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FRAMES OF DILATIONS

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In this paper the frames of dilation for $L_2(0,\infty)$ are introduced. The elements of this frames appears by the action of the special class of the dilation operators on a single element $\phi \in L_2(0,\infty)$. We find some sufficient or necessary conditions for Bessel condition. The problem that when this sequences are frame or frame sequence in $L_2(0,\infty)$ is open.

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1. Introduction

Given a separable Hilbert space $\mathcal{H}$ with inner product $< .,. >$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |< f, f_k >|^2 \leq B\|f\|^2, \tag{1}$$

where A, B are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. If $A = B$, then $\{f_k\}_{k=1}^{\infty}$ is called a tight frame[1, 2, 4, 7]. A sequence $\{f_k\}_{k=1}^{\infty}$ in Hilbert space $\mathcal{H}$ is called a frame sequence in $\mathcal{H}$ if it is a frame for Hilbert space $\overline{span}\{f_k\}_{k=1}^{\infty}$. For more information concerning frames refer to[3, 5, 6]

For any $x \in \mathbb{R}$ and $y \in \mathbb{R} - \{0\}$ the translation operator T_x and dilation operator D_y on $L_2(\mathbb{R})$ are defined by

$$(T_x f)(t) = f(t - x), \quad (D_y f)(x) = \frac{1}{\sqrt{|y|}} f(\frac{x}{y}).$$

Frames of translates are frame sequences in $L_2(\mathbb{R})$ of the form $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$, where $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a sequence in $\mathbb{R}$ and $\phi \in L_2(\mathbb{R})$ [2].
The following example was a motivation to use the dilation operator on $L_2(0,\infty)$ instead of translation operator in the definition of the frames of translates.

Example 1.1. *The sequence $\{D_{a^j}\chi_{(1,a)}\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$ with Bessel bound $a - 1$, where $a > 1$, D_{a^j} is the dilation operator by a^j and $\chi_{(1,a)}$ is the*

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characteristic function on interval $(1, a)$*. Indeed, for all* $f \in L_2(0, \infty)$ *we have*

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\left\langle f, D_{a^j}\chi_{(1,a)}\right\rangle|^2 &= \sum_{j\in\mathbb{Z}} a^j |\int_1^a f(a^j x)dx|^2 \\
&= \sum_{j\in\mathbb{Z}} a^{-j} |\int_{a^j}^{a^{j+1}} f(x)dx|^2 \\
&\leq \sum_{j\in\mathbb{Z}} a^{-j} \int_{a^j}^{a^{j+1}} |f(x)|^2 dx \int_{a^j}^{a^{j+1}} 1^2 dx \\
&= (a-1)\sum_{j\in\mathbb{Z}} \int_{a^j}^{a^{j+1}} |f(x)|^2 dx \\
&= (a-1)\int_0^\infty |f(x)|^2 dx \\
&= (a-1)\|f\|^2.
\end{aligned}$$

Also the sequence $\{D_{a^j}\chi_{(1,a)}\}_{j\in\mathbb{Z}}$ *is a tight frame sequence in* $L_2(0,\infty)$ *with frame bound* $a-1$*. Indeed, if* $f \in span\{D_{a^j}\chi_{(1,a)}\}_{j\in\mathbb{Z}}$*, then* $f = \sum_{i\in I} c_i D_{a^i}\chi_{(1,a)}$*, where* I *is a finite subset of* $\mathbb{Z}$ *and* $c_i \in \mathbb{R}$ *for all* $i \in I$*. Thus by using the fact* $D_y^* = D_{\frac{1}{y}}$*,* $D_x D_y = D_{xy}$ *and* $D_{a^i}\chi_{(1,a)}(x) = 0$ *for all* $i \neq 0$*,* $x \in (1,a)$ *we have*

