ON NEW EXISTENCE RESULTS OF FRACTIONAL DIFFERENTIAL INCLUSIONS VIA SET-VALUED JS-CONTRACTIONS

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Dedicated to the memory of my father Nurettin Işık (1958-1991)

In the present paper, we introduce a new type of multivalued JS-contractions by omitting one of the conditions on the auxiliary function and establish some fixed point theorems for such contractions on complete metric spaces. We derive many existing fixed point results in the literature by means of our results. We support the results obtained herein with some nontrivial examples. We also apply our results to prove the existence of solutions for fractional differential inclusions.

Keywords: JS-contraction, multivalued map, fixed point, fractional differential inclusion, nonlocal boundary condition.

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1. Introduction

Fixed point theory is one of the most significant and beneficial instruments in mathematical analysis on account of the fact that it purveys sufficient and necessary conditions of finding the existence and uniqueness of a solution of mathematical and practical problems which can be reduced to an equivalent fixed point problem. In particular, Banach contraction principle, which states that every contraction on a complete metric space has a unique fixed point, has a variety of applications in many branches of mathematics and other disciplines. This fundamental principle has been generalized in two main directions; either by generalizing the domain of the mapping or by weakening the contractive condition or sometimes even both. Some of those were studied by Berinde [5], Chatterja [7], Ćirić [8, 9], Hardy and Rogers [11], Kannan [14], Reich [20], Suzuki [21] and Zamfirescu [24]. In other respects, Nadler [17] extended Banach contraction principle from self-maps to multivalued mappings by using the notion of Hausdorff metric. The theory of multivalued mappings has various applications in optimal control theory, convex optimization, integral inclusions, fractional differential inclusions, economics and game theory. Recently, Jleli and Samet [13] introduced a new type of contractive self-maps known as JS-contraction and proved some fixed point theorems for such contractions by using a new technique of proof via the properties of the function. After then, several researchers extended the results in [13] to multivalued mappings in different directions, see for example, Nastasi and Vetro [18], Pansuwan *et al.* [19] and Vetro [22].

The aim of the present paper is to introduce a new class of contractions for multivalued mappings by weakening the conditions on the auxiliary function and to establish some fixed point theorems for such contractions on complete metric spaces. The obtained results improve and extend existing fixed point results in [8, 13, 17, 20, 22, 24] and many others. We

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supply some examples to illustrate the effectiveness of the new theory. Also, we apply our results to provide the existence of solutions for Caputo type fractional differential inclusions.

2. Preliminaries and Background

Here, we recollect some basic definitions, lemmas, notations and some known theorems which are helpful for the understanding of this paper. In the sequel, we will indicate the set of all non-negative real numbers and the set of all natural numbers by the letters $\mathbb{R}^+$ and $\mathbb{N}$, respectively. Let (Λ, d) be a metric space and denote the family of nonempty, closed and bounded subsets of Λ by $\mathcal{CB}(\Lambda)$. For $U, V \in \mathcal{CB}(\Lambda)$, define $H\colon \mathcal{CB}(\Lambda) \times \mathcal{CB}(\Lambda) \to \mathbb{R}^+$ by

$$H(U,V) = \max\left\{\sup_{u\in U} d(u,V),\ \sup_{v\in V} d(v,U)\right\}$$

where $d(u,V) = \inf\{d(u,\nu)\colon\ \nu \in V\}$. Such a function H is called the Pompeiu-Hausdorff metric induced by d, for more details, see [6]. Also, denote the family of nonempty and closed subsets of Λ by $\mathcal{CL}(\Lambda)$ and the family of nonempty and compact subsets of Λ by $\mathcal{K}(\Lambda)$. Note that $H\colon \mathcal{CL}(\Lambda) \times \mathcal{CL}(\Lambda) \to [0,\infty]$ is a generalized Pompeiu-Hausdorff metric, that is, $H(U,V) = \infty$ if $\max\{\sup_{u\in U}\ d(u,V),\ \sup_{v\in V}\ d(v,U)\}$ does not exist in $\mathbb{R}$.

Lemma 2.1 ([22])**.** *Let (Λ, d) be a metric space and $U, V \in \mathcal{CL}(\Lambda)$ with $H(U,V) > 0$. Then, for each $r > 1$ and for each $u \in U$, there exists $v = v(u) \in V$ such that $d(u,v) < rH(U,V)$.*

Following the results in [13], Vetro [22] established fixed point results for multivalued mappings.

Definition 2.1 ([13, 22])**.** Let (Λ, d) be a metric space. A map $\Upsilon\colon \Lambda \to \mathcal{CL}(\Lambda)$ is called a *JS-contraction* if there exist $r \in (0,1)$ and $\theta \in \mathcal{J}$ such that

$$\theta(H(\Upsilon\eta, \Upsilon\zeta)) \le [\theta(d(\eta,\zeta))]^r, \tag{1}$$

for all $\eta, \zeta \in \Lambda$ with $H(\Upsilon\eta, \Upsilon\zeta) > 0$, where $\mathcal{J}$ is the set of functions $\theta\colon (0,\infty) \to (1,\infty)$ satisfying the following conditions:

($\theta 1$) θ is non-decreasing;
($\theta 2$) for each sequence $\{h_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\ \theta(h_n) = 1$ if and only if $\lim_{n\to\infty}\ h_n = 0$;
($\theta 3$) there exist $\alpha \in (0,1)$ and $\beta \in (0,\infty]$ such that $\lim_{h\to 0^+} \dfrac{\theta(h)-1}{h^\alpha} = \beta$.

The following functions $\theta_i\colon (0,\infty) \to (1,\infty)$ for $i \in \{1,2\}$, are elements of $\mathcal{J}$. Furthermore, substituting in (1) these functions, we obtain some contractions known in the literature: for all $\eta, \zeta \in \Lambda$ with $H(\Upsilon\eta, \Upsilon\zeta) > 0$,

$$\theta_1(h) = e^{\sqrt{h}}, \qquad H(\Upsilon\eta, \Upsilon\zeta) \le r^2 d(\eta,\zeta),$$

$$\theta_2(h) = e^{\sqrt{he^h}}, \qquad \frac{H(\Upsilon\eta, \Upsilon\zeta)}{d(\eta,\zeta)} e^{H(\Upsilon\eta,\Upsilon\zeta) - d(\eta,\zeta)} \le r^2.$$

Theorem 2.1 ([22])**.** *Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{K}(\Lambda)$ be a JS-contraction. Then Υ has a fixed point, that is, there exists a point $\upsilon \in \Lambda$ such that $\upsilon \in \Upsilon\upsilon$.*

Note that Theorem 2.1 is invalid, if we take $\mathcal{CB}(\Lambda)$ instead of $\mathcal{K}(\Lambda)$. In [22], Vetro showed that Theorem 2.1 is still true for $\Upsilon\colon \Lambda \to \mathcal{CB}(\Lambda)$, whenever $\theta \in \mathcal{J}$ is right continuous.

