U.P.B. Sci. Bull., Series A, Vol. 85, Iss. 4, 2023 ISSN 1223-7027

GENOCCHI OPERATIONAL MATRIX METHOD AND THEIR APPLICATIONS FOR SOLVING FRACTIONAL WEAKLY SINGULAR TWO-DIMENSIONAL PARTIAL VOLTERRA INTEGRAL EQUATION

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The present paper introduces numerical techniques in order to approximate the solution of fractional weakly singular partial Volterra integral equation in two dimensions. For this purpose, an operational matrix was developed on the basis of the Genocchi polynomials for the purpose of fractional integration. By employing the 2-D Genocchi polynomials and the fractional derivatives of the unknown functions, the 2-D fractional singular integro-differential equations were converted into a nonlinear system. One may present an approximate solution to the original problem by solving the same system. In the last stage, the effectiveness and validity of the technique were demonstrated by providing a number of examples.

Keywords: Two-dimensional Genocchi polynomials (2D-GPs), Fractional integral equation, Fractional derivative, Operational matrix.

1. Introduction

Even though fractional calculus and the foundation of its theory date back over three centuries, it has been in recent times that academics and scholars have paid attention to this theory. A great number of scientific discoveries and real-world phenomena can be modeled directly via fractional models [1, 2]. In addition, numerous scientific fields, such as physics, economics, and even biology, are faced with dilemmas associated with Volterra integral equations. [3, 4]. In addition, fractional integral differential equations play key roles in a variety of scientific disciplines. There are still numerous numerical techniques that are applicable in order to approximate solutions to such problems, even if obtaining exact solutions to such problems is too complicated or absent. The techniques of applying numerical approaches were utilized primarily for the purpose of fractional integro-differential equations. In scientific investigations, one may use integro-differential equations of fractional order in order to model different phenomena, including electromagnetics, signal processing, viscoelasticity, economics, etc. [6, 7, 10].

Piecewise functions, in particular orthogonal functions, are employed in the approximation theory in order to reach an approximately exact solution of equations characterized by high precision. The use of operational matrices may simplify complicated problems and decrease the required calculation time. As a result, it is an efficient technique in order to reach a solution characterized by lower time and costs. Such as CAS wavelets [23], B-spline [29], Tau method [21], Legendre scaling function [25], Triangular functions [33, 27], Laguerre series [30], Hybrid Bernstein Block-Pulse functions [9], Bernstein operational matrix [31], Jacobi operational matrix [32], Homotopy analysis method [22], Boubaker Hybrid Functions [26], Collocation method [24], Wavelets method [12], Boubaker polynomials [28], Block-pulse

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functions (BPFs) [8, 11].
A great number of engineering and scientific fields employ Genocchi polynomials to solve equations and approximate them. For example, E. Hashemizadeh et al. utilized Genocchi polynomials in order to solve Volterra integral equations weakly singular kernels [20]. A. Kanwal et al. have employed the Genocchi polynomials technique in order to solve the diffusion wave equation [19]. By employing Genocchi polynomials, Singh and Saha Ray tried to approximate the stochastic Itô-Volterra integral equation [18]. In addition, Genocchi polynomials have been used to solve the fractional Rosenau-Hyman equation [17]. By employing two-dimensional Genocchi polynomials, Dehestani discovered the approximate solution of partial integro-differential equations [16] and utilized Genocchi wavelet in order to discover the solution of fractional differential equations with delay [15]. In addition, Isah studied the Genocchi wavelet operational matrix for the purpose of fractional differential equations [13]. Pana et al. used Genocchi polynomials to find the approximate solution of nonlinear type of fractional differential equations [14]. A great number of earlier studies have employed the concept of the singular integro-differential equation, as well as the fractional class of the same equations. A number of orthogonal functions have been used to solve the fractional singular integro-differential equations. For instance, Nemati et al. presented a fast algorithm on the basis of the Chebyshev polynomials in order to seek the approximate solution for the FIDEs featuring weakly singular kernels [35]. Singh et al. could find the approximate solution for the FSIDEs featuring operational matrix technique [36]. By employing the two-dimensional operational technique, Behera and Saha Ray could solve weakly singular partial integro-differential equations in 2020 [37]. In addition, Ghanbari et al. reached the numerical solution to a particular set of FIDEs featuring weakly singular kernels [38].
In our current work, the fractional weakly singular 2-D partial Volterra integral equation (FWS2DPVIEs) is considered as the form:

$$u_{\varsigma,\iota}(\varsigma,\iota)+u_{\iota}(\varsigma,\iota)=u(\varsigma,\iota)+g(\varsigma,\iota) + \int_0^{\varsigma}\int_0^{\iota}\frac{H(\varsigma,\iota,s,y,u(s,y))}{(\varsigma-s)^{1-\theta_1}(\iota-y)^{1-\theta_2}}dyds + \int_0^{\varsigma}\int_0^{\iota}H(\varsigma,\iota,s,y,u(s,y))dyds,$$

with the following boundary condition

$$u(\varsigma,0) = u_0(\varsigma), \tag{1.1}$$

$$u(0,\iota) = u_0(\iota), \tag{1.2}$$

where $0<\theta_1<1, 0<\theta_2<1$, and $u(\varsigma,\iota)$ is an unknown function which should be determined. The known functions $H(\varsigma,\iota,s,y,u(s,y))$, $g(\varsigma,\iota)$, $u_0(\varsigma)$ and $u_0(\iota)$ are defined on interval $\Omega=[0,1]\times[0,1]$.

The rest of the present paper has been organized as follows. A number of basic characteristics of fractional calculus will be introduced in the second section of the present paper. Genocchi polynomials and function approximation and error analysis will be presented in section 3. In section 4, the operational matrix of fractional differentiation and integration has been presented and introduced. The numerical simulation will be carried out on the basis of an operational matrix suggested in section 5. The technique will be presented by providing a number of schematic examples in section 6, and for the purpose of better interpretation, the numerical results will be compared with those of the earlier papers. A short conclusion will be finally presented.

2. **Preliminaries**

The basic definitions and specifications of the fractional derivative and integral have been presented in the following.

Definition 2.1. *One may consider a real function $f(\varsigma)$, $\varsigma > 0$ within the C_μ space, $\mu \in R$, if a real number $p > \mu$ exists in such a way that $f(x) = \varsigma^p f_1(x)$, in which $f_1 \in C[0, \infty)$. It is evident that $C_\mu \in C_\beta$ if $\beta < \mu$.*

Definition 2.2. *One may consider a function $f(\varsigma)$, $\varsigma > 0$ within the C_μ^n space if and only if $f^{(n)} \in C_\mu$, $n \in N$.*

Definition 2.3. *One may define the Riemann-Liouville fractional integral operator I^{θ_1} of order $\theta_1 \geq 0$, of a function $f \in C_\mu, \mu \geq 1$ as*

$$(I^{\theta_1})f(\varsigma) \quad = \quad \begin{cases} \frac{1}{\Gamma(\theta_1)} \int_0^\varsigma \frac{f(s)}{(\varsigma - s)^{1-\theta_1}} ds, & s > 0, \\ f(x), & \theta_1 = 0, \end{cases}$$

for $\theta_2 \geq -1$, the property of the operator I^{θ_1} that is needed in this article as

$$I^{\theta_1} \varsigma^{\theta_2} = \frac{\Gamma(\theta_2 + 1)}{\Gamma(\theta_2 + \theta_1 + 1)} \varsigma^{\theta_1 + \theta_2}.$$

Definition 2.4. *The Caputo fractional derivative ${}^cD^{\theta_1}$ of order θ_1 is defined as*

$$({}^cD^{\theta_1} f)(\varsigma) = \frac{1}{\Gamma(n - \theta_1)} \int_0^\varsigma \frac{f^{(n)}(s)}{(\varsigma - s)^{\theta_1 + 1 - n}} ds, \quad s > 0,$$

for $n - 1 < \theta_1 \leq n$, $n \in N$ and $f \in C_{-1}^n$, where $D = \frac{d}{d\varsigma}$ and $\Gamma(.)$ is the Gamma function.

