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ACCURACY OF THE ANALYTICAL AND FINITE ELEMENT MODELS FOR CIRCULAR BENDING PLATES

Alice MARINESCU[1][1a], Ștefan SOROHAN[2][2], Traian CICONE[3][1b]

The present work had two main purposes. The first one consisted in assessing the limits of applicability of analytical models for thin circular bending plates of constant thickness under two boundary conditions (simply supported and clamped), subjected to constant pressure on a single face. The analytical results were compared with more accurate results, obtained using adequate finite element models for a large range of input parameters. Small and large displacement options were considered within the analyses. The second objective was computing the errors between the analytical and finite element models – useful when deciding which model to use for a particular analysis.

Keywords: circular plates, bending, non-linear, FEM.

1. Introduction

Since its earliest applications in the 1940s, the finite element method (FEM) has gained over the past couple of decades a great notoriety, especially for problems requiring intricate structural analyses, such as those related to the engineering design of structures. Normally, the problem-solving process by the means of FEs requires complicated mathematical models, but due to the evolution of dedicated software with user-friendly interfaces, the usage of FEM has been made accessible on a larger scale.

In a recently published paper [1], focused on analysing the behaviour of a compliant hydrostatic thrust bearing, the authors proposed a simplified analytical model, based on the *thin circular plate model*, also known as the Kirchhoff – Love model. The model assumes that a mid-surface plane can be used to represent a three-dimensional plate in a two-dimensional form. The following assumptions are considered within this theory [2]:

- in-plane deformations are null at small displacements;
- the straight lines normal to the mid-surface remain normal to the mid-surface after deformation;

[1] Eng, Dept. of Machine Elements and Tribology, University POLITEHNICA of Bucharest, Romania, e-mail: alice.marinescu@upb.ro

[2] Prof., Strength of Materials Dept., University POLITEHNICA of Bucharest, Romania stefan.sorohan@upb.ro

[3] Prof., Dept. of Machine Elements and Tribology, University POLITEHNICA of Bucharest, Romania, e-mail: , traian.cicone@upb.ro

- the normal stresses in the direction transverse to the plate are neglected.

Under these assumptions, the stresses and deformations of thin, plane, circular plates can be computed analytically for various constraints and loading cases [3].

These simplifying hypotheses are valid only for large values of the thickness-to-radius ratio, $\bar{t} = R/t$. However, for very thin plates and/or at high loads, the deflections of the thin plate are significant, the limits of applicability of the linear analytical model become questionable and a nonlinear (large deformations) approach should be considered. These limits, as well as the resulting errors, in terms of deflections and stresses, are further presented in this paper.

2. General aspects

Even though the subject is not novel, an extended bibliographical research suggested the lack of unanimous opinions regarding the limits of applicability of the thin plate model. Moreover, there could not be found any values of the relative errors produced when using this model.

According to literature, there is a lower limit of the relative thickness of the plate, expressed by the plate thickness-to-radius ratio, $\bar{t}$, which varies between different bibliographic sources. Moreover, between the thin plate and thick plate models, transition zones were defined using terms such as "moderately thick" or "moderately thin", as presented below. Within these intervals, the thin plate model can still be used, with acceptable errors. However, the magnitude of these errors cannot be found in literature.

For instance, according to Steele et al. [4], plates could be classified as:

- very thin, when $\bar{t} > 50$;
- moderately thin, when $10 < \bar{t} < 50$;
- thick, when $1.5 < \bar{t} < 10$;
- very thick, when $\bar{t} < 1.5$.

According to Szilard [5], the accuracy of the thin plate theory is proportional to the square of the plate thickness. The thin plate model is valid as long as the smaller lateral dimension of the plate is at least 10 times larger than the thickness of the plate *t*, which gives for circular plates $\bar{t} > 5$. For larger values of the plate thickness (moderately thick plates), the plate thickness-to-radius ratio is considered still acceptable, as long as it takes values in the interval: $2.5 < \bar{t} < 5$. If the deflections are small, they are underestimated with respect to the Kirchhoff-Love theory.