3. Main Results

We will not be need the condition ($\theta 2$) in our results. Thence, we denote by $\mathfrak{J}$ the set of all fuctions θ satisfying the conditions ($\theta 1$) and ($\theta 3$). We can define the functions which belong to the set $\mathfrak{J}$ but not to $\mathcal{J}$ as shown in the following examples.

Example 3.1. *Define* $\theta\colon (0,\infty)\to(1,\infty)$ *with* $\theta(h)=e^{\sqrt{h}+1}$. *Evidently* θ *satisfies* $(\theta 1)$ *and since* $\lim_{h\to 0^+}(e^{\sqrt{h}+1}-1)/h^{\alpha}=\infty$ *for* $\alpha\in(0,1)$, *also* $(\theta 3)$. *However,* θ *does not satisfy the condition* $(\theta 2)$. *Indeed, consider* $h_n=\frac{1}{n}$ *for all* $n\in\mathbb{N}$, *then* $\lim_{n\to\infty}\ h_n=0$ *and* $\lim_{n\to\infty}\ \theta(h_n)=e\neq 1$. *Consequently,* $\theta\in\mathfrak{J}$ *while* $\theta\notin\mathfrak{I}$.

Example 3.2. *Let* $a>1$ *and* $\theta(h)=a+\ln(\sqrt{h}+1)$. *It can easily be seen that* θ *satisfies the conditions* $(\theta 1)$ *and* $(\theta 3)$. *But if we take* $h_n=\frac{1}{n}$ *for all* $n\in\mathbb{N}$, *then* $\lim_{n\to\infty} h_n=0$ *and* $\lim_{n\to\infty}\ \theta(h_n)=a>1$. *Hence,* $\theta\in\mathfrak{J}$ *and* $\theta\notin\mathfrak{I}$.

The next lemma will help us to make up for the lack of the condition $(\theta 2)$ in the proofs.

Lemma 3.1. *Let* $\theta\colon(0,\infty)\to(1,\infty)$ *be a non-decreasing function and* $\{h_n\}\subset(0,\infty)$ *a decreasing sequence such that* $\lim_{n\to\infty}\ \theta(h_n)=1$. *Then, we have* $\lim_{n\to\infty}\ h_n=0$.

Proof. Since the sequence $\{h_n\}$ is decreasing, there exists $h\geq 0$ such that $\lim_{n\to\infty} h_n=h$. Suppose that $h>0$. Considering the fact that θ is non-decreasing and $h_n\geq h$, we get $\theta(h_n)\geq\theta(h)$, for all $n\geq 0$. Taking the limit as $n\to\infty$ in the last inequality, we deduce $1=\lim_{n\to\infty}\ \theta(h_n)\geq\theta(h)$ which contradicts by the definition of θ, hence $h=0$. □

Now, following the lines in [10], we denote by $\mathcal{P}$ the set of all continuous mappings $\varrho\colon(\mathbb{R}^+)^5\to\mathbb{R}^+$ satisfying the following conditions:

($\varrho 1$) $\varrho(1,1,1,2,0),\varrho(1,1,1,0,2),\varrho(1,1,1,1,1)\in(0,1]$;

($\varrho 2$) ϱ is sub-homogeneous, that is, for all $(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\in(\mathbb{R}^+)^5$ and $\delta\geq 0$, we have

$$\varrho(\delta\eta_1,\delta\eta_2,\delta\eta_3,\delta\eta_4,\delta\eta_5)\leq\delta\varrho(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5);$$

($\varrho 3$) ϱ is a non-decreasing function, that is, for $\eta_i,y_i\in\mathbb{R}^+,\ \eta_i\leq\zeta_i,\ i=1,\dots,5$, we have

$$\varrho(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\leq\varrho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)$$

and if $\eta_i,\zeta_i\in\mathbb{R}^+,\ \eta_i<\zeta_i,\ i=1,\dots,4$, then

$$\varrho(\eta_1,\eta_2,\eta_3,\eta_4,0)<\varrho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,0)\quad\text{and}\quad\varrho(\eta_1,\eta_2,\eta_3,0,\eta_4,)<\varrho(\zeta_1,\zeta_2,\zeta_3,0,\zeta_4).$$

Then we have the next result.

Lemma 3.2. *If* $\varrho\in\mathcal{P}$ *and* $\upsilon,\nu\in\mathbb{R}^+$ *are such that*

$$\upsilon<\max\left\{\varrho(\nu,\nu,\upsilon,\nu+\upsilon,0),\varrho(\nu,\nu,\upsilon,0,\nu+\upsilon),\varrho(\nu,\upsilon,\nu,\nu+\upsilon,0),\varrho(\nu,\upsilon,\nu,0,\nu+\upsilon)\right\},$$

then $\upsilon<\nu$.

Proof. Without loss of generality, we can suppose that $\upsilon<\varrho(\nu,\nu,\upsilon,\nu+\upsilon,0)$. If $\nu\leq\upsilon$, then

$$\upsilon<\varrho(\nu,\nu,\upsilon,\nu+\upsilon,0)\leq\varrho(\upsilon,\upsilon,\upsilon,2\upsilon,0)\leq\upsilon\varrho(1,1,1,2,0)\leq\upsilon$$

which is a contradiction. Thus, we deduce that $\upsilon<\nu$. □

We are now ready to give the following definition.

Definition 3.1. *Let* (Λ,d) *be a metric space. A multivalued mapping* $\Upsilon\colon\Lambda\to\mathcal{CL}(\Lambda)$ *is called a JS-*ϱ*-contraction, if there exist* $\theta\in\mathfrak{J},\ \varrho\in\mathcal{P}$ *and* $r\in(0,1)$ *such that*

$$\theta(H(\Upsilon\eta,\Upsilon\zeta))\leq[\theta(\varrho(d(\eta,\zeta),d(\eta,\Upsilon\eta),d(\zeta,\Upsilon\zeta),d(\eta,\Upsilon\zeta),d(\zeta,\Upsilon\eta)))]^r,\tag{2}$$

for all $\eta,\zeta\in\Lambda$ *with* $H(\Upsilon\eta,\Upsilon\zeta)>0$.