Definition 2.5. *The Caputo partial fractional derivative of $u(\varsigma, \iota)$ with respect to ς of order $\theta_1 > 0$ is define as*

$$({}^cD_\varsigma^{\theta_1} u)(\varsigma, \iota) = \frac{\partial^{\theta_1} u(\varsigma, \iota)}{\partial \varsigma^{\theta_1}} \begin{cases} \frac{1}{\Gamma(n-\theta_1)} \int_0^\varsigma \frac{\partial^n u(s,t)}{\partial s^n} \frac{ds}{(\varsigma - s)^{\theta_1 + 1 - n}}, & n - 1 < \theta_1 < n, \quad n \in N. \\ \frac{\partial^n u(\varsigma, \iota)}{d\varsigma^n}, & \theta_1 = n, \end{cases}$$

Lemma 2.1. [5]. *If $n - 1 < \theta_1 \leq n$, $n \in \mathbb{N}$, then $D_\varsigma^{\theta_1} I^{\theta_1} u(\varsigma, \iota) = u(\varsigma, \iota)$, and:*

$$I^{\theta_1} D_\varsigma^{\theta_1} u(\varsigma, \iota) = u(\varsigma, \iota) - \sum_{k=0}^{n-1} \frac{\partial^k u(0^+, \iota)}{\partial \varsigma^k} \frac{\varsigma^k}{k!}, \quad \varsigma > 0.$$

Definition 2.6. [34]. *Let $(\theta_1, \theta_2) \in (0, \infty) \times (0, \infty), \theta = (0, 0), \Omega := [0, a] \times [0, b]$, and $u \in L^1(\Omega)$. The left-sided mixed Riemann-Liouville integral of order (θ_1, θ_2) of u is defined by*

$$(I_\theta^{(\theta_1, \theta_2)} u)(\varsigma, \iota) = \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^x \int_0^t (x - s)^{(\theta_1 - 1)} (t - y)^{(\theta_2 - 1)} u(s, t) dy ds. \tag{2.1}$$

In particular

1. $(I_\theta^{(\theta_1, \theta_2)} u)(\varsigma, \iota) = u(\varsigma, \iota)$,
2. $(I_\theta^{(\sigma)} u)(\varsigma, \iota) = \int_0^x \int_0^y u(s, t) dt ds, (\varsigma, \iota) \in \Omega, \sigma = (1, 1)$,
3. $(I_\theta^{(\theta_1, \theta_2)} u)(x, 0) = (I_\theta^{(\theta_1, \theta_2)})(0, t) = 0, x \in [0, a], y \in [0, b]$,
4. $I_\theta^{\theta_1, \theta_2} \varsigma^\lambda \iota^\omega = \frac{\Gamma(1+\lambda) \times \Gamma(1+\omega)}{\Gamma(1+\lambda+\theta_1) \times \Gamma(1+\omega+\theta_2)} \varsigma^{\lambda+\theta_1} \iota^{\omega+\theta_2}, (\varsigma, \iota) \in \Omega, \lambda, \omega \in (-1, \infty)$.

3. Genocchi polynomials (GPs)

The GPs can be written as [17]

$$\frac{2\iota e^{\varsigma\iota}}{e^\iota + 1} = \sum_{n=0}^\infty G_n(\varsigma) \frac{\iota^n}{n!}, (|\iota| < \pi),$$

where GPs of order n defined on the interval $[0, 1]$ as follows:

$$G_n(\varsigma) = \sum_{p=0}^n \begin{pmatrix} n \\ p \end{pmatrix} g_{n-p} \varsigma^p,$$

where $g_k = 2B_k - 2^{k+1}B_k$ is the Genocchi number, and B_n is the well-known Bernoulli number.

Some properties of GPs are as follows:

$$(1) \quad G_n(1) + G_n(0) = 0, \quad n > 1,$$

$$(2) \quad \frac{dG_n(\varsigma)}{d\varsigma} = nG_{n-1}(\varsigma), \quad n \geqslant 1,$$

$$(3) \quad \frac{d^k G_n(\varsigma)}{d\varsigma^k} = \begin{cases} 0, & n \leqslant k. \\ k! \begin{pmatrix} n \\ p \end{pmatrix} G_{n-k}(\varsigma), & n > k, k, m \in N \cup 0 \end{cases}$$

$$(4) \quad \int_0^1 G_n(\varsigma) G_m(\varsigma) d\varsigma = \frac{2(-1)^n n! m!}{(n+m)!} g_{n+m}, \quad n, m \geqslant 1.$$

$$(5) \quad \int_a^b G_n(\varsigma) d\varsigma = \frac{G_{n+1}(b) - G_{n+1}(a)}{n+1}.$$

Let $\Psi(\varsigma)$ denote a set of orthonormal GPs as follows:

$$\Psi(\varsigma) = [\Psi_1(\varsigma), \Psi_2(\varsigma), ..., \Psi_n(\varsigma)]^T, \tag{3.1}$$

where $\Psi(\varsigma)$ for $i = 0, 1, ..., N$, is one dimensional GPs. There polynomials can be written in the following matrix form

$$\Psi(\varsigma) = PT_N(\varsigma), \tag{3.2}$$

where

$$P = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} g_1 & 0 & \cdots & 0 \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} g_2 & \begin{pmatrix} 2 \\ 1 \end{pmatrix} g_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} n \\ 0 \end{pmatrix} g_n & \begin{pmatrix} n \\ 1 \end{pmatrix} g_{n-1} & \cdots & \begin{pmatrix} n \\ n-1 \end{pmatrix} g_1 \end{pmatrix}_{n \times n},$$

and

$$T_N(\xi) = \begin{pmatrix} 1 \\ \varsigma \\ \vdots \\ \varsigma^N \end{pmatrix}.$$

3.1. Function approximation

A function $u(\varsigma) \in L^2([0,1])$ can be expanded in terms of GPs as follows:

$$u(\varsigma) \approx u_0(\varsigma) = \sum_{i=0}^{n} c_i \psi_i(\varsigma) = C^T \Psi(\varsigma) = \Psi^T(\varsigma) C, \tag{3.3}$$

where

$$C = [c_0, c_1, ..., c_n,]^T, \tag{3.4}$$

then

$$C = Q^{-1} \langle u(\varsigma), \Psi(\varsigma) \rangle, \tag{3.5}$$

where Q is an $n \times n$ matrix and is defined as

$$Q = \langle \Psi(\varsigma), \Psi(\varsigma) \rangle = \int_a^b \Psi(\varsigma) \Psi^T(\varsigma) d\varsigma. \tag{3.6}$$

