omega^2 = \frac{1}{L^4}\frac{I\left(q_0 + \mathrm{i}\omega q_1\right)}{\rho A\left(1+\mathrm{i}\omega p_1\right)}\left(n\pi\right)^4 . \tag{45}$$

It is found that Eq. (45) corresponds to the expression presented by Lei et al. [15].

Combining Eqs. (9) and (43), the dimensionless mode function of the viscoelastic beam with an open crack is presented as follows

$$W_n^*(\xi) = W_n^{(1)*}(\xi) H\left(\xi_1 - \xi\right) + W_n^{(2)*}(\xi) H\left(\xi - \xi_1\right). \tag{46}$$

Where $H(x)$ is the Heaviside function [4], and

$$\begin{cases} W_n^{(1)*}(\xi) = \Omega_n \left[\sin\left(n\pi\xi\right) + \dfrac{\left(1-\xi_1\right)\xi}{\mu_1^*}\left(n\pi\right)^2 \sin\left(n\pi\xi_1\right)\right], & (0 \le \xi \le \xi_1) \\ W_n^{(2)*}(\xi) = \Omega_n \left[\sin\left(n\pi\xi\right) + \dfrac{\left(1-\xi\right)\xi_1}{\mu_1^*}\left(n\pi\right)^2 \sin\left(n\pi\xi_1\right)\right]. & (\xi_1 \le \xi \le 1) \end{cases} \tag{47}$$

In addition, the similar methodology can be utilized to analyze vibration of the viscoelastic beam with an arbitrary number of cracks.

4. Numerical results and discussion

4.1 Validation of the present methods

Let $E_1 \to \infty$ and $d_1 \to 0$, the present model is degenerated into the Kelvin-Voigt intact model. Suppose the geometric and physical parameters are $L = 1\ \mathrm{m}$, $b = 0.2\ \mathrm{m}$, $h = 0.0015\ \mathrm{m}$, $\rho = 7800\ \mathrm{kg/m^3}$, $E_2 = 2\times10^{11}\ \mathrm{N/m^2}$, $E_1/E_2 = 9999$ and $\eta_2 = 6.8\times10^{-4} E_2$. And the beam is uniformly meshed by 20 finite elements ($K = 20$). The first five eigenfrequencies are shown in table 1. It can be seen that the results of the present methods are in excellent agreement with those of references [4,11].

Table 1

First five eigenfrequencies of the simply-supported Kelvin-Voigt beam

	FEM	AAM	Ref. [4]	Ref. [11]
1st	3.4440+0.0253i	3.4424+0.0253i	3.4439+0.0253i	3.444+0.025i
2nd	13.7702+0.4054i	13.7640+0.4047i	13.7702+0.4054i	13.771+0.405i
3rd	30.9292+2.0524i	30.9147+2.0486i	30.9283+2.0523i	30.930+2.052i
4th	54.7273+6.4876i	54.6993+6.4749i	54.7215+6.4862i	54.724+6.486i
5th	84.6534+15.8437i	84.6031+15.8089i	84.6325+15.8356i	84.636+15.836i

4.2 Analysis of natural frequency of a viscoelastic cracked beam

For a standard linear solid beam under the simple-supported boundary conditions, we suppose that the geometric parameters of the rectangular beam are $L = 1\ \mathrm{m}$, $\rho = 500\ \mathrm{kg/m^3}$ and $L/h = 20$. The material parameters are $E_1 = 14\ \mathrm{GPa}$, $E_2 = 39.68\ \mathrm{GPa}$ and $\eta_2 = 6.9\times10^3\ \mathrm{GPa\cdot h}$.

Considering the crack effects, a simple-supported viscoelastic beam with N symmetrically distributed cracks is considered. For the sake of simplicity, the crack location is $\xi_j = j/\left(N+1\right)$ ($j = 1,\cdots,N$), crack depth is d_j/h, and the real part (natural frequency) and imaginary part (decrement coefficient) of the k-th eigenfrequency ω_k are defined by $\mathrm{Re}(\omega_k)$ and $\mathrm{Im}(\omega_k)$, respectively.