Remark 3.1. *Let* (Λ,d) *be a metric space. If* $\Upsilon\colon\Lambda\to\mathcal{CL}(\Lambda)$ *is a JS-*ϱ*-contraction, then by (2), we get*

$$\begin{aligned}\ln\theta(H(\Upsilon\eta,\Upsilon\zeta))&\leq r\ln\theta(\varrho(d(\eta,\zeta),d(\eta,\Upsilon\eta),d(\zeta,\Upsilon\zeta),d(\eta,\Upsilon\zeta),d(\zeta,\Upsilon\eta)))\\&<\ln\theta(\varrho(d(\eta,\zeta),d(\eta,\Upsilon\eta),d(\zeta,\Upsilon\zeta),d(\eta,\Upsilon\zeta),d(\zeta,\Upsilon\eta))).\end{aligned}$$

Since θ is non-decreasing, we obtain

$$H(\Upsilon\eta,\Upsilon\zeta) < \varrho(d(\eta,\zeta), d(\eta,\Upsilon\eta), d(\zeta,\Upsilon\zeta), d(\eta,\Upsilon\zeta), d(\zeta,\Upsilon\eta)),$$

for all $\eta,\zeta \in \Lambda$ with $\Upsilon\eta \neq \Upsilon\zeta$. This implies that

$$H(\Upsilon\eta,\Upsilon\zeta) \leq \varrho(d(\eta,\zeta), d(\eta,\Upsilon\eta), d(\zeta,\Upsilon\zeta), d(\eta,\Upsilon\zeta), d(\zeta,\Upsilon\eta)), \text{ for all } \eta,\zeta \in \Lambda.$$

for all $\eta,\zeta \in \Lambda$.

The first result of this study is the following.

Theorem 3.1. *Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{K}(\Lambda)$ a JS-ϱ-contraction. Then Υ has a fixed point.*

Proof. Let η_0 be an arbitrary point of Λ and $\eta_1 \in \Upsilon\eta_0$. If $\eta_0 = \eta_1$ or $\eta_1 \in \Upsilon\eta_1$, then η_1 is a fixed point of Υ and so the proof is completed. Because of this, assume that $\eta_0 \neq \eta_1$ and $\eta_1 \notin \Upsilon\eta_1$, then $d(\eta_1, \Upsilon\eta_1) > 0$ and hence $H(\Upsilon\eta_0, \Upsilon\eta_1) > 0$. Since $\Upsilon\eta_1$ is compact, there exists $\eta_2 \in \Upsilon\eta_1$ such that $d(\eta_1,\eta_2) = d(\eta_1, \Upsilon\eta_1)$. Bearing in mind that the functions θ and ϱ are non-decreasing, by (2), we have

$$\begin{aligned}\theta(d(\eta_1,\eta_2)) &= \theta(d(\eta_1,\Upsilon\eta_1)) \leq \theta(H(\Upsilon\eta_0,\Upsilon\eta_1))\\ &\leq [\theta(\varrho(d(\eta_0,\eta_1), d(\eta_0,\Upsilon\eta_0), d(\eta_1,\Upsilon\eta_1), d(\eta_0,\Upsilon\eta_1), d(\eta_1,\Upsilon\eta_0)))]^r\\ &\leq [\theta(\varrho(d(\eta_0,\eta_1), d(\eta_0,\eta_1), d(\eta_1,\eta_2), d(\eta_0,\eta_1)+d(\eta_1,\eta_2), 0))]^r. \qquad (3)\end{aligned}$$

By Remark 3.1, this inequality implies that

$$d(\eta_1,\eta_2) < \varrho(d(\eta_0,\eta_1), d(\eta_0,\eta_1), d(\eta_1,\eta_2), d(\eta_0,\eta_1)+d(\eta_1,\eta_2), 0).$$

From Lemma 3.2, we get that $d(\eta_1,\eta_2) < d(\eta_0,\eta_1)$. Thus, using the properties of θ and ϱ in (3), we infer

$$\begin{aligned}\theta(d(\eta_1,\eta_2)) &\leq [\theta(\varrho(d(\eta_0,\eta_1), d(\eta_0,\eta_1), d(\eta_1,\eta_2), d(\eta_0,\eta_1)+d(\eta_1,\eta_2), 0))]^r\\ &< [\theta(\varrho(d(\eta_0,\eta_1), d(\eta_0,\eta_1), d(\eta_0,\eta_1), 2d(\eta_0,\eta_1), 0))]^r\\ &\leq [\theta(d(\eta_0,\eta_1)\varrho(1,1,1,2,0))]^r \leq [\theta(d(\eta_0,\eta_1))]^r.\end{aligned}$$

Following the previous procedures, we can assume that $\eta_1 \neq \eta_2$ and $\eta_2 \notin \Upsilon\eta_2$. Then $d(\eta_2,\Upsilon\eta_2) > 0$, and so $H(\Upsilon\eta_1,\Upsilon\eta_2) > 0$. Since $\Upsilon\eta_2$ is compact, there exists $\eta_3 \in \Upsilon\eta_2$ such that $d(\eta_2,\eta_3) = d(\eta_2,\Upsilon\eta_2)$. Considering ($\theta$1), ($\varrho$3) and (2), we get

$$\begin{aligned}\theta(d(\eta_2,\eta_3)) &= \theta(d(\eta_2,\Upsilon\eta_2)) \leq \theta(H(\Upsilon\eta_1,\Upsilon\eta_2))\\ &\leq [\theta(\varrho(d(\eta_1,\eta_2), d(\eta_1,\Upsilon\eta_1), d(\eta_2,\Upsilon\eta_2), d(\eta_1,\Upsilon\eta_2), d(\eta_2,\Upsilon\eta_1)))]^r\\ &\leq [\theta(\varrho(d(\eta_1,\eta_2), d(\eta_1,\eta_2), d(\eta_2,\eta_3), d(\eta_1,\eta_2)+d(\eta_2,\eta_3), 0))]^r, \qquad (4)\end{aligned}$$

follows by Remark 3.1 that

$$d(\eta_2,\eta_3) < \varrho(d(\eta_1,\eta_2), d(\eta_1,\eta_2), d(\eta_2,\eta_3), d(\eta_1,\eta_2)+d(\eta_2,\eta_3), 0).$$

Again from Lemma 3.2, we obtain that $d(\eta_2,\eta_3) < d(\eta_1,\eta_2)$. Thereby, using the properties of θ and ϱ in (4), we deduce

$$\begin{aligned}\theta(d(\eta_2,\eta_3)) &\leq [\theta(\varrho(d(\eta_1,\eta_2), d(\eta_1,\eta_2), d(\eta_2,\eta_3), d(\eta_1,\eta_2)+d(\eta_2,\eta_3), 0))]^r\\ &< [\theta(\varrho(d(\eta_1,\eta_2), d(\eta_1,\eta_2), d(\eta_1,\eta_2), 2d(\eta_1,\eta_2), 0))]^r\\ &\leq [\theta(d(\eta_1,\eta_2)\varrho(1,1,1,2,0))]^r \leq [\theta(d(\eta_1,\eta_2))]^r.\end{aligned}$$