To expanded $u(\varsigma,\iota)$ in terms of 2D-GPs, first, we define 2D-GPs as follows:

$$u(\varsigma,\iota)=\Psi_i(\varsigma)\Psi_j(\iota). \tag{3.7}$$

Now, we define

$$\Psi(\varsigma,\iota)=[\Psi_{0,0}(\varsigma,\iota),\Psi_{0,1}(\varsigma,\iota),...,\Psi_{0,m}(\varsigma,\iota),\Psi_{0,1}(\varsigma,\iota),...,\Psi_{n,m}(\varsigma,\iota)]^T, \tag{3.8}$$

obviously, we can results $\Psi(\varsigma,\iota)$ as the following form:

$$\Psi(\varsigma,\iota)=\Psi(\varsigma)\otimes\Psi(\iota), \tag{3.9}$$

where $\otimes$ is the kronecker product and

$$\Psi(\varsigma)=[\Psi_0(\varsigma),\Psi_1(\varsigma),...,\Psi_n(\varsigma)]^T,\ \Psi(\iota)=[\Psi_0(\iota),\Psi_1(\iota),...,\Psi_n(\iota)]^T. \tag{3.10}$$

Now, suppose that $u(\varsigma,\iota)\in L^2([0,1)\times[0,1))$. Clearly, we can expand $u(\varsigma,\iota)$ in terms of 2D-GPs

$$u(\varsigma,\iota)\backsimeq\sum_{i=0}^{n}\sum_{j=0}^{n}c_{ij}\psi_{ij}(\varsigma,\iota)=C^T\Psi(\varsigma,\iota)=\Psi^T(\varsigma,\iota)C, \tag{3.11}$$

where

$$C=[c_{0,0},c_{0,1},...,c_{0,n},c_{1,0},...,c_{1,n},...,c_{n,0},...,c_{n,m}]^T.$$

To get the approximate function of $[u(\varsigma,\iota)]^p$, now we have

$$\begin{aligned}[u(\varsigma,\iota)]^2 &\simeq (\Psi(\varsigma,\iota)U)(\Psi^T(\varsigma,\iota)U)=(U^T\Psi(\varsigma,\iota))(\Psi^T(\varsigma,\iota)U)\\ &= U^T\hat{U}\Psi(\varsigma,\iota)=\Psi^T(\varsigma,\iota)C_2,\end{aligned}$$

$$\begin{aligned}[u(\varsigma,\iota)]^3 &\simeq (\Psi^T(\varsigma,\iota)U)(\Psi^T(\varsigma,\iota)C_2)=(U^T\Psi(\varsigma,\iota))(\Psi^T(\varsigma,\iota)C_2)\\ &= U^T\hat{C}_2\Psi(\varsigma,\iota)=\Psi^T(\varsigma,\iota)C_3,\\ &\ \vdots\\ [u(\varsigma,\iota)]^P &\simeq (\Psi^T(\varsigma,\iota)U)(\Psi^T(\varsigma,\iota)C_{p-1})=(U^T\Psi(\varsigma,\iota))(\Psi^T(\varsigma,\iota)C_{p-1})\\ &= U^T\hat{C}_{p-1}\Psi(\varsigma,\iota)=\Psi^T(\varsigma,\iota)C_p,\end{aligned}$$

where $C_2=(U^T\hat{U})^T$, $C_3=(U^T\hat{C}_2)^T$ and $C_p=(U^T\hat{C}_{p-1})^T$.

4. **Operational matrices**

Here, we obtain the operational matrix of GPs. To do this, we have:

$$\int_0^{\varsigma}\Psi(t)dt=\Upsilon\Psi(\varsigma), \tag{4.1}$$

where

$$\Upsilon=\left\langle\int_0^{\varsigma}\Psi(t)dt,\Psi(\varsigma)\right\rangle.Q^{-1},$$

is an $n\times n$ OM of integration. By using Eq. (4.1), the OM of integration based on 2D-GPs for variable ς is obtained as follows

$$\begin{aligned}\int_0^{\varsigma}\Psi(\varsigma,y)d\varsigma &= \int_0^{\varsigma}(\Psi(\varsigma)\otimes\Psi(y))d\varsigma=(\int_0^{\varsigma}\Psi(\varsigma)d\varsigma)\otimes\Psi(y)\\ &= (\Upsilon\Psi(\varsigma))\otimes(I\Psi(\iota))=(\Upsilon\otimes I)(\Psi(\varsigma)\otimes\Psi(y))\\ &= \hat{\Upsilon}_{\varsigma}\Psi(\varsigma,y),\end{aligned} \tag{4.2}$$

where I is a identify matrix. In a similar way, for ι variable, we get

$$\begin{aligned}\int_0^{\iota}\Psi(x,\iota)d\iota &= \int_0^{\iota}(\Psi(x)\otimes\Psi(\iota))d\iota = \Psi(x)\otimes(\int_0^{\iota}\Psi(\iota)d\iota)\\ &= I\Psi(x)\otimes(\Upsilon\Psi(\iota)) = (I\otimes\Upsilon)(\Psi(x)\otimes\Psi(\iota))\\ &= \hat{\Upsilon}_{\iota}\Psi(x,\iota). \end{aligned} \tag{4.3}$$

Now, for mixed variable, we conclude that

$$\begin{aligned}\int_0^{\varsigma}\int_0^{\iota}\Psi(x,y)dxdy &= \int_0^{\varsigma}\int_0^{\iota}(\Psi(x)\otimes\Psi(y))dxdy = (\int_0^{\varsigma}\Psi(x)dx)\otimes(\int_0^{\iota}\Psi(y)dy)\\ &= (\Upsilon_{\varsigma}\Psi(\varsigma))\otimes(\Upsilon_{\iota}\Psi(\iota)) = (\Upsilon_{\varsigma}\otimes\Upsilon_{\iota})(\Psi(\varsigma)\otimes\Psi(\iota))\\ &= \hat{\Upsilon}_{\varsigma\iota}\Psi(\varsigma,\iota), \end{aligned} \tag{4.4}$$

where $\hat{\Upsilon}_{\varsigma}$ are matrixes of order $(n+1)^2$ and $\hat{\Upsilon}_{\varsigma\iota}$ is an $(n+1)^2\times(n+1)^2$ matrix with the following form:

$$\hat{\Upsilon}_{\varsigma\iota} = \begin{pmatrix} \Upsilon & O & \dots & O\\ O & \Upsilon & \dots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \dots & \Upsilon \end{pmatrix}.$$

Let $k(\varsigma,\iota,s,y)$ be a function of four variables on $([0,1)\times[0,1)\times[0,1)\times[0,1))$. It can be approximated with respect to 2D-GPs as follows:

$$k(\varsigma,\iota,s,y) = \Psi^T(\varsigma,\iota).K.\Psi(s,y), \tag{4.5}$$

where $\Psi(\varsigma,\iota)$ and $\Psi(s,y)$ are 2D-GPs vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively, and K is a $(4m_1m_2)\times(4m_3m_4)$ 2D-GPs coefficients matrix.