As stated in Ventsel et al. [6], plates can be divided into three categories:

- thick plates, when $\bar{t} < 8 \ldots 10$;
- membranes, when $\bar{t} > 80 \ldots 100$;
- thin plates, when $8 \ldots 10 < \bar{t} < 80 \ldots 100$.

Reddy et al. [7] give a very minimalistic description in regard to the thin plates limitations, stating that the thin plate model is valid for plates having the thickness-to-side (radius) ratio $\bar{t}$ > 15.

A second limitation of the thin plate model is given by the maximum deflection-to-plate thickness ratio, $\bar{\delta}_{max} = \delta_{max}/t$, which will be further referred to as dimensionless deflection. This limitation is related to whether the model is linear (small deflections) or nonlinear (large deflections). Even in this case, there is no clear information on the percentual differences between the two models – linear and nonlinear.

As stated in Szilard [5], in order to apply the small deflection theory, the dimensionless deformation should be $\bar{\delta}_{max} < 0.1$. On the other hand, Striz [8] has shown that this limit can be extended up to 0.3 for clamped plates or 0.5 for simply supported plates.

According to [4], the Kirchhoff linear plate theory yields sufficiently accurate results for maximum stresses if $\bar{\delta}_{max} < 0.2$. Obviously, this limitation depends on the applied load, thus it is of interest to express it in terms of a dimensionless load. Hence a load parameter K, function of the pressure p, Young Modulus E, and radius-to-plate thickness ratio $\bar{t}$, was introduced:

$$K = \frac{p}{E}\left(\frac{R}{t}\right)^4 = \frac{p}{E}\bar{t}^4 \quad (1)$$

Striz et al. [8] give limits for this load parameter, as well as comparisons with nonlinear models, but without any given indications for the accuracy in terms of relative errors.

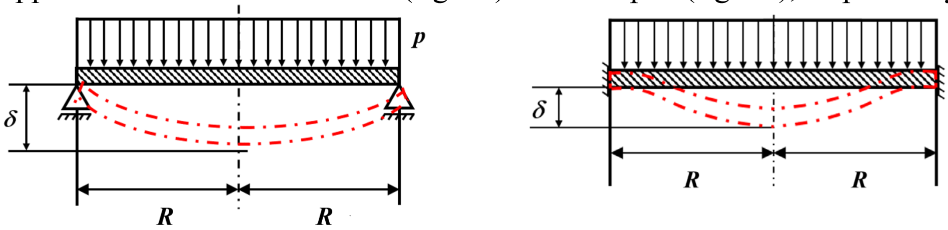
The inexistence of analytical solutions for the exact problem of both linear and nonlinear elasticity introduces a supplementary burden in the problem of defining a reference for evaluating the errors of the thin plate model.

Due to the time period when the first such analyses have been performed, characterized by the lack of powerful computational machines and usage of rudimental finite elements, the references considered reliable within these analyses can no longer be considered “exact” with respect to today’s standards, due to the emergence of new and more performant types of finite elements.

The current paper targets to find the validity limits and the accuracy of the analytical thin plate models in comparison with FE simulations, when the plates are subjected to a uniformly distributed pressure.

3. Analytical models

For both the analytical and finite element models, the analyses were performed for thin, circular plates, made of a homogeneous and isotropic material, subjected to uniform transverse pressure. Since the plates were loaded with uniformly distributed pressures, this led to an axi-symmetric problem. The analyses were focused on two different geometrical constraints of the plates: simply supported on the outer diameter (fig. 1a) and clamped (fig. 1b), respectively.



(a) (b)

Fig. 1 Thin circular plate: (a) Simply supported; (b) Clamped

Furthermore, for each of the two boundary conditions, the corresponding analytical models were compared with finite element models (considering both linear and nonlinear approaches), depending on the ratio between the maximum dimensionless deflection, $\bar{\delta}_{max}$.

3.1. Linear analytical solutions

The current analysis is restrained to the linear thin plate model, which has closed-form analytical solutions. The plates were assumed to be perfectly flat and maintaining a uniform and constant thickness prior to and after bending occurred. The analytical solutions for deformations and stresses, produced by a uniformly distributed pressure, could be found, for instance, in [3].