Repeating this process, we can constitute a sequence $\{\eta_n\} \subset \Lambda$ such that $\eta_n \neq \eta_{n+1} \in \Upsilon\eta_n$ and

$$1 < \theta(d(\eta_n,\eta_{n+1})) < [\theta(d(\eta_{n-1},\eta_n))]^r, \qquad (5)$$

for all $n \in \mathbb{N}$. Letting $\sigma_n := d(\eta_n, \eta_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, from (5), we get

$$1 < \theta(\sigma_n) < [\theta(\sigma_0)]^{r^n}, \qquad \text{for all } n \in \mathbb{N}, \tag{6}$$

which implies that $\lim_{n\to\infty} \theta(\sigma_n) = 1$. On the other side, by the inequality (5), we know that the sequence $\{\sigma_n\}$ is decreasing and hence we can apply Lemma 3.1 to get $\lim_{n\to\infty} \sigma_n = 0$. Now, we claim that $\{\eta_n\}$ is a Cauchy sequence, for this, consider the condition ($\theta 3$). From ($\theta 3$), there exist $\alpha \in (0, 1)$ and $\beta \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\sigma_n) - 1}{(\sigma_n)^\alpha} = \beta. \tag{7}$$

Take $\lambda \in (0, \beta)$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$[\sigma_n]^\alpha \le \lambda^{-1}[\theta(\sigma_n) - 1], \qquad \text{for all } n > n_0.$$

Using (6) and the above inequality, we deduce

$$n[\sigma_n]^\alpha \le \lambda^{-1} n([\theta(\sigma_0)]^{r^n} - 1), \qquad \text{for all } n > n_0.$$

This implies that

$$\lim_{n\to\infty} n[\sigma_n]^\alpha = \lim_{n\to\infty} n[d(\eta_n, \eta_{n+1})]^\alpha = 0.$$

Thence, there exists $n_1 \in \mathbb{N}$ such that

$$d(\eta_n, \eta_{n+1}) \le \frac{1}{n^{1/\alpha}}, \qquad \text{for all } n > n_1. \tag{8}$$

Let $m > n > n_1$. Then, using the triangular inequality and (8), we have

$$d(\eta_n, \eta_m) \le \sum_{j=n}^{m-1} d(\eta_j, \eta_{j+1}) \le \sum_{j=n}^{m-1} \frac{1}{j^{1/\alpha}} \le \sum_{j=n}^{\infty} \frac{1}{j^{1/\alpha}}$$

and hence $\{\eta_n\}$ is a Cauchy sequence in Λ. From the completeness of (Λ, d), there exists $\upsilon \in \Lambda$ such that $\eta_n \to \upsilon$ as $n \to \infty$. We now show that υ is a fixed point of Υ. Suppose that $d(\upsilon, \Upsilon\upsilon) > 0$. Taking Remark 3.1 into account, we have

$$\begin{aligned}
d(\upsilon, \Upsilon\upsilon) &\le d(\upsilon, \eta_{n+1}) + d(\eta_{n+1}, \Upsilon\upsilon) \\
&\le d(\upsilon, \eta_{n+1}) + H(\Upsilon\eta_n, \Upsilon\upsilon) \\
&\le d(\upsilon, \eta_{n+1}) + \varrho(d(\eta_n, \upsilon), d(\eta_n, \Upsilon\eta_n), d(\upsilon, \Upsilon\upsilon), d(\eta_n, \Upsilon\upsilon), d(\upsilon, \Upsilon\eta_n)) \\
&\le d(\upsilon, \eta_{n+1}) + \varrho(d(\eta_n, \upsilon), d(\eta_n, \eta_{n+1}), d(\upsilon, \Upsilon\upsilon), d(\eta_n, \upsilon) + d(\upsilon, \Upsilon\upsilon), d(\upsilon, \eta_{n+1})).
\end{aligned}$$

Passing to limit as $n \to \infty$ in the above inequality, we obtain

$$d(\upsilon, \Upsilon\upsilon) \le \varrho(0, 0, d(\upsilon, \Upsilon\upsilon), 0 + d(\upsilon, \Upsilon\upsilon), 0),$$

which implies by Lemma 3.2 that

$$0 < d(\upsilon, \Upsilon\upsilon) < 0,$$

which is a contradiction. Hence $d(\upsilon, \Upsilon\upsilon) = 0$. Since $\Upsilon\upsilon$ is closed, we deduce that $\upsilon \in \Upsilon\upsilon$. □

In the next theorem, we replace $\mathcal{K}(\Lambda)$ with $\mathcal{CB}(\Lambda)$ by considering an additional condition for the function θ.

Theorem 3.2. *Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{CB}(\Lambda)$ a JS-ϱ-contraction with right continuous function $\theta \in \mathfrak{J}$. Then Υ has a fixed point.*

Proof. Let $\eta_0 \in \Lambda$ and $\eta_1 \in \Upsilon\eta_0$. If $\eta_0 = \eta_1$ or $\eta_1 \in \Upsilon\eta_1$, then η_1 is a fixed point of Υ. Herewith, we assume that $\eta_0 \neq \eta_1$ and $\eta_1 \notin \Upsilon\eta_1$, and hence $d(\eta_1, \Upsilon\eta_1) > 0$. From (2), we get

$$\begin{aligned}\theta(d(\eta_1, \Upsilon\eta_1)) &\leq \theta(H(\Upsilon\eta_0, \Upsilon\eta_1))\\ &\leq [\theta(\varrho(d(\eta_0, \eta_1), d(\eta_0, \Upsilon\eta_0), d(\eta_1, \Upsilon\eta_1), d(\eta_0, \Upsilon\eta_1), d(\eta_1, \Upsilon\eta_0)))]^r\\ &\leq [\theta(\varrho(d(\eta_0, \eta_1), d(\eta_0, \eta_1), d(\eta_1, \Upsilon\eta_1), d(\eta_0, \eta_1) + d(\eta_1, \Upsilon\eta_1), 0))]^r,\end{aligned}$$

and so

$$d(\eta_1, \Upsilon\eta_1) < \varrho(d(\eta_0, \eta_1), d(\eta_0, \eta_1), d(\eta_1, \Upsilon\eta_1), d(\eta_0, \eta_1) + d(\eta_1, \Upsilon\eta_1), 0).$$