4.1. Operational matrix of product

The 2D-OGPs operational matrix of the product is obtained in the present section. If R represents a column vector as below,

$$R = [R_{00},R_{01},...,R_{0n},R_{10},R_{11},...,R_{1n},...,R_{n0},R_{n1},...,R_{nn}]^T. \tag{4.6}$$

The $\hat{R}$ matrix satisfying the relation below is known as the operational matrix pertaining to the product of two 2D-OGPs vectors.

$$\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)R = \hat{R}\Psi(\varsigma,\iota). \tag{4.7}$$

In order to present an explicit presentation of $\hat{R}$, the following routine is adopted. The application of Eq (3.2) gives us the following

$$\begin{aligned}\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)R &= (\Psi(\varsigma)\otimes\Psi(\iota))\Psi^T(\varsigma,\iota)R = (PT_n(\varsigma))\otimes(PT_n(\iota))\Psi^T(\varsigma,\iota)R\\ &= (P\otimes P)(T_n(\varsigma)\otimes T_n(\iota))\Psi^T(\varsigma,\iota)R\\ &= \tilde{P}T_n(\varsigma,\iota)\Psi^T(\varsigma,\iota)R, \end{aligned} \tag{4.8}$$

where $\tilde{P} = P\otimes P$ and

$$T_n(\varsigma,\iota) = [1,\iota,...,\iota^n,\varsigma,\varsigma\iota,...,\varsigma\iota^n,...,\varsigma^n,\varsigma^n\iota,...,\varsigma^n\iota^n]^T. \tag{4.9}$$

Thus,

$$\begin{aligned}\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)R &= \tilde{P}[\sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\psi_{ij}(\varsigma,\iota), \sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\iota\psi_{ij}(\varsigma,\iota), ..., \sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\iota^n\psi_{ij}(\varsigma,\iota)\\ &\sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\varsigma\psi_{ij}(\varsigma,\iota), ..., \sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\varsigma\iota^n\psi_{ij}(\varsigma,\iota), ..., \sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\varsigma^n\psi_{ij}(\varsigma,\iota)\\ &, ..., \sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\varsigma^n\iota^n\psi_{ij}(\varsigma,\iota)]^T. \end{aligned} \tag{4.10}$$

We approximate the functions $\varsigma^l\iota^s\psi_{ij}(\varsigma,\iota)$ by using 2D-OGPs as follows

$$\varsigma^l\iota^s\psi_{ij}(\varsigma,\iota) \simeq \sum_{p=0}^{n}\sum_{q=0}^{n}\gamma_{pq}^{ijls}\Psi_{pq}(\varsigma,\iota) = \eta_{ijls}^T\Psi(\varsigma,\iota), \quad l,s = 0,1,...,n. \tag{4.11}$$

where

$$\eta_{ijls} = [\gamma_{00}^{ijls}, \gamma_{01}^{ijls}, ..., \gamma_{0n}^{ijls}, ..., \gamma_{10}^{ijls}, ..., \gamma_{1n}^{ijls}, ..., \gamma_{n0}^{ijls}, ..., \gamma_{nn}^{ijls}]^T, \tag{4.12}$$

and η_{ijls} is calculated as follows

$$\eta_{ijls} = \int_0^1\int_0^1 \varsigma^l\iota^s\psi_{ij}(\varsigma,\iota)\Psi(\varsigma,\iota)d\varsigma d\iota. \tag{4.13}$$

From Eq (4.11), we have

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\varsigma^l\iota^s\psi_{ij}(\varsigma,\iota) &= \sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\sum_{p=0}^{n}\sum_{q=0}^{n}\gamma_{pq}^{ijls}\psi_{pq}(\varsigma,\iota) = \sum_{p=0}^{n}\sum_{q=0}^{n}\psi_{pq}(\varsigma,\iota)\sum_{i=0}^{n}\sum_{j=0}^{n} r_{ij}\gamma_{pq}^{ijls}\\ &= \Psi^T(\varsigma,\iota)[\eta_{00ls}, \eta_{01ls}, ..., \eta_{0nls}, ..., \eta_{n0ls}, \eta_{n1ls}, ..., \eta_{nnls}]R\\ &= \Psi^T(\varsigma,\iota)\eta_{ls}, \end{aligned} \tag{4.14}$$

where $\eta_{ls} = [\eta_{00ls}, \eta_{01ls}, ..., \eta_{0nls}, ..., \eta_{n0ls}, \eta_{n1ls}, ..., \eta_{nnls}]R$. Now, we define $(n+1)^2 \times (n+1)^2$ matrix η as follows

$$\eta = [\eta_{00}, \eta_{01}, ..., \eta_{0n}, ..., \eta_{n0}, \eta_{n1}, ..., \eta_{nn}].$$

By inserting Eq (4.14) into Eq (4.10), we get

$$\begin{aligned}\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)R &= \tilde{P}[\Psi^T(\varsigma,\iota)\eta_{00}, \Psi^T(\varsigma,\iota)\eta_{01}, ..., \Psi^T(\varsigma,\iota)\eta_{0n}, ..., \Psi^T(\varsigma,\iota)\eta_{n0},\\ &\quad \Psi^T(\varsigma,\iota)\eta_{n1}, ..., \Psi^T(\varsigma,\iota)\eta_{nn}]^T\\ &= \tilde{P}(\Psi^T(\varsigma,\iota)[\eta_{00}, \eta_{01}, ..., \eta_{0n}, ..., \eta_{n0}, \eta_{n1}, ..., \eta_{nn}])^T\\ &= \tilde{P}\eta^T\Psi(\varsigma,\iota) = \hat{R}\Psi(\varsigma,\iota), \end{aligned} \tag{4.15}$$

and

$$\Psi(\varsigma,\iota)D\Psi^T(\varsigma,\iota) = \tilde{D}\Psi(\varsigma,\iota), \tag{4.16}$$

where D is m-vector with elements equal to the diagonal entries of D.

4.2. Operational matrix of a 2D fractional integration of 2D-GPs

We define an m-set of 2D-BPF as

$$b_{i_1,i_2}(\varsigma,\iota) = \begin{cases} 1, & (i_1-1)h_1 \leqslant \varsigma < i_1h_1 \text{ and } (i_2-1)h_2 \leqslant \iota < i_2h_2,\\ & i = 0,1,2,\cdots,(m-1),\\ 0, & \text{otherwise}. \end{cases} \tag{4.17}$$

The function $b_{i_1,i_2}(\varsigma,\iota)$ are orthogonal and disjoint, that is

$$b_{i_1,i_2}(\varsigma,\iota)b_{j_1,j_2}(\varsigma,\iota) = \begin{cases} b_{i_1,i_2}(\varsigma,\iota), & i_1 = j_1 \text{ and } i_2 = j_2,\\ 0, & \text{otherwise}. \end{cases} \tag{4.18}$$