Dimensionless parameters, function of the input data (i.e. Young modulus, Poisson ratio, pressure, yield stress, plate thickness, plate radius), have been used in order to reduce the number of variables and cover a larger interval of values. Subsequently, for the sake of a better understanding, the equations of the classical analytical models, which are briefly presented herein, were chiselled and rewritten using particular notations.

The maximum deflection, written in dimensionless form for the simply supported circular plate, is:

$$\bar{\delta}_{s\_max} = K\frac{3}{16}(1-\upsilon)(5+\upsilon) \quad (2)$$

and for the clamped case:

$$\bar{\delta}_{c\_max} = K\frac{3}{16}(1 - \upsilon^2) \qquad (3)$$

Where $\bar{\delta}_{s\_max} = \frac{\delta_{s\_max}}{t}$ and $K = \frac{p}{E}\left(\frac{R}{t}\right)^2$.

The relative bending stresses in the centre of the plate, in radial and circumferential directions, which ought to be theoretically equal, were computed using equation (4) for the simply supported case:

$$\bar{\sigma}_{b\_s} = K\frac{3}{8}(3 + \upsilon) \qquad (4)$$

and equation (5) for the clamped case:

$$\bar{\sigma}_{b\_c} = K\frac{3}{8}(1 + \upsilon) \qquad (5)$$

where $\bar{\sigma}_b$ is the dimensionless bending stress, whose "s" and "c" subscripts stand for simply supported and clamped, respectively.

3.2. Non-linear analytical solutions

The nonlinear analysis of thin plates is more complex and includes a supplementary assumption for the boundary conditions, which could be either movable or immovable [2]; the corresponding results for stresses and deformations are quite different. In the present paper, the analysis was limited to considering immovable edges.

However, the analytical solutions for nonlinear models for thin plates are approximate, since they are based on restrictive hypotheses. Timoshenko et al. [2] give in their book approximate solutions for thin, clamped plates, loaded with uniform pressures that produce large deflections (nonlinear model). According to their solutions, the maximum deflection is reduced with respect to the linear model, with a factor depending on the square relative deflection:

$$\bar{\delta}_{max} = \frac{3}{16}K(1 - \upsilon^2)\frac{1}{1+0.488\bar{\delta}_{max}^2} \qquad (6)$$

where the correction factor on the right-hand side represents the effect of the middle-surface stretching on the deflection. This effect corroborates the increase of rigidity with the increase in deflection, due to the fact that the latter is no longer proportional with the bending load intensity. From eq. (6), it results that for $\bar{\delta}_{max}$=0.5, the value of the load parameter *K* will increase by 11% for the non-linear case. However, this equation is valid only for a transitional zone between linear and nonlinear loading conditions.

A more general implicit formulation of stresses and deformations was obtaine by a series method proposed by K. Federhofer and H. Egger and can be found in Timoshenko et al. [2]. According to their solution, the bending stresses and deformations can be calculated with the following equations:

$$\bar{\delta}_{\max} + A\bar{\delta}_{\max}^{3} = B\,K \tag{7}$$

$$\bar{\sigma}_{\max} = \beta\bar{\delta}_{\max}^{2} \tag{8}$$

where *A, B* and β are constants given in table 1

Table 1

	A	B	β
Simply supported immovable edges	1.852	0.696	1.778
Clamped immovable edges	0.471	0.171	2.86

Normally, the radial and circumferential stresses are expressed as a sum of bending and membrane stresses, but since the current analytical approach gives an approximate solution, only the bending stress component shall be further taken into account for the sake of simplicity.

4. Finite element models

The finite element model was implemented by running the Mechanical APDL module of the ANSYS® [9] software package, in order to compute the deflections and bending stresses in the centre of the plates, through both linear and nonlinear analyses.