Then Lemma 3.2 gives that $d(\eta_1, \Upsilon\eta_1) < d(\eta_0, \eta_1)$. Thus, we obtain

$$\begin{aligned}\theta(d(\eta_1, \Upsilon\eta_1)) &\leq \theta(H(\Upsilon\eta_0, \Upsilon\eta_1))\\ &\leq [\theta(\varrho(d(\eta_0, \eta_1), d(\eta_0, \eta_1), d(\eta_1, \Upsilon\eta_1), d(\eta_0, \eta_1) + d(\eta_1, \Upsilon\eta_1), 0))]^r\\ &< [\theta(\varrho(d(\eta_0, \eta_1), d(\eta_0, \eta_1), d(\eta_0, \eta_1), 2d(\eta_0, \eta_1), 0))]^r\\ &\leq [\theta(d(\eta_0, \eta_1)\varrho(1, 1, 1, 2, 0))]^r\\ &\leq [\theta(d(\eta_0, \eta_1))]^r,\end{aligned}$$

and hence

$$\theta(H(\Upsilon\eta_0, \Upsilon\eta_1)) < [\theta(d(\eta_0, \eta_1))]^r.$$

By the property of right continuity of $\theta \in \mathfrak{J}$, there exists a real number $h_1 > 1$ such that

$$\theta(h_1 H(\Upsilon\eta_0, \Upsilon\eta_1)) \leq [\theta(d(\eta_0, \eta_1))]^r. \tag{9}$$

From

$$d(\eta_1, \Upsilon\eta_1) \leq H(\Upsilon\eta_0, \Upsilon\eta_1) < h_1 H(\Upsilon\eta_0, \Upsilon\eta_1),$$

by Lemma 2.1, there exists $\eta_2 \in \Upsilon\eta_1$ such that $d(\eta_1, \eta_2) \leq h_1 H(\Upsilon\eta_0, \Upsilon\eta_1)$. Thus, by (9), we infer that

$$\theta(d(\eta_1, \eta_2)) \leq \theta(h_1 H(\Upsilon\eta_0, \Upsilon\eta_1)) \leq [\theta(d(\eta_0, \eta_1))]^r.$$

Continuing in this manner, we build two sequences $\{\eta_n\} \subset \Lambda$ and $\{h_n\} \subset (1, \infty)$ such that $\eta_n \neq \eta_{n+1} \in \Upsilon\eta_n$ and

$$1 < \theta(d(\eta_n, \eta_{n+1})) \leq \theta(h_n H(\Upsilon\eta_{n-1}, \Upsilon\eta_n)) \leq [\theta(d(\eta_{n-1}, \eta_n))]^r,$$

for all $n \in \mathbb{N}$. Hence,

$$1 < \theta(d(\eta_n, \eta_{n+1})) \leq [\theta(d(\eta_0, \eta_1))]^{r^n}, \quad \text{for all } n \in \mathbb{N},$$

which gives that

$$\lim_{n\to\infty} \theta(d(\eta_n, \eta_{n+1})) = 1.$$

The rest of the proof is analogous with the proof of Theorem 3.1. □

The following corollaries express us that we can obtain various types of contractive multivalued mappings by using JS-ϱ-contraction.

Corollary 3.1. ([17]) *Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{CB}(\Lambda)$ (resp. $\mathcal{K}(\Lambda)$) a JS-contraction of Nadler type, that is, there exist $\theta \in \mathfrak{J}$ and $r \in (0, 1)$ such that*

$$\theta(H(\Upsilon\eta, \Upsilon\zeta)) \leq [\theta(d(\eta, \zeta))]^r, \quad \text{for all } \eta, \zeta \in \Lambda \text{ with } H(\Upsilon\eta, \Upsilon\zeta) > 0.$$

Then Υ has a fixed point.

Proof. Consider $\varrho \in \mathcal{P}$ given by $\varrho(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \eta_1$. Then Υ is a JS-ϱ-contraction and the result follows from Theorem 3.2 (resp. Theorem 3.1). □

Corollary 3.2. *([20]) Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{CB}(\Lambda)$ (resp. $\mathcal{K}(\Lambda)$) a JS-contraction of Reich type, that is, there exist $\theta \in \mathfrak{J}$, $r \in (0,1)$ and non-negative real numbers α, β, γ with $\alpha + \beta + \gamma \leq 1$ such that*

$$\theta(H(\Upsilon\eta, \Upsilon\zeta)) \leq [\theta(\alpha d(\eta, \zeta) + \beta d(\eta, \Upsilon\eta) + \gamma d(\zeta, \Upsilon\zeta))]^r,$$

for all $\eta, \zeta \in \Lambda$ with $H(\Upsilon\eta, \Upsilon\zeta) > 0$. Then Υ has a fixed point.

Proof. Consider $\varrho \in \mathcal{P}$ given by $\varrho(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3$. Then Υ is a JS-ϱ-contraction and the result follows from Theorem 3.2 (resp. Theorem 3.1). □

Corollary 3.3. *([8]) Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{CB}(\Lambda)$ (resp. $\mathcal{K}(\Lambda)$) a JS-contraction of Ćirić type I, that is, there exist $\theta \in \mathfrak{J}$ and $r \in (0,1)$ such that*

$$\theta(H(\Upsilon\eta, \Upsilon\zeta)) \leq [\theta(\max\{d(\eta, \zeta), d(\eta, \Upsilon\eta), d(\zeta, \Upsilon\zeta), \frac{1}{2}[d(\eta, \Upsilon\zeta) + d(\zeta, \Upsilon\eta)]\})]^r,$$

for all $\eta, \zeta \in \Lambda$ with $H(\Upsilon\eta, \Upsilon\zeta) > 0$. Then Υ has a fixed point.

Proof. Consider $\varrho \in \mathcal{P}$ given by $\varrho(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \max\{\eta_1, \eta_2, \eta_3, \frac{\eta_4+\eta_5}{2}\}$. Then Υ is a JS-ϱ-contraction and the result follows from Theorem 3.2 (resp. Theorem 3.1). □

Corollary 3.4. *([24]) Let (Λ, d) be a complete metric space and $\Upsilon\colon \Lambda \to \mathcal{CB}(\Lambda)$ (resp. $\mathcal{K}(\Lambda)$) a Zamfirescu type JS-contraction, that is, there exist $\theta \in \mathfrak{J}$ and $r \in (0,1)$ such that*

$$\theta(H(\Upsilon\eta, \Upsilon\zeta)) \leq [\theta(\max\{d(\eta, \zeta), \frac{1}{2}[d(\eta, \Upsilon\eta) + d(\zeta, \Upsilon\zeta)], \frac{1}{2}[d(\eta, \Upsilon\zeta) + d(\zeta, \Upsilon\eta)]\})]^r,$$

for all $\eta, \zeta \in \Lambda$ with $H(\Upsilon\eta, \Upsilon\zeta) > 0$. Then Υ has a fixed point.