The 2D-GPs can be expanded in terms to m-set of 2D-BPFs as

$$\Psi(\varsigma,\iota)=\Phi_{m\times m}B_m(\varsigma,\iota), \tag{4.19}$$

where $B_m(\varsigma,\iota)=(b_{0,0}(\varsigma,\iota),b_{0,1}(\varsigma,\iota),\cdots,b_{0,n}(\varsigma,\iota),\cdots,b_{1,0}(\varsigma,\iota),\cdots b_{m,m}(\varsigma,\iota))^T$[23], and Φ is an $mn\times mn$ product operational matrix. Next, we derive the Genocchi polynomials OMFI. The OMFI as follows

$$\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}\int_0^x\int_0^t(\varsigma-s)^{\theta_1-1}(\iota-y)^{\theta_2-1}\Psi(s,y)dyds\simeq F^{\theta_1,\theta_2}\Psi(\varsigma,\iota) \tag{4.20}$$

$$F^{\theta_1,\theta_2}=\frac{1}{m^{\theta_1}m^{\theta_2}}\frac{1}{\Gamma(\theta_1+2)\Gamma(\theta_2+2)}\begin{bmatrix}1&\varsigma_1&\varsigma_2&\varsigma_3&\dots&\varsigma_{m-1}\\0&1&\varsigma_1&\varsigma_2&\dots&\varsigma_{m-1}\\0&0&1&\varsigma_1&\dots&\varsigma_{m-3}\\\vdots&\vdots&\ddots&\ddots&&\vdots\\0&0&\dots&0&1&\varsigma_1\\0&0&0&\dots&0&1\end{bmatrix}\otimes\begin{bmatrix}1&\eta_1&\eta_2&\eta_3&\dots&\eta_{m-1}\\0&1&\eta_1&\eta_2&\dots&\eta_{m-1}\\0&0&1&\eta_1&\dots&\eta_{m-3}\\\vdots&\vdots&\ddots&\ddots&&\vdots\\0&0&\dots&0&1&\eta_1\\0&0&0&\dots&0&1\end{bmatrix}, \tag{4.21}$$

$\varsigma_\varrho=(\varrho+1)^{\theta_1+1}-2\rho^{\theta_1+1}+(\varrho-1)^{\theta_1+1}$, and $\eta_\varrho=(\varrho+1)^{\theta_2+1}-2\varrho^{\theta_2+1}+(\varrho-1)^{\theta_2+1}$. In the following, fractional integration of the 2D-BPFs vector has been presented.

$$(I^{\theta_1,\theta_2}B_m)(\varsigma,\iota)\approx F^{\theta_1,\theta_2}B_m(\varsigma,\iota). \tag{4.22}$$

Now, we derive the Genocchi polynomials OMFI.

$$(I^{\theta_1,\theta_2}\Psi)(\varsigma,\iota)\approx P^{\theta_1,\theta_2}_{m\times m}\Psi(\varsigma,\iota), \tag{4.23}$$

where matrix $P^{\theta_1,\theta_2}_{m\times m}$ is called the Genocchi polynomials OMFI. Using Eqs.(4.19) and (4.22), we have

$$(I^{\theta_1,\theta_2}\Psi)(\varsigma,\iota)\approx(I^{\theta_1,\theta_2}\Phi_{m\times m}B_m)(\varsigma,\iota)=\Phi_{m\times m}(I^{\theta_1,\theta_2}B_m)(\varsigma,\iota)\approx\Phi_{m\times m}F^{\theta_1,\theta_2}B_m(\varsigma,\iota). \tag{4.24}$$

By Eqs.(4.23) and (4.24) we get

$$P^{\theta_1,\theta_2}_{m\times m}\Psi(\varsigma,\iota)=\Phi_{m\times m}F^{\theta_1,\theta_2}B_m(\varsigma,\iota)=\Phi_{m\times m}F^{\theta_1,\theta_2}\Phi^{-1}_{m\times m}\Psi(\varsigma,\iota). \tag{4.25}$$

So, the Genocchi polynomials OMFI, $P^{\theta_1,\theta_2}_{m\times m}$, is given by

$$P^{\theta_1,\theta_2}_{m\times m}=\Phi_{m\times m}F^{\theta_1,\theta_2}\Phi^{-1}_{m\times m}. \tag{4.26}$$

4.3. Operational matrix of differentiation

The derivative operational matrix of OGPs with respect to ς, is obtained as follows

$$\begin{aligned}\Psi_\varsigma(\varsigma)&=\frac{\partial}{\partial\varsigma}\Psi(\varsigma)=\frac{\partial}{\partial\varsigma}(PT_n(\varsigma))=P(\frac{\partial}{\partial\varsigma}(T_n(\varsigma)))\\&=PLT_n(\varsigma)=PLP^{-1}\Psi(\varsigma)=\tilde{L}\Psi(\varsigma),\end{aligned} \tag{4.27}$$

where $\tilde{L}=PLP^{-1}$, according to Eq (4.27), differentiation of vector $\Psi(\varsigma,\iota)$ defined in Eq (3.10) respect to ς and ι, is approximated as

$$\begin{aligned}\Psi_\varsigma(\varsigma,\iota)=\Psi_\varsigma(\varsigma)\otimes\Psi(\iota)&\simeq(\tilde{L}\Psi(\varsigma))\otimes(I\Psi(\iota))=(\tilde{L}\otimes I)(\Psi(\varsigma))\otimes\Psi(\iota)\\&=\hat{L}_\varsigma\Psi(\varsigma,\iota),\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\Psi_\iota(\varsigma,\iota)=\Psi(\varsigma)\otimes\Psi_\iota(\iota) &\simeq (I\Psi(\varsigma))\otimes(\tilde{L}\Psi(\iota))=(I\otimes\tilde{L})(\Psi(\varsigma))\otimes\Psi(\iota)\\ &= \hat{L}_\iota\Psi(\varsigma,\iota), \end{aligned} \tag{4.29}$$

where, $\hat{L_\varsigma}=\tilde{L}\otimes I$ and $\hat{L_\iota}=I\otimes\tilde{L}$ are $(n+1)^2\times(n+1)^2$ matrices.

5. **The numerical scheme for the model**

Consider the nonlinear FWS2DPVIEs:

$$\begin{aligned}u_{\varsigma,\iota}(\varsigma,\iota)+u_\iota(\varsigma,\iota)=u(\varsigma,\iota) &+ g(\varsigma,\iota)+\int_0^\varsigma\int_0^\iota\frac{H(\varsigma,\iota,s,y,u(s,y))}{(\varsigma-s)^{1-\theta_1}(\iota-y)^{1-\theta_2}}dyds\\ &+\int_0^\varsigma\int_0^\iota H(\varsigma,\iota,s,y,u(s,y))dyds,\end{aligned} \tag{5.1}$$

for solving Eq (5.1), we start by approximating the functions $u_{\varsigma,\iota}(\varsigma,\iota)$, $u_\iota(\varsigma,\iota)$, $u(\varsigma,\iota)$, $g(\varsigma,\iota)$, $u(\varsigma,0)$ and $u(0,\iota)$ interms of 2D-OGPs as follows

$$u_{\varsigma,\iota}(\varsigma,\iota)=C^T\Psi(\varsigma,\iota), \tag{5.2}$$

$$g(\varsigma,\iota)=G^T\Psi(\varsigma,\iota), \tag{5.3}$$

$$u_\iota(0,\iota)=Z^T\Psi(\varsigma,\iota), \tag{5.4}$$

$$u(\varsigma,0)=U^T\Psi(\varsigma,\iota), \tag{5.5}$$

$$u(0,\iota)=V^T\Psi(\varsigma,\iota), \tag{5.6}$$

$$u(0,0)=D^T\Psi(\varsigma,\iota). \tag{5.7}$$

By integrating form both sides of Eqs (5.2) with respect to ς and by using Eqs (4.25) and (5.4), we have

$$\begin{aligned}u_\iota(\varsigma,\iota) &= u_\iota(0,\iota)+C^T\int_0^\varsigma\Psi(t,\iota)dt=Z^T\Psi(\varsigma,\iota)+C^T\hat{\Upsilon}_\varsigma(\varsigma,\iota)\\ &= (Z^T+C^T\hat{\Upsilon}_\iota)\Psi(\varsigma,\iota)=Q^T\Psi(\varsigma,\iota).\end{aligned} \tag{5.8}$$