To analyze the natural frequency of a viscoelastic beam constituted by SLS model with a single crack($N=1$), we denote $\mathrm{Re}(\omega_{\mathrm{EAM},n})$, $\mathrm{Re}(\omega_{\mathrm{FEM},n})$ and $\mathrm{Re}(\omega_{\mathrm{AAM},n})$ as the real part of the n-th eigenfrequency based on the results of EAM, FEM and AAM, respectively. Then, the error comparisons between the first three natural frequencies obtained by EAM, FEM and AAM with the crack depth d_1/h and crack location ξ_1 are shown in tables 2~4. Here, the beam is uniformly meshed by 20 finite elements ($K=20$). As can be seen, the errors of the first three natural frequencies obtained by EAM and FEM are extremely small, which indicates that the results of EAM and FEM are in excellent agreement with each other.

However, in tables 3 and 4, as the crack depth increases from $d_1/h=0.2$ to 0.4 and 0.6, the errors between the 2nd natural frequency by EAM and AAM are 11.95% and 38.5%, respectively, while the corresponding error values of 3rd natural frequency increase to be 44.83% and 73.01%. Obviously, the 2nd and 3rd natural frequencies obtained by the approximate analytical method show a significant error. There was a similar conclusion of the elastic cracked beam reported by Bakhtiari-Nejad et al. [13]. The possible reason of the error can be interpreted that the mode function Eq. (43) is the linear correction function. Therefore, by comparing with the results of EAM, the present FEM can provide higher accuracy and applicability for the viscoelastic cracked beam, while the present AAM is only advised to predict 1st natural frequency.

Table 2

Error comparisons between the 1st natural frequencies obtained by EAM, FEM and AAM

	$\left|\mathrm{Re}\left(\omega_{\mathrm{FEM},1}-\omega_{\mathrm{EAM},1}\right)/\mathrm{Re}\left(\omega_{\mathrm{EAM},1}\right)\right|\times 100\%$			$\left|\mathrm{Re}\left(\omega_{\mathrm{AAM},1}-\omega_{\mathrm{EAM},1}\right)/\mathrm{Re}\left(\omega_{\mathrm{EAM},1}\right)\right|\times 100\%$		
ξ_1	d_1/h=0.2	d_1/h=0.4	d_1/h=0.6	d_1/h=0.2	d_1/h=0.4	d_1/h=0.6
0.1	0	0	0.002	0	0..001	0.005
0.2	0.001	0.004	0	0.001	0.007	0.02
0.3	0	0.001	0.004	0	0.005	0.025
0.4	0.001	0	0	0.001	0.002	0.007
0.5	0.003	0	0	0.003	0	0.001

Table 3

Error comparisons between the 2nd natural frequencies obtained by EAM, FEM and AAM

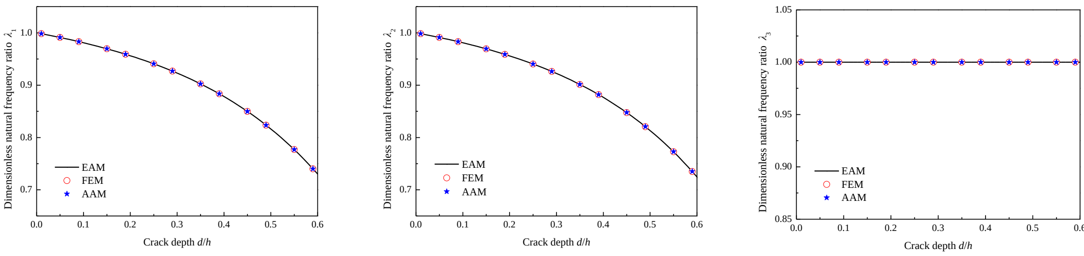
	$\left|\mathrm{Re}\left(\omega_{\mathrm{FEM},2}-\omega_{\mathrm{EAM},2}\right)/\mathrm{Re}\left(\omega_{\mathrm{EAM},2}\right)\right|\times 100\%$			$\left|\mathrm{Re}\left(\omega_{\mathrm{AAM},2}-\omega_{\mathrm{EAM},2}\right)/\mathrm{Re}\left(\omega_{\mathrm{EAM},2}\right)\right|\times 100\%$		
ξ_1	d_1/h=0.2	d_1/h=0.4	d_1/h=0.6	d_1/h=0.2	d_1/h=0.4	d_1/h=0.6
0.1	0.001	0.001	0	0.103	0.93	5.94
0.2	0	0.001	0	0.92	6.80	26.03
0.3	0.001	0.001	0.001	1.73	11.95	38.50
0.4	0	0.001	0	0.98	8.07	34.31
0.5	0.001	0.001	0.001	0	0	0