Proof. Consider $\varrho \in \mathcal{P}$ given by $\varrho(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \max\{\eta_1, \frac{\eta_2+\eta_3}{2}, \frac{\eta_4+\eta_5}{2}\}$. Then Υ is a JS-ϱ-contraction and the result follows from Theorem 3.2 (resp. Theorem 3.1). □

4. **An Application**

Consider the following nonlocal integral boundary value problem of Caputo type fractional differential inclusion:

$$\begin{cases} {}^{\mathcal{C}}\mathcal{D}^{\delta}_{h_0}\eta(h) \in \mathcal{F}(h, \eta(h)), & h \in Q = [h_0, H], \quad n-1 < \delta < n, \\ \eta^{(j)}(\sigma) = c_j + \int\limits_{h_0}^{\sigma} \rho_j(s, \eta(s))ds, & j = 0, 1, \ldots, n-1, \quad \sigma \in (h_0, H), \end{cases} \tag{10}$$

where $\mathcal{F}\colon Q \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $\rho_j\colon Q \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $c_j \in \mathbb{R}$ and ${}^{\mathcal{C}}\mathcal{D}^{\delta}_{h_0}$ denotes the Caputo fractional derivative of order δ, $n = [\delta] + 1$, $[\delta]$ denotes the integer part of the real number δ.

We now recall some basic definitions of fractional calculus [15] and multivalued analysis [12]. We also refer the reader to [1, 2, 3, 4, 16, 23] for more details.

Let $\Lambda := \mathcal{C}(Q, \mathbb{R})$ be the Banach space of all continuous real valued functions defined on Q endowed with the norm defined by $\|\eta\| = \sup\{|\eta(h)| : h \in Q\}$. By $\mathcal{L}^1(Q, \mathbb{R})$, we denote the Banach space of all measurable functions $\eta : Q \to \mathbb{R}$ which are Lebesgue integrable endowed with the norm

$$\|\eta\|_{\mathcal{L}^1} = \int_{h_0}^{H} |\eta(h)|\, dh.$$

Definition 4.1. *The Riemann-Liouville fractional integral of order δ for a function $g \in \Lambda$ is given by*

$$I^{\delta}g(h) = \frac{1}{\Gamma(\delta)} \int_{h_0}^{h} (h-s)^{\delta-1} g(s)ds, \quad \delta > 0,$$

provided the right hand-side is point-wise defined on $[h_0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\delta)=\int_0^\infty h^{\delta-1}e^{-h}dh$.

Definition 4.2. *Let $g:[h_0,\infty)\to\mathbb{R}$ be such that $g\in\mathcal{AC}^n(Q,\mathbb{R})$, then the Caputo derivative of fractional order δ for g is defined by*

$$^{C}\mathcal{D}^{\delta}_{h_0}g(h)=\frac{1}{\Gamma(n-\delta)}\int_{h_0}^{h}(h-s)^{n-\delta-1}g^{(n)}(s)ds,\quad n-1<\delta<n,\quad n=[\delta]+1,$$

where $\mathcal{AC}^n(Q,\mathbb{R})$ is the space of all real valued functions $g(h)$ which have absolutely continuous derivative up to order $(n-1)$ on Q.

Definition 4.3. *A multivalued mapping $\mathcal{F}\colon Q\to\mathcal{K}(\mathbb{R})$ is called measurable if for every $\zeta\in\mathbb{R}$, the function*

$$h\to d(\zeta,\mathcal{F}(h))=\inf\{|\zeta-\xi|:\xi\in\mathcal{F}(h)\}$$

is measurable.

Definition 4.4. *Let $\mathcal{F}\colon Q\times\mathbb{R}\to\mathcal{K}(\mathbb{R})$ be a multivalued map and $\upsilon\in\Lambda$, then the set of selections of $\mathcal{F}(\cdot,\cdot)$, denoted by $\mathcal{S}_{\mathcal{F},\upsilon}$, is of lower semi-continuous type if*

$$\mathcal{S}_{\mathcal{F},\upsilon}=\{\theta\in\mathcal{L}^1(Q,\mathbb{R})\colon\theta(h)\in\mathcal{F}(h,\upsilon(h)),\ \textit{for almost each } h\in Q\}$$

is lower semi-continuous with nonempty closed and decomposable values.

In this section, we present an application of Theorem 3.1 in establishing the existence of solutions for problem (10). To define the solution of problem (10), let us consider its linear variant given by

$$\begin{cases} ^{\mathcal{C}}\mathcal{D}^{\delta}_{h_0}\eta(h)=\widetilde{g}(h), & h\in Q,\\ \eta^{(j)}(\sigma)=c_j+\int\limits_{h_0}^{\sigma}\rho_j(s)ds, & j=0,1,\ldots,n-1,\quad \sigma\in Q,\end{cases}\tag{11}$$

where $\eta\in\mathcal{AC}^n(Q,\mathbb{R})$, $\widetilde{g}\in\mathcal{AC}(Q,\mathbb{R})$ and $\rho_j\in\Lambda$.

Lemma 4.1. *([2]) The fractional nonlocal boundary value problem (11) is equivalent to the integral equation*

$$\eta(h)=I^{\delta}\widetilde{g}(h)+\sum_{j=0}^{n-1}\frac{(h-\sigma)^j}{j!}(c_j+\int_{h_0}^{\sigma}\rho_j(s)ds-I^{\delta-j}\widetilde{g}(\sigma)),\quad h\in Q.$$

Our hypotheses are on the following data :

(A) Let $\mathcal{F}\colon Q\times\mathbb{R}\to\mathcal{K}(\mathbb{R})$ be such that $\mathcal{F}(\cdot,\eta)\colon Q\to\mathcal{K}(\mathbb{R})$ is measurable for each $\eta\in\mathbb{R}$;

(B) for almost all $h\in Q$ and $\eta,\tilde{\eta}\in\mathbb{R}$ with $\lambda\in\mathcal{C}(Q,(0,\infty))$

$$H(\mathcal{F}(h,\eta),\mathcal{F}(h,\tilde{\eta}))\le\lambda(h)\,|\eta-\tilde{\eta}|$$

and $d(0,\mathcal{F}(h,0))\le\lambda(h)$;

(C) there exist functions $\mu_j\in\mathcal{C}(Q,(0,\infty))$ such that

$$|\rho_j(h,\eta)-\rho_j(h,\tilde{\eta})|\le\mu_j(h)\,|\eta-\tilde{\eta}|\,,$$

for $h\in Q,\ j=0,1,\ldots,n-1$ and $\eta,\tilde{\eta}\in\mathbb{R}$;

(D) there exists $\tau\in(0,\infty)$ such that

$$\phi_1\,\|\lambda\|+\phi_2\le e^{-\tau},$$

where

$$\phi_1=\left\{\frac{2}{\Gamma(\delta+1)}+\sum_{j=1}^{n-1}\frac{1}{j!\,\Gamma(\delta-j+1)}\right\}(H-h_0)^{\delta}$$

and

$$\phi_2 = \sum_{j=0}^{n-1} \frac{(H - h_0)^j \|\mu_j\|}{j!}.$$

We are now ready to present main result of this section.