Similarly, by integrating from both sides of Eqs (5.8) with respect to ι and using Eqs. (4.4) and (5.5)-(5.7), we have

$$\begin{aligned}u(\varsigma,\iota) &= u(\varsigma,0)+u(0,\iota)-u(0,0)+C^T\int_0^\varsigma\int_0^\iota\Psi(t,s)dsdt\\ &\simeq U^T\Psi(\varsigma,\iota)+V^T\Psi(\varsigma,\iota)-D^T\Psi(\varsigma,\iota)+C^T\hat{\Upsilon_{\varsigma,\iota}}\Psi(\varsigma,\iota)\\ &= (U^T+V^T-D^T+C^T\hat{\Upsilon_{\varsigma\iota}})\Psi(\varsigma,\iota)=R^T\Psi(\varsigma,\iota).\end{aligned} \tag{5.9}$$

By differentiating from both sides of Eqs (5.8) respect to ι and using Eq (4.29), we get

$$u_{\iota\iota}(\varsigma,\iota)=Q^T\Psi_\iota(\varsigma,\iota)=Q^T\hat{L}_\iota\Psi(\varsigma,\iota). \tag{5.10}$$

Now, we calculate integral part

$$\begin{aligned}\int_0^\varsigma\int_0^\iota\frac{k(\varsigma,\iota,s,y)[u(s,y)]^p}{(\varsigma-s)^{1-\theta_1}(\iota-y)^{1-\theta_2}}dyds &=\int_0^\varsigma\int_0^\iota(\varsigma-s)^{\theta_1-1}(\iota-y)^{\theta_2-1}\Psi^T(\varsigma,\iota)K\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)C_pdyds\\ &=\Psi^T(\varsigma,\iota)K\int_0^\varsigma\int_0^\iota(\varsigma-s)^{\theta_1-1}(\iota-y)^{\theta_2-1}\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)C_pdyds\\ &=\Psi^T(\varsigma,\iota)K\hat{C_p}\int_0^\varsigma\int_0^\iota(\varsigma-s)^{\theta_1-1}(\iota-y)^{\theta_2-1}\Psi(\varsigma,\iota)dyds\\ &=\Psi^T(\varsigma,\iota)K\hat{C_p}\Gamma(\theta_1)\Gamma(\theta_2)F^{\theta_1,\theta_2}\Psi(\varsigma,\iota)\\ &=\tilde{\Omega}^T\Psi(\varsigma,\iota),\end{aligned}$$

here $\tilde{\Omega} = \Gamma(\theta_1)\Gamma(\theta_2)K\hat{C}_p F^{\theta_1,\theta_2}$ is an $4m_1m_2$-vector with elements equal to the diagonal entries of the following matrix $\Omega = \Gamma(\theta_1)\Gamma(\theta_2)K\hat{C}_p F^{\theta_1,\theta_2}$. By using Eq. (4) and (4.7), we have

$$\begin{aligned}\int_0^{\varsigma}\int_0^{\iota} H(\varsigma,\iota,s,y,u(s,y))dyds &= \int_0^{\varsigma}\int_0^{\iota} \Psi^T(\varsigma,\iota)K\Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)C_p dyds \\ &= \Psi^T(\varsigma,\iota)K\int_0^{\varsigma}\int_0^{\iota} \Psi(\varsigma,\iota)\Psi^T(\varsigma,\iota)C_p dyds \\ &= \Psi^T(\varsigma,\iota)K\hat{C}_p\int_0^{\varsigma}\int_0^{\iota}\Psi(\varsigma,\iota)dyds \\ &= \Psi^T(\varsigma,\iota)K\hat{C}_p\hat{\Upsilon}_{\varsigma\iota}\Psi(\varsigma,\iota) \\ &= \widetilde{K\hat{C}_p}\hat{\Upsilon}_{\varsigma\iota}\Psi(\varsigma,\iota). \end{aligned} \quad (5.11)$$

By inserting the Eqs (5.2)-(5.11) into Eq (5.1), we have

$$\begin{aligned} C^T\Psi(\varsigma,\iota) + Q^T\Psi(\varsigma,\iota) &= R^T\Psi(\varsigma,\iota) + G^T\Psi(\varsigma,\iota) + \tilde{\Omega}^T\Psi(s,t) \\ &\quad + K\widetilde{\hat{\Upsilon}\hat{C}_p}\hat{\Upsilon}_{\varsigma\iota}\Psi(\varsigma,\iota), \end{aligned} \quad (5.12)$$

by setting

$$O = K\widetilde{\hat{\Upsilon}\hat{C}_p\hat{\Upsilon}}_{\varsigma\iota},$$

we will have

$$C^T + Q^T = R^T + G^T + \tilde{\Omega}^T + O^T. \quad (5.13)$$

As a nonlinear system of an algebraic equation, Eq. (5.13) can be determined by solving the same system via Newton's or other iterative techniques. As a result, an approximate solution is

$$u(\varsigma,\iota) = C^T\Psi(\varsigma,\iota), \quad (5.14)$$

can be computed for Eq (5.1).

6. Numerical examples

The accuracy and applicability of the suggested Genocchi polynomials operational matrix technique are tested in the present section for fractional weakly singular partial Volterra integral equation using a number of numerical examples, and the tables and figures display the obtained results. Our suggested technique is examined in the following for the fractional weakly singular partial Volterra integral equation in a variety of states.

Example 6.1. *The below FWS2DPVIE is considered*

$$\begin{aligned} u_{\varsigma,\iota}(\varsigma,\iota) + u_{\iota}(\varsigma,\iota) = u(\varsigma,\iota) + g(\varsigma,\iota) &+ \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}\int_0^{\varsigma}\int_0^{\iota}(\varsigma-s)^{\frac{1}{2}}(\iota-y)^{\frac{3}{2}}[u(s,y)]^2 dyds \\ &+ \int_0^{\varsigma}\int_0^{\iota}(4\iota\varsigma)[u(s,y)]^2 dyds, \end{aligned} \quad (6.1)$$

with

$$\begin{aligned} g(\varsigma,\iota) &= 2\varsigma + 2\iota + \varsigma^2 + 2\varsigma\iota - \varsigma^2\iota - \varsigma\iota^2 - \frac{8}{3\pi}(\frac{4096}{1091475}\varsigma^{\frac{11}{2}}\iota^{\frac{9}{2}} + \frac{128}{3675}\varsigma^{\frac{7}{2}}\iota^{\frac{7}{2}}sy^2 \\ &\quad + \frac{4}{15}\varsigma^{\frac{3}{2}}\iota^{\frac{5}{2}}sy^4) - \frac{4}{15}\iota^4\varsigma^6 - \frac{4}{3}\iota^3\varsigma^4 sy^2 - 4\iota^2\varsigma^2 sy^4, \end{aligned} \quad (6.2)$$

the exact solution is calculated and presented by $u(\varsigma,\iota) = \varsigma^2\iota + \varsigma\iota^2$, and supplementary conditions of $u(\varsigma,0) = 0, u(0,\iota) = 0$. The numerical results calculated by the purposed method is shown in Table 1. The absolute errors (AEs) functions of $n = 2, n = 5$ and $\theta = 0.95$ have been plotted in Figures 1-2.