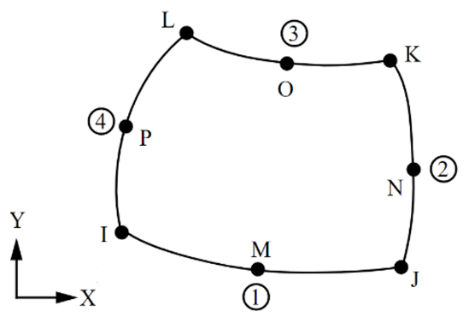
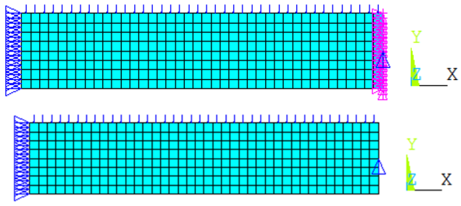


Fig. 2 PLANE183 finite element

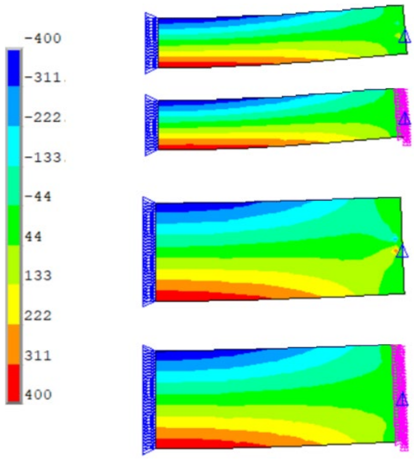
PLANE183 2D axi-symmetric quad elements (fig. 2), defined by 8 nodes, each node having two degrees of freedom – translations in the radial and axial directions – were used for setting the limits of applicability of the classical, analytical model.

In the first instance, the analysis was performed using the previously mentioned finite element in a static linear hypothesis, followed by a nonlinear analysis for the same finite element. A maximum allowable stress σ= 400MPa in the centre of the plates was chosen as reference for further computations. The PLANE183 element in nonlinear analysis was used as a reference when computing errors both for the linear finite elements and for the analytical models.

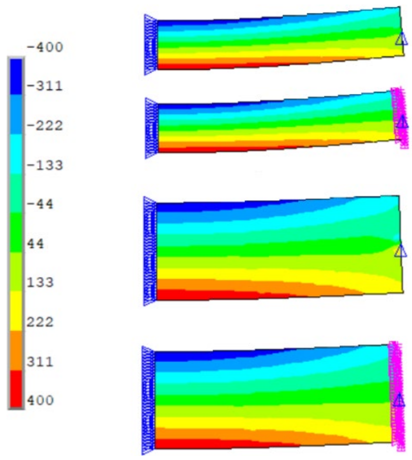
The boundary conditions applied for the simply supported plate implied constraining the plate in radial direction along the axis of symmetry, and in axial direction, on the plate edge. A rigid region was implemented along the line where the plate has been constrained in axial direction, in order to emulate the analytical boundary conditions (fig 3a). However, there exists a simpler alternative to the latter boundary condition, that implies constraining the plate in axial direction only in a point situated in the mid-plane (fig. 3b).



(a)
(b)
Fig. 3 Boundary conditions for the simply supported plate



(a)
(b)
(c)
(d)
Fig. 4 Radial stress distributions:
(a) thin plate; (b) thin plate with rigid region
(c) thick plate; (d) thick plate with rigid region



(a)
(b)
(c)
(d)
Fig. 5 Circumferential stress distributions:
(a) thin plate; (b) thin plate with rigid region
(c) thick plate; (d) thick plate with rigid region

Tests were performed for two plates having the same radius and different thicknesses (i.e. a *thin* and a *thick* plate), in order to verify whether the manner of applying the second BC for the simply supported case affects the results when subjecting the plate to bending. When comparing the results, one noticed that there were no significant changes in the radial (fig.4) and circumferential (fig.5) stress distributions, regardless of the manner of applying the boundary conditions, as long as the size of the finite elements is approximately *t*/8. However, if a very fine mesh is used, the simplified second BC placed according to fig. 3b can be used, especially for non-linear analyses.