Theorem 4.1. *Assume that the conditions* $(A) - (D)$ *hold. Then the fractional differential inclusion problem* (10) *has at least one solution on* Λ.

Proof. Using Lemma 4.1, define an operator $\Upsilon_{\mathcal{F}} \colon \Lambda \to \mathcal{P}(\Lambda)$ by

$$\Upsilon_{\mathcal{F}}(\eta) = \Bigg\{ \upsilon \in \Lambda \colon \upsilon(h) = \int_{h_0}^{h} \frac{(h-s)^{\delta-1}}{\Gamma(\delta)} g(s) ds \\ + \sum_{j=0}^{n-1} \frac{(h-\sigma)^j}{j!} \left(c_j + \int_{h_0}^{\sigma} \rho_j(s, \eta(s)) ds - \int_{h_0}^{\sigma} \frac{(\sigma - s)^{\delta - j - 1}}{\Gamma(\delta - j)} g(s) ds \right) \Bigg\}$$

for $g \in \mathcal{S}_{\mathcal{F},\eta}$. Note that the set $\mathcal{S}_{\mathcal{F},\eta}$ is nonempty for each $\eta \in \Lambda$ by assumption (A), so $\mathcal{F}$ has a measurable selection (see Theorem 3.6 in [12]). Also, $\Upsilon_{\mathcal{F}}(\eta)$ is compact for each $\eta \in \Lambda$. This is obvious since $\mathcal{S}_{\mathcal{F},\eta}$ is compact ($\mathcal{F}$ has compact values), and therefore we omit its proof. We now prove that $\Upsilon_{\mathcal{F}}$ is a JS-ϱ-contraction. Let $\eta, \tilde{\eta} \in \mathcal{C}(Q, \mathbb{R})$ and $\upsilon_1 \in \Upsilon_{\mathcal{F}}(\eta)$. Then there exists $\theta_1(h) \in \mathcal{F}(h, \eta(h))$ such that for all $h \in Q$, we obtain

$$\upsilon_1(h) = \int_{h_0}^{h} \frac{(h-s)^{\delta-1}}{\Gamma(\delta)} \theta_1(s) ds \\ + \sum_{j=0}^{n-1} \frac{(h-\sigma)^j}{j!} \left(c_j + \int_{h_0}^{\sigma} \rho_j(s, \eta(s)) ds - \int_{h_0}^{\sigma} \frac{(\sigma - s)^{\delta - j - 1}}{\Gamma(\delta - j)} \theta_1(s) ds \right).$$

By the assumption (B), we have

$$H(\mathcal{F}(h, \eta), \mathcal{F}(h, \tilde{\eta})) \leq \lambda(h) \left| \eta(h) - \tilde{\eta}(h) \right|.$$

So, there exists $k^\star \in \mathcal{F}(h, \tilde{\eta}(h))$ such that

$$\left| \theta_1(h) - k^\star \right| \leq \lambda(h) \left| \eta(h) - \tilde{\eta}(h) \right|, \quad h \in Q.$$

Define the operator $\Omega \colon Q \to \mathcal{P}(\mathbb{R})$ by

$$\Omega(h) = \{ k^\star \in \mathbb{R} \colon \left| \theta_1(h) - k^\star \right| \leq \lambda(h) \left| \eta(h) - \tilde{\eta}(h) \right| \}.$$

Since $\Omega(h) \cap \mathcal{F}(t, \tilde{\eta}(t))$ is measurable (see Proposition 3.4 in [12]), there exists a function $\theta_2(h)$ which is a measurable selection for Ω. Hence, $\theta_2(h) \in \mathcal{F}(h, \tilde{\eta}(h))$ and for all $h \in Q$,

$$\left| \theta_1(h) - \theta_2(h) \right| \leq \lambda(h) \left| \eta(h) - \tilde{\eta}(h) \right|.$$

Now, we define

$$\upsilon_2(h) = \int_{h_0}^{h} \frac{(h-s)^{\delta-1}}{\Gamma(\delta)} \theta_2(s) ds \\ + \sum_{j=0}^{n-1} \frac{(h-\sigma)^j}{j!} \left(c_j + \int_{h_0}^{\sigma} \rho_j(s, \tilde{\eta}(s)) ds - \int_{h_0}^{\sigma} \frac{(\sigma - s)^{\delta - j - 1}}{\Gamma(\delta - j)} \theta_2(s) ds \right).$$

It follows that, for all $h \in Q$

$$\begin{aligned}|\upsilon_1(h)-\upsilon_2(h)| \leq & \int_{h_0}^{h} \frac{(h-s)^{\delta-1}}{\Gamma(\delta)} |\theta_1(s)-\theta_2(s)|\, ds \\ & + \sum_{j=0}^{n-1} \frac{(h-\sigma)^j}{j!} \int_{h_0}^{\sigma} \frac{(\sigma-s)^{\delta-j-1}}{\Gamma(\delta-j)} |\theta_1(s)-\theta_2(s)|\, ds \\ & + \sum_{j=0}^{n-1} \frac{(h-\sigma)^j}{j!} \int_{h_0}^{\sigma} |\rho_j(s,\eta(s))-\rho_j(s,\tilde{\eta}(s))|\, ds \\ \leq & \left\{ \left\{ \frac{1}{\Gamma(\delta+1)} + \sum_{j=0}^{n-1} \frac{1}{j!\,\Gamma(\delta-j+1)} \right\} (H-h_0)^{\delta} \|\lambda\| \right. \\ & \left. + \sum_{j=0}^{n-1} \frac{(H-h_0)^j \|\mu_j\|}{j!} \right\} \|\eta-\tilde{\eta}\|,\end{aligned}$$

and so

$$\begin{aligned}|\upsilon_1(h)-\upsilon_2(h)| \leq & \left\{ \left\{ \frac{2}{\Gamma(\delta+1)} + \sum_{j=1}^{n-1} \frac{1}{j!\,\Gamma(\delta-j+1)} \right\} (H-h_0)^{\delta} \|\lambda\| \right. \\ & \left. + \sum_{j=0}^{n-1} \frac{(H-h_0)^k \|\mu_j\|}{j!} \right\} \|\eta-\tilde{\eta}\|.\end{aligned}$$