TABLE 1. The AEs for Example 6.1.

(ς, ι)	$n = 3$	$n = 5$	$n = 7$
$(0.1, 0.1)$	2.365214×10^{-4}	8.254100×10^{-5}	7.202079×10^{-7}
$(0.2, 0.2)$	3.365874×10^{-4}	1.965402×10^{-4}	1.702565×10^{-7}
$(0.3, 0.3)$	1.355381×10^{-3}	9.030514×10^{-5}	3.092219×10^{-6}
$(0.4, 0.4)$	9.196630×10^{-4}	7.121174×10^{-5}	3.416147×10^{-7}
$(0.5, 0.5)$	2.000506×10^{-4}	7.477421×10^{-4}	2.544555×10^{-7}
$(0.6, 0.6)$	1.696282×10^{-3}	1.035208×10^{-6}	7.619858×10^{-8}
$(0.7, 0.7)$	2.960114×10^{-4}	2.955547×10^{-5}	1.440310×10^{-7}
$(0.8, 0.8)$	6.775898×10^{-5}	2.985022×10^{-5}	3.255507×10^{-7}
$(0.9, 0.9)$	1.754210×10^{-4}	7.411200×10^{-4}	1.611203×10^{-7}

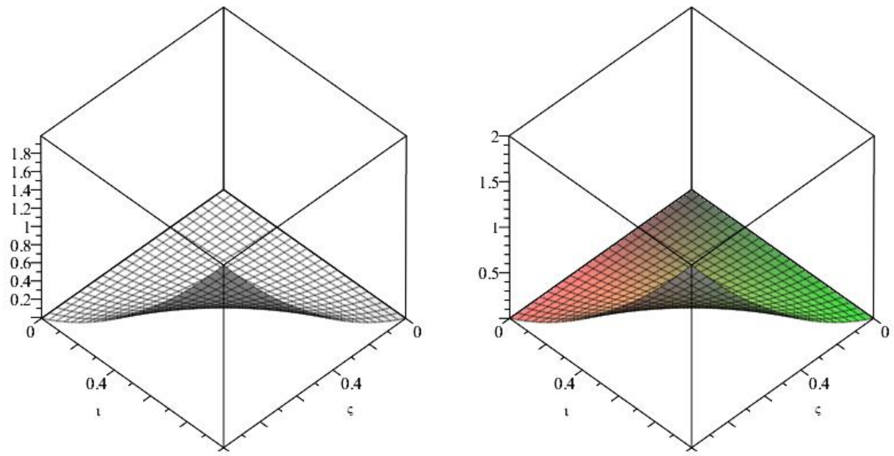

FIGURE 1. Comparing numerical (Right) and exact (Left) solutions, $u(\iota, \varsigma)$, with $n = 2$ for example 6.1.

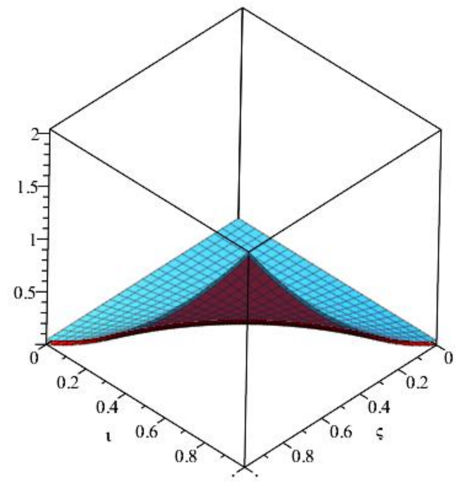

FIGURE 2. Exact and approximation solutions with $n = 5$ of Example 6.1.

Example 6.2. *In the next example, the below FWS2DPVIE is considered*

$$u_{\varsigma,\iota}(\varsigma,\iota)+u_{\iota}(\varsigma,\iota)=u(\varsigma,\iota)+g(\varsigma,\iota)+\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}\int_0^{\varsigma}\int_0^{\iota}(\varsigma-s)^{\frac{1}{2}}(\iota-y)^{\frac{1}{2}}\exp(\varsigma)\sqrt{\iota}[u(s,y)]dyds$$
$$+\int_0^{\varsigma}\int_0^{\iota}\exp(\iota)\sqrt{\varsigma}[u(s,y)]^2dyds, \qquad (6.3)$$

with

$$g(\varsigma,\iota) = -3\iota^2-\varsigma^3+\iota^3-\frac{256}{945\pi}(\varsigma^{\frac{3}{2}}\exp(\varsigma)\iota^5+\varsigma^{\frac{9}{2}}\exp(\varsigma)\iota^2)$$
$$-\frac{1}{7}\exp(\iota)\varsigma^{\frac{3}{2}}\iota^7+\frac{1}{8}\exp(\iota)\varsigma^{\frac{9}{2}}\iota^4-\frac{1}{7}\exp(\iota)\varsigma^{\frac{15}{2}}\iota. \qquad (6.4)$$

One may calculate and present the exact solution using $u(\varsigma,\iota)=\varsigma^3-\iota^3$, *and supplementary conditions* $u(\varsigma,0)=\varsigma^3, u(0,\iota)=-\iota^3$. *The numerical results acquired using the suggested technique are presented in Table 2. Figures 3-4 plots the absolute error function for* $n=2, n=5$ *and* $\theta=0.95$.

TABLE 2. The AEs for Example 6.2.

(ς,ι)	$n=3$	$n=5$	$n=7$
$(0.1,0.1)$	5.369872×10^{-3}	1.141100×10^{-4}	1.107079×10^{-6}
$(0.2,0.2)$	5.741470×10^{-4}	8.001412×10^{-4}	9.482565×10^{-6}
$(0.3,0.3)$	1.356777×10^{-3}	7.032304×10^{-5}	3.092219×10^{-6}
$(0.4,0.4)$	8.195790×10^{-3}	4.129834×10^{-5}	4.006147×10^{-7}
$(0.5,0.5)$	2.000506×10^{-4}	7.477421×10^{-4}	6.576505×10^{-6}
$(0.6,0.6)$	9.754202×10^{-3}	1.035874×10^{-5}	4.600258×10^{-7}
$(0.7,0.7)$	1.060184×10^{-4}	7.955487×10^{-5}	6.456310×10^{-6}
$(0.8,0.8)$	6.025898×10^{-4}	7.098722×10^{-5}	4.298707×10^{-7}
$(0.9,0.9)$	4.054210×10^{-4}	8.498100×10^{-4}	6.871203×10^{-7}

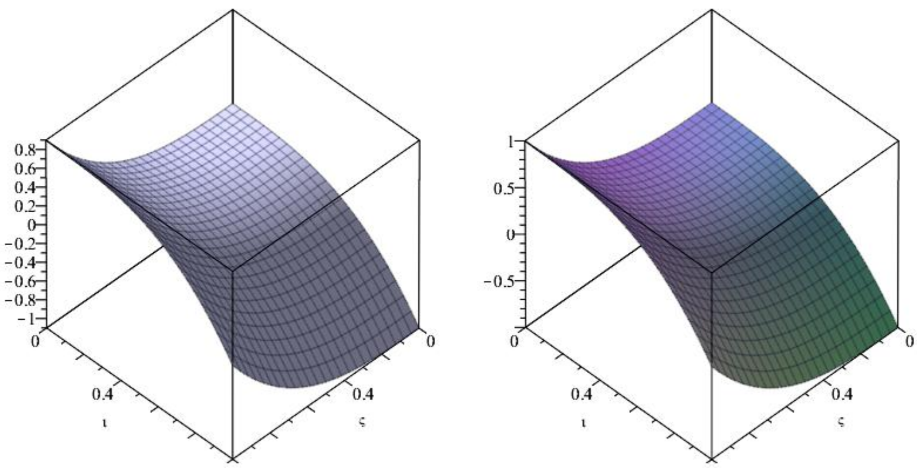

FIGURE 3. Comparing numerical (Right) and exact (Left) solutions, $u(\iota,\varsigma)$, with $n=2$ for Example 6.2.