Table 4

Error comparisons between the 3rd natural frequencies obtained by EAM, FEM and AAM

	$\left\|\mathrm{Re}\left(\omega_{\mathrm{FEM,3}}-\omega_{\mathrm{EAM,3}}\right)/\mathrm{Re}\left(\omega_{\mathrm{EAM,3}}\right)\right\|\times 100\%$			$\left\|\mathrm{Re}\left(\omega_{\mathrm{AAM,3}}-\omega_{\mathrm{EAM,3}}\right)/\mathrm{Re}\left(\omega_{\mathrm{EAM,3}}\right)\right\|\times 100\%$		
ξ_1	d_1/h=0.2	d_1/h=0.4	d_1/h=0.6	d_1/h=0.2	d_1/h=0.4	d_1/h=0.6
0.1	0.003	0.003	0.003	1.15	8.69	32.65
0.2	0.003	0.003	0.003	5.00	27.28	60.13
0.3	0.003	0.004	0.003	1.07	9.19	39.74
0.4	0.003	0.003	0.003	4.60	27.62	63.69
0.5	0.003	0.003	0.003	11.78	44.83	73.01

To consider the effects of crack, we suppose that ω_{0n} and ω_n are the *n*-th eigenfrequency of the viscoelastic intact beam and cracked beam, respectively, then $\lambda_n = \mathrm{Re}(\omega_n)/\mathrm{Re}(\omega_{0n})$ is the *n*-th natural frequency ratio. In the case of a viscoelastic beam with two symmetric cracks, the crack depths are equal to each other. Fig. 2 shows the first three natural frequencies of the cracked beam by EAM, FEM and AAM. In the computation, the beam is uniformly meshed by 21 finite elements. It can be seen clearly that the first three natural frequency ratios of the three methods are in excellent agreement with each other for different crack depth. In Addition, when the cracks are located at the nodes of vibration, i.e. $\xi_1 = 1/3$ and $\xi_2 = 2/3$, the 3rd natural frequency ratio is $\lambda_3 = 1$, which reveals that the 3rd natural frequency is independent of the crack depth, in fig. 2(c).



(a) 1st frequency ratio (b) 2nd frequency ratio (c) 3rd frequency ratio

Fig. 2. Comparisons between the first three frequencies ratio of the simply-supported beam with two symmetric cracks obtained by EAM, FEM and AAM

5. Conclusions

In this paper, the finite element model and approximate analytical expressions to analyze the viscoelastic cracked beam with open cracks are derived to overcome the weakness of solving eigenvalue problem. In numerical computations, the accuracy and applicability of the present methods (FEM, AAM) are compared with those of the exact analytical method (EAM), and the effects of the crack location, crack depth, and crack number on the vibration properties of the viscoelastic cracked beam are demonstrated. Some conclusions arising from the numerical results can be summarized as follows: (1) Results of the present finite element method are in excellent agreement with those of the exact analytical method, while the approximate analytical method is advised to predict 1st natural frequency with some acceptable deviations. (2) At the nodes of vibration, the natural frequency is independent of the crack depth.

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