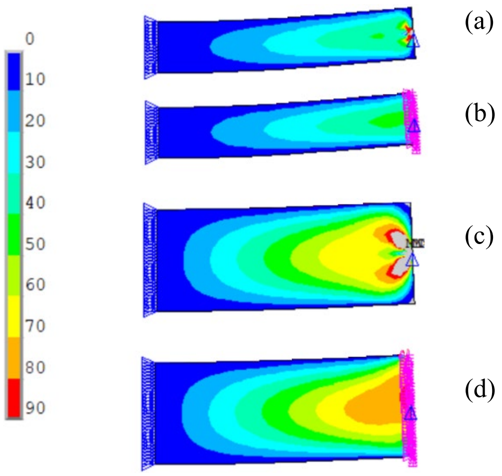


Fig. 6 Shear stress distributions: (a) thin plate; (b) thin plate with rigid region
(c) thick plate; (d) thick plate with rigid region

The shear stress distributions (fig. 6) presented sensible differences when applying the boundary conditions differently. The shear stress distribution is used when computing the deflection of thick plates. However, the current analysis did not count for a thorough analysis of shear stresses, these having been neglected.

For the clamped plate, the boundary conditions were applied as follows: in radial direction, along the axis of symmetry, and at the edge of the plate, on both translations (fig. 7).

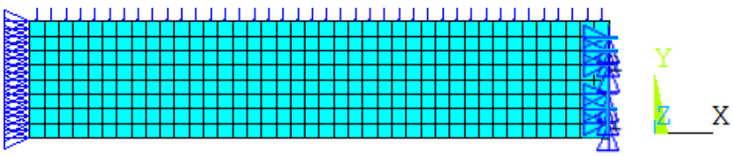


Fig. 7 Boundary conditions for the clamped plate.

Just as in the previous examples, tests were performed for two clamped plates, having the same radius and different thicknesses. When analysing fig. 8 and fig. 9, once could conclude that the radial and circumferential stress distributions are overlapping almost perfectly, even though slight singularities are present, since the maximum stress for clamped plates appears within the constraint areas. Grey areas can be noticed at the corners of both fig. 8a and fig. 8b; these are due to the fact that the imposed allowable stress of 400MPa in the centre of the plate was exceeded. Moreover, in the case of fig. 8b, these areas appear in the middle of the plate as well, which means that the analytically estimated load was more suitable for the thin plate model.

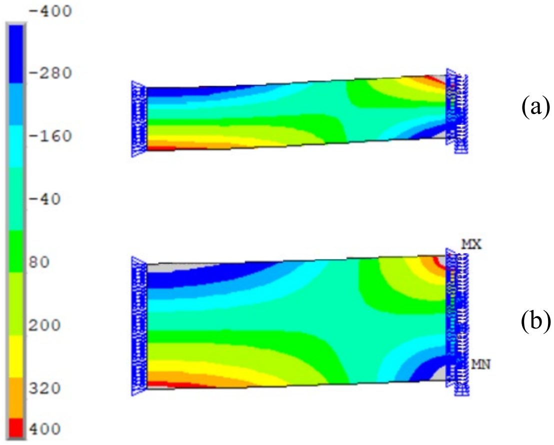


Fig. 8 Radial stress distributions: (a) thin plate (b) thick plate

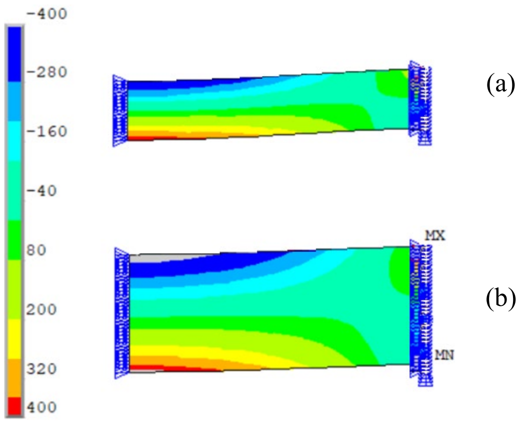


Fig. 9 Circumferential stress distributions: (a) thin plate (b) thick plate

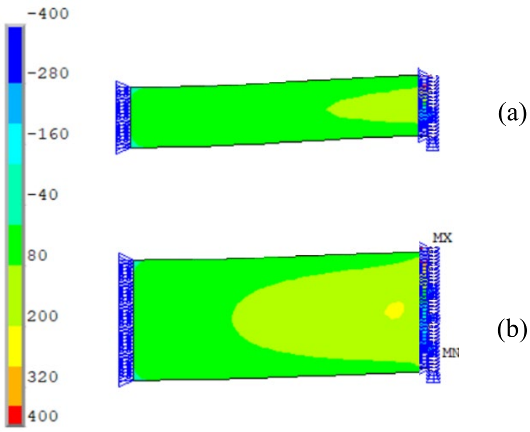


Fig.10 Shear stress distributions: (a) thin clamped plate; (b) thick clamped plate

The shearing effect (fig. 10) was observed to be milder than in the case of simply supported plates.