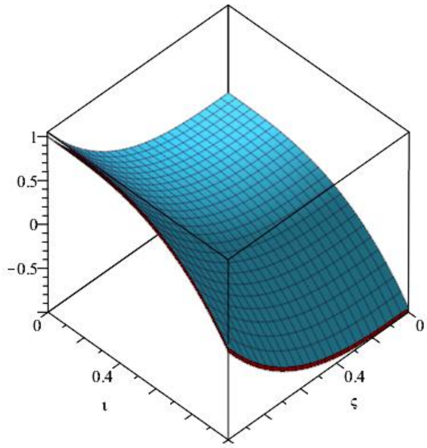

FIGURE 4. Exact and approximation solutions with $n = 5$ of Example 6.2.

Example 6.3. *In the final example, the below FWS2DPVIE is considered*

$$u_{\varsigma,\iota}(\varsigma,\iota) + u_{\iota}(\varsigma,\iota) = u(\varsigma,\iota) + g(\varsigma,\iota) + \frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{11}{2})}\int_0^{\varsigma}\int_0^{\iota}(\varsigma-s)^{\frac{5}{2}}(\iota-y)^{\frac{9}{2}}[u(s,y)]^2 dyds + \int_0^{\varsigma}\int_0^{\iota}(2\iota\varsigma)^3[u(s,y)]^3 dyds, \quad (6.5)$$

with

$$g(\varsigma,\iota) = 2\varsigma + 2\iota + \varsigma^2 + 2\varsigma\iota - \varsigma^2\iota - \varsigma\iota^2 - \frac{65536}{21070924875\pi}\varsigma^{\frac{11}{2}}\iota^{\frac{15}{2}} - \frac{1}{4}\iota^6\varsigma^6,$$

the exact solution is calculated and presented by $u(\varsigma,\iota) = \varsigma\iota$, and supplementary conditions of $u(\varsigma,0) = 0, u(0,\iota) = 0$. The numerical results calculated by the purposed method is shown in Table 3. The absolute error functions of $n = 2, n = 5$ and $\theta = 0.95$ have been plotted in Figures. 5-6.

TABLE 3. The AEs for Example 6.3.

(ς,ι)	$n=4$	$n=6$	$n=8$
$(0.1, 0.1)$	4.301514×10^{-5}	4.004101×10^{-7}	9.214079×10^{-8}
$(0.2, 0.2)$	1.365171×10^{-5}	1.965402×10^{-7}	4.702145×10^{-9}
$(0.3, 0.3)$	1.649781×10^{-4}	3.075314×10^{-6}	3.092219×10^{-6}
$(0.4, 0.4)$	7.153330×10^{-5}	2.145874×10^{-7}	3.476147×10^{-8}
$(0.5, 0.5)$	2.045546×10^{-5}	4.475221×10^{-7}	2.544555×10^{-7}
$(0.6, 0.6)$	1.641082×10^{-4}	1.718208×10^{-6}	7.619858×10^{-8}
$(0.7, 0.7)$	2.941257×10^{-5}	2.183547×10^{-7}	1.103110×10^{-7}
$(0.8, 0.8)$	6.524798×10^{-5}	2.002522×10^{-7}	2.242107×10^{-8}
$(0.9, 0.9)$	2.772410×10^{-4}	7.418520×10^{-5}	1.625703×10^{-8}

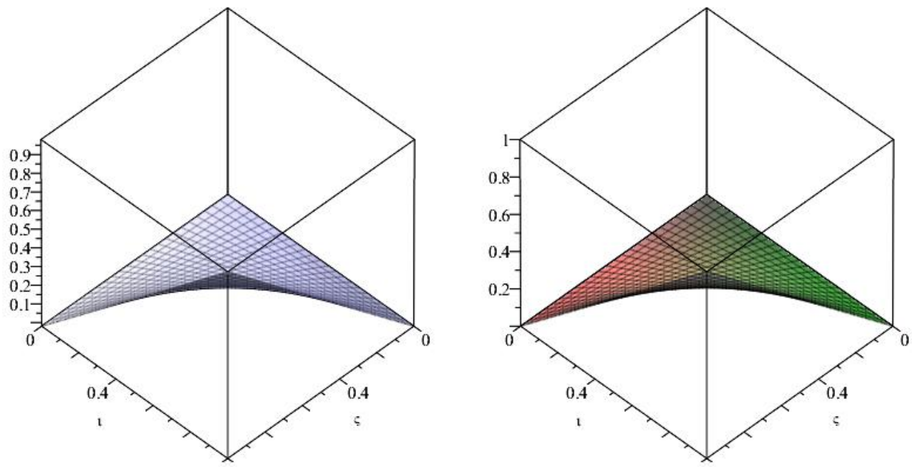

FIGURE 5. Comparing numerical (Right) and exact (Left) solutions, $u(\iota, \varsigma)$, with $n = 2$ for Example 6.3.

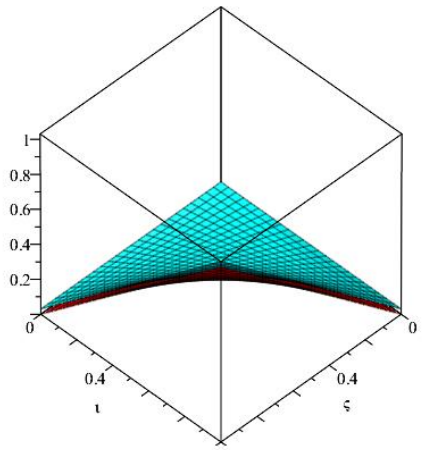

FIGURE 6. Exact and approximation solutions with $n = 5$ of Example 6.3.

7. **Conclusion**

The present paper has studied the use of GPs for the purpose of solving fractional weakly singular 2-D partial Volterra integral equation. The advantages of the technique include the lower costs required for setting up the system of equations without the need for using projection techniques, e.g., the collocation, Galerkin, etc., and the very lower computational costs of operations. From the computational perspective, this is one of the benefits of the technique that makes it very cheap and simple. Also, using a number of examples, its applicability and accuracy were verified. According to the numerical results, the acquired solutions feature good accuracy. In addition, one may run the present technique by increasing $m2$ and $m1$ up to the points where the results settle down to a desirable accuracy.

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