Thus, we obtain

$$\|\upsilon_1-\upsilon_2\| \leq (\phi_1 \|\lambda\| + \phi_2) \|\eta-\tilde{\eta}\| \leq e^{-\tau} \|\eta-\tilde{\eta}\|.$$

Now, by just interchanging the role of η and $\tilde{\eta}$, we reach to

$$H(\Upsilon_{\mathcal{F}}(\eta), \Upsilon_{\mathcal{F}}(\tilde{\eta})) \leq e^{-\tau} \|\eta-\tilde{\eta}\|. \tag{12}$$

Consider $\varrho \in \mathcal{P}$ and $\theta \in \mathfrak{J}$ given by $\varrho(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5) = \eta_1$ and $\theta(h) = e^{\sqrt{h}}$, respectively. Then, by (12), we infer

$$e^{\sqrt{H(\Upsilon_{\mathcal{F}}(\eta),\Upsilon_{\mathcal{F}}(\tilde{\eta}))}} \leq e^{\sqrt{e^{-\tau}\|\eta-\tilde{\eta}\|}} \leq \left[e^{\sqrt{\|\eta-\tilde{\eta}\|}}\right]^r,$$

which implies that

$$\begin{aligned}\theta(H(\Upsilon_{\mathcal{F}}(\eta), \Upsilon_{\mathcal{F}}(\tilde{\eta}))) \leq [\theta(\varrho(&\|\eta-\tilde{\eta}\|, \|\eta-\Upsilon_{\mathcal{F}}(\eta)\|, \|\tilde{\eta}-\Upsilon_{\mathcal{F}}(\tilde{\eta})\|, \\ &\|\eta-\Upsilon_{\mathcal{F}}(\tilde{\eta})\|, \|\tilde{\eta}-\Upsilon_{\mathcal{F}}(\eta)\|))]^r,\end{aligned}$$

for all $\eta, \tilde{\eta} \in \Lambda$, where $r = \sqrt{e^{-\tau}}$. Since $\tau > 0$, then $r \in (0,1)$. This means that $\Upsilon_{\mathcal{F}}$ is a JS-ϱ-contraction. Consequently, by Theorem 3.1, $\Upsilon_{\mathcal{F}}$ has a fixed point $\eta \in \Lambda$ which is a solution of the problem (10). □

Example 4.1. *Consider the fractional differential inclusion problem given by*

$$\begin{cases} {}^{\mathcal{C}}\mathcal{D}_0^{6.7}\eta(h) \in \mathcal{F}(h,\eta(h)), & h \in [0,1], \\ \eta^{(j)}(0.5) = 1 + \int\limits_0^{0.5} \frac{s^j}{3(j+1)} e^{-\eta(s)} ds, & j = 0,1,\ldots,6, \end{cases} \tag{13}$$

where $h_0 = 0$, $H = 1$, $\delta = 6.7$, $\sigma = 0.5$, $c_j = 1$, $\rho_j(h,\eta(h)) = \frac{h^j}{3(j+1)} e^{-\eta(h)}$ *and* $\mathcal{F}\colon [0,1]\times\mathbb{R} \to$

$\mathcal{P}(\mathbb{R})$ *is a multivalued mapping given by*

$$\mathcal{F}(h,\eta)=\left[0,\ \frac{h\,|\eta(h)|}{8\,(1+|\eta(h)|)}\right].$$

Note that

$$h\to\mathcal{F}(h,\eta)=\left[0,\ \frac{h\,|\eta(h)|}{8\,(1+|\eta(h)|)}\right]$$

is measurable for each $\eta\in\mathbb{R}$, *since both the lower and upper functions are measurable on* $[0,1]\times\mathbb{R}$. *Also*

$$|\rho_j(h,\eta)-\rho_j(h,\tilde{\eta})|\leq\frac{h^j}{3\,(j+1)}\left|e^{-\eta(h)}-e^{-\tilde{\eta}(h)}\right|.$$

Here $\mu_j(h)=\frac{h^j}{3\,(j+1)}$ *and so*

$$\|\mu_j\|=\frac{1}{3\,(j+1)},\ \textit{for } j=0,1,\ldots,6.$$

On the other hand, we infer that

$$\sup\{|\zeta|:\zeta\in\mathcal{F}(h,\eta)\}\leq\frac{h\,|\eta(h)|}{8\,(1+|\eta(h)|)}\leq\frac{1}{8},$$

for each $(h,\eta)\in[0,1]\times\mathbb{R}$, *and*

$$\begin{aligned}H(\mathcal{F}(h,\eta),\mathcal{F}(h,\tilde{\eta}))&=\left(\left[0,\ \frac{h\,|\eta(h)|}{8\,(1+|\eta(h)|)}\right],\left[0,\ \frac{h\,|\tilde{\eta}(h)|}{8\,(1+|\tilde{\eta}(h)|)}\right]\right)\\&\leq\frac{h}{8}\,|\eta-\tilde{\eta}|\,.\end{aligned}$$

Here $\lambda(h)=\frac{h}{8}$ *with* $\|\lambda\|\approx 0.125$. *Besides, we find that*

$$\phi_1=\frac{2}{\Gamma(7.7)}+\frac{1}{\Gamma(6.7)}+\frac{1}{\Gamma(5.7)}+\frac{1}{\Gamma(4.7)}+\frac{1}{\Gamma(3.7)}+\frac{1}{\Gamma(2.7)}+\frac{1}{\Gamma(1.7)}\approx 2.07,$$

$$\phi_2=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}\cdot\frac{1}{2}+\frac{1}{12}\cdot\frac{1}{6}+\frac{1}{15}\cdot\frac{1}{24}+\frac{1}{18}\cdot\frac{1}{120}+\frac{1}{21}\cdot\frac{1}{720}=0.5727,$$

and so

$$\phi_1\,\|\lambda\|+\phi_2\ \approx\ (2.07)\cdot(0.125)+0.5727=0.83145\leq e^{-\tau}$$

where $\tau\in\left(0,\frac{1}{6}\right]$.

Thus, all conditions of Theorem 4.1 are satisfied. The compactness of $\mathcal{F}$ *together with the above calculations lead to the existence of solution of the problem* (13) *by Theorem 4.1.*

5. **Conclusions**

In this paper, a new type of contractions has been proposed for multivalued mappings by weakening the conditions on θ and by using auxilary functions. New fixed point theorems have been derived for multivalued mappings on complete metric spaces by means of this new class of contractions, which generalize the results in [8, 13, 17, 20, 22, 24] and many others in the literature. To support of effectiveness and usability of new theory have been furnished several examples. Finally, sufficient conditions have been investigated to ensure the existence of solutions for the nonlocal integral boundary value problem of Caputo type fractional differential inclusions by using the results obtained herein.

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