3. Results and discussions

3.1 Mesh accuracy

In previous approaches from literature employing a finite element analysis of circular plates [10], there has been established a clear correlation between the size of the mesh and the accuracy of the model, that is, the convergence of FEM with the analytical solution. As expected, a higher accuracy requires a compelling compromise - the smaller the mesh size, the longer the computational time required.

The accuracy analysis implied using PLANE183 finite elements, implemented linearly and non-linearly for both clamped and simply supported plates. The analysis comprised two stages.

Thus, the first stage consisted in a successive refinement of the mesh elements after each new program run. This refinement consisted in doubling the number of elements both across the thickness and along the radius of the 2D model. Meshes having aspect ratios (AR) of 1 and 5 were implemented for a value of the relative plate thickness $\bar{t} = 50$, situated at the extreme boundary of the thin plate model range ($\bar{t} \in (2 \dots 50)$).

The relative errors were computed with the following formula:

$$\varepsilon = \left|\frac{a_{ref} - a_i}{a_{ref}}\right| \tag{9}$$

where a_{ref} represents the value obtained for the finest mesh with respect to a certain aspect ratio, and a_i takes successively the values corresponding to the subsequent, coarser meshes.

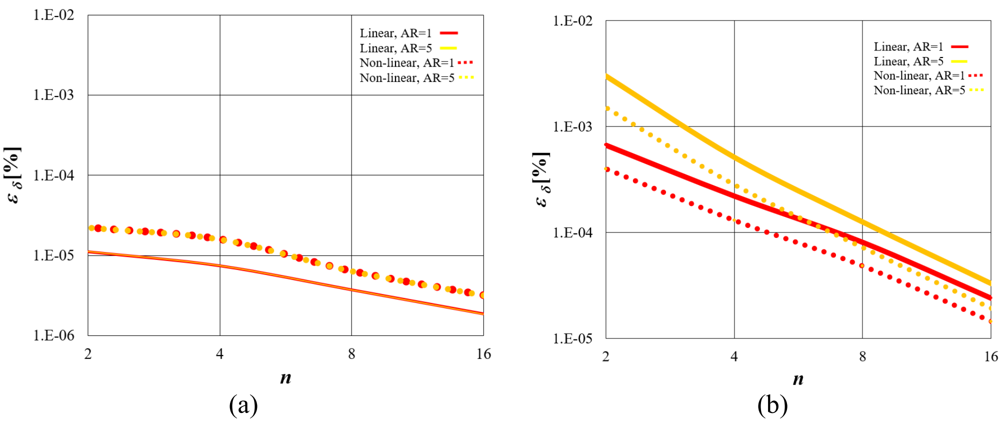


Fig. 11 Relative error of maximum deflection function of the number of elements along the thickness direction

(a) Simply supported plate; (b) Clamped plate

Figure 11 displays the errors for deflections (ε_δ) with respect to the number of elements *n,* across the thickness of the plate. An overlapping of the curves for the two aspect ratios (fig. 11a) can be observed for the case of the simply supported plates.

For the clamped plate (fig. 11b), the curves do not overlap, and the errors are slightly higher than for the simply supported case; even so, since the errors are in a reasonable range, the previous statement remains valid.

The deviations from linearity from fig. 11b are produced by local singularities.

By comparing these results, one arrived to the conclusion that the factor which influenced the accuracy of the results was the number of elements across the thickness of the plate, regardless of the aspect ratio of the finite elements employed, or the number of elements along the radius.

Thus, one could conclude that reasonable errors (under 10^{-4}) could be obtained for a number of at least 8 elements per plate thickness for both the simply supported and clamped plates.

For the sake of using less computational time, one could opt for a higher aspect ratio, corresponding to less elements along the radius, while considering using a reasonable number of elements along the plate thickness, as stated previously.

3.2 Comparison between the analytical and FE models

The limitations of the analytical models were emphasised by an analysis relying on the dimensionless parameters $\bar{t}$, $\bar{\delta}$ and *K*. The simulations were made for two levels of pressure loading selected to produce maximum bending stresses around 100MPa and 400MPa, respectively, for values of the Young modulus E= 210GPa and Poisson ratio ν= 0.3. As expected, the values computed with respect to the two loading cases overlapped while being represented in dimensionless form. Therefore, the following graphical representations were traced only with respect to the stress of 400 MPa.

The graphs shown in fig. 12 represents synthetically the most important result of this work: the maximum deformation and the maximum normal stress function of the load (in dimensionless form) predicted by linear and non-linear FE analyses, compared with the analytical solution.

When putting in contrast the analytical and the FE models, one could note that, under the small deflections assumption, there are different limitations for deflections and stresses for the same constraint.

Thus, at a close inspection of the graphs from fig. 12, one could conclude that, for deflections (fig. 12a), the limits where the targeted models overlap are around the values $\bar{\delta} = 0.5$ (K= 0.75) for the simply supported case, and $\bar{\delta} = 0.3$

(K= 2) for the clamped case. For the stresses (fig. 12b), the limits are situated approximately at $\bar{\sigma} = 1$ (K= 0.85), for the simply supported plate and $\bar{\sigma} = 1.6$ (K= 3.5), for the clamped plate.

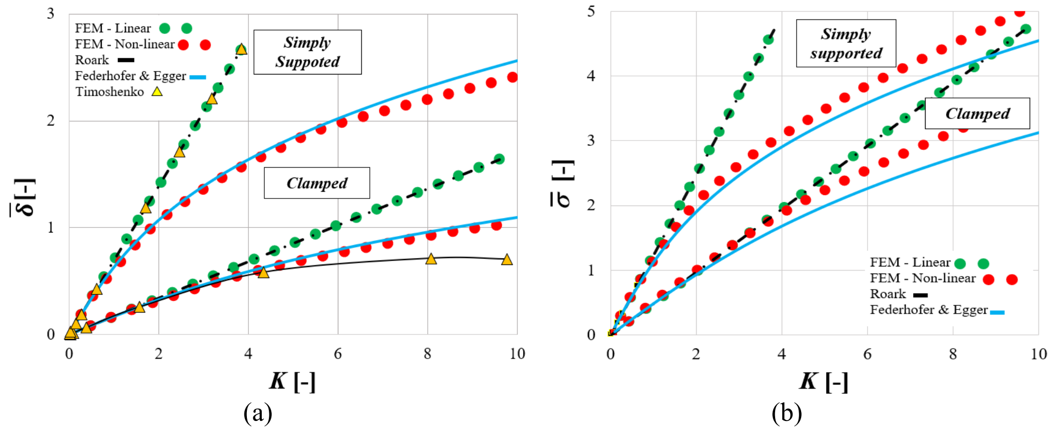


Fig. 12 Comparisons between the linear and nonlinear FE and the analytical model for clamped and simply supported plates for: (a) deflections; (b) stresses

The graphs from figure 13 depict the errors computed for deflections and stresses, for the analytical model – fig. 13 (a), (c) – and for the linear finite element model – fig. 13 (b), (d). The non-linear finite element model was used as a reference model when computing the errors. The increase in the errors situated on the extremity of left-hand sides, both for the case of deflections – fig. 13 (a), (b) – and for stresses – fig. 13 (c), (d) – are due to the fact that the values obtained exceed the assumptions of the small plate model. On the right-hand sides of the two graphs, the errors are increasing due to the fact that the plate falls in the large deflection model. Therefore, one could conclude that the transition zone between the linear and nonlinear models is in the zones where the curves come closer together or intersect.

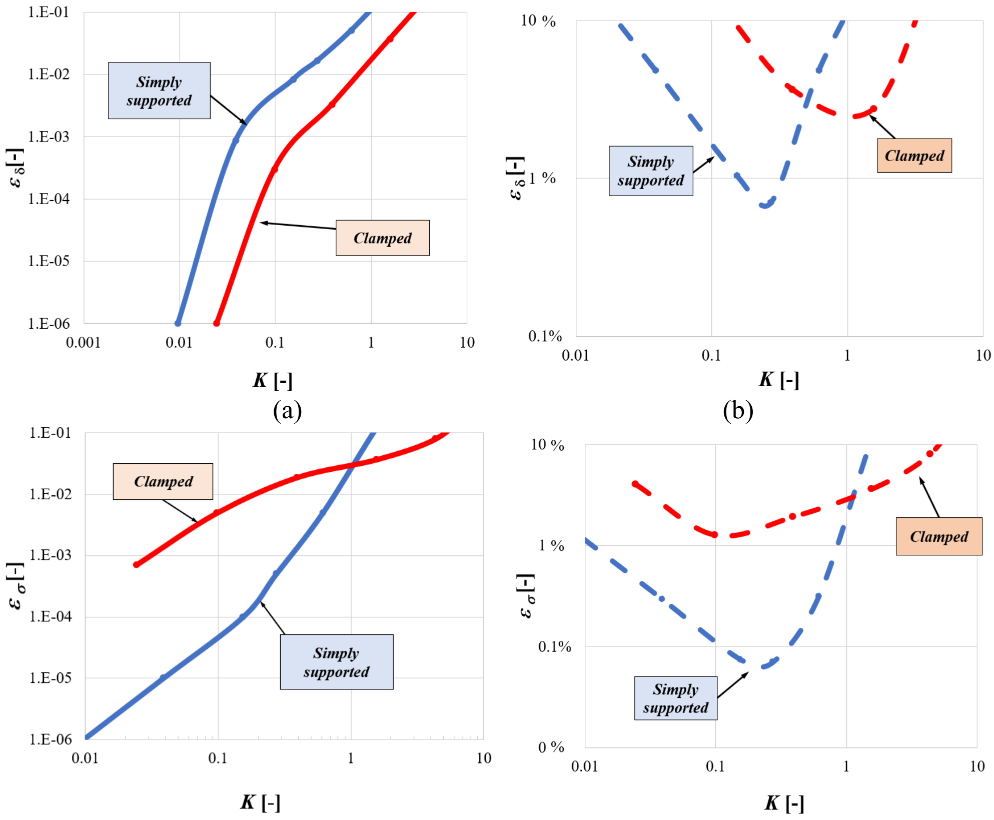


Fig. 13 Errors for: (a), (b) deflections; (c), (d) stresses

4. Conclusions

For optimization problems, an analytical approach involving simplified analytical relations could be more beneficial than using a more complex, numerical approach. Using an analytical approach could help reducing considerably the working time and the level of dificulty of the problem-solving algorithm. Therefore, in order to have reasonable errors of the analytical model with respect to its FEM equivalent, it is of paramount importance knowing the limits of the input values.

The current paper managed settling the limits of applicability for thin plates, both from a relative thickness and a deflection point of view (i.e. whether the plate is in the small deflections or large deflections range), as well as assessing the accuracy limits of the approximate analytical model with respect to the more precise, finite element model. Moreover, the paper provides errors for the maximum deflections and the stresses in the centre of the plates, which are extremely useful when establishing the transition zones of plates with respect to the radius-to-plate thickness ratio.

The limits of accuracy of the thin plate model are broader when computing stresses, in comparison to the case when deflections are targeted.

It was observed that the analytical model for the simply supported plate gives errors for the deflections less than 1% with respect to the nonlinear FE model for values of K within the interval $0.15 \div 0.33$. The analytical model for the camped plate has higher values of these errors, which exceed 1% regardless of the values of the load parameter K. If one was to extend the limit of these errors up to a typical engineering error of 3%, the clamped plate model could be used for values of K within the interval $0.4 \div 1.75$, while the reliable interval for the simply supported model would be extended to a K between $0.054 \div 0.53$.

If keeping in mind the same 3% error limitation as in the previous case when analysing the accuracy of the analytical model for the simply supported plate for stresses, the parameter K ought to take values in the interval 0.01...0.75. For the clamped plate, the parameter K should be situated between 0.036...0.98.

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