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\left\langle f, D_{a^j}\chi_{(1,a)}\right\rangle|^2 &= \sum_{j\in\mathbb{Z}} |\left\langle \sum_{i\in I} c_i D_{a^i}\chi_{(1,a)}, D_{a^j}\chi_{(1,a)}\right\rangle|^2 \\
&= \sum_{j\in\mathbb{Z}} |\sum_{i\in I} c_i \left\langle D_{a^{i-j}}\chi_{(1,a)}, \chi_{(1,a)}\right\rangle|^2 \\
&= \sum_{j\in\mathbb{Z}} |\sum_{i\in I} c_i \int_0^\infty D_{a^{i-j}}\chi_{(1,a)}(x)\chi_{(1,a)}(x)dx|^2 \\
&= \sum_{j\in\mathbb{Z}} |\sum_{i\in I} c_i \int_1^a \frac{1}{\sqrt{a^{i-j}}}\chi_{(1,a)}(a^{j-i}x)dx|^2 \\
&= \sum_{j\in I} |\sum_{i\in I} c_i \int_1^a \frac{1}{\sqrt{a^{i-j}}}\chi_{(1,a)}(a^{j-i}x)dx|^2 \\
&+ \sum_{j\in\mathbb{Z}\setminus I} |\sum_{i\in I} c_i \int_1^a \frac{1}{\sqrt{a^{i-j}}}\chi_{(1,a)}(a^{j-i}x)dx|^2 \\
&= \sum_{j\in I} |\sum_{i\in I} c_i \int_1^a \frac{1}{\sqrt{a^{i-j}}}\chi_{(1,a)}(a^{j-i}x)dx|^2 + o
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\left\langle f, D_{a^j}\chi_{(1,a)}\right\rangle|^2 &= \sum_{j\in I} |c_j \int_1^a \chi_{(1,a)}(x)dx + \sum_{i\in I, i\neq j} c_i \int_1^a \frac{1}{\sqrt{a^{i-j}}}\chi_{(1,a)}(a^{j-i}x)dx|^2 \\
&= \sum_{j\in I} |c_j \|\chi_{(1,a)}\|^2 + 0|^2 \\
&= (a-1)\sum_{j\in I} |c_j|^2 \|\chi_{(1,a)}\|^2 \\
&= (a-1)\left\langle \sum_{i\in I} c_i D_{a^i}\chi_{(1,a)}, \sum_{j\in I} c_j D_{a^j}\chi_{(1,a)}\right\rangle \\
&= (a-1)\|f\|^2.
\end{aligned}$$

2. **Main Results**

Definition 2.1. *A frame of dilation is a frame sequence in $L_2(0,\infty)$ of the form $\{D_{\lambda_j}\phi\}_{j\in\mathbb{Z}}$, where $\{\lambda_j\}_{j\in\mathbb{Z}}$ is a sequence in $(0,\infty)$ and $\phi \in L_2(0,\infty)$.*
If $\lambda_j = a^j$, where $a > 1$, then $\{D_{\lambda_j}\phi\}_{j\in\mathbb{Z}}$ is called regular .

A sufficient condition for Bessel condition of a regular frame of dilation is given in the next proposition.We define the function Φ on $(0,\infty)$ by $\Phi(x) = \sum_{j\in\mathbb{Z}} a^j|\phi(a^jx)|^2$, for any $\phi \in L_2(0,\infty)$ and $a > 1$. It is easy to show that $\int_1^a \Phi(x)dx = \|\phi\|^2$ and hence $\Phi \in L_1(1,a)$.

Proposition 2.1. *If $supp(\phi) \subset \cup_{i\in I}[a^i, a^{i+1}]$, for some finite subset I of $\mathbb{Z}$, then $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$ with Bessel bound $|I|\|\phi\|^2$.*

Proof. For all $f \in L_2(0,\infty)$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\left\langle f, D_{a^j}\phi\right\rangle|^2 &= \sum_{j\in\mathbb{Z}} a^j | \int_{supp(\phi)} f(a^jx)\overline{\phi(x)}dx|^2 \\
&\le \sum_{j\in\mathbb{Z}} a^j \int_{supp(\phi)} |f(a^jx)|^2dx \int_{supp(\phi)} |\phi(x)|^2dx \\
&\le \|\phi\|^2 \sum_{j\in\mathbb{Z}}\sum_{i\in I} \int_{a^i}^{a^{i+1}} |f(a^jx)|^2 a^j dx \\
&= \|\phi\|^2 \sum_{i\in I}\sum_{j\in\mathbb{Z}} \int_{a^{i+j}}^{a^{i+j+1}} |f(x)|^2 dx \\
&= |I|\|\phi\|^2\|f\|^2.
\end{aligned}$$

□

Corollary 2.1. *If $supp(\phi) \subset \cup_{i\in I}[c_i, d_i]$, for some finite subset I of $\mathbb{Z}$ such that $\frac{d_i}{c_i} \le a$ for all $i \in I$, then $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$ with Bessel bound $|I|\|\phi\|^2$.*

Proof. The intervals $[c_i a^j, d_i a^j)$ are disjoint for any $i \in I$ and $j \in \mathbb{Z}$, since $\frac{d_i}{c_i} \leq a$. Now for all $f \in L_2(0,\infty)$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\langle f, D_{a^j}\phi\rangle|^2 &= \sum_{j\in\mathbb{Z}} a^j \Big| \int_{supp(\phi)} f(a^j x)\overline{\phi(x)}dx\Big|^2 \\
&\leq \sum_{j\in\mathbb{Z}} a^j \int_{supp(\phi)} |f(a^j x)|^2 dx \int_{supp(\phi)} |\phi(x)|^2 dx \\
&\leq \|\phi\|^2 \sum_{j\in\mathbb{Z}} \sum_{i\in I} \int_{c_i}^{d_i} |f(a^j x)|^2 a^j dx \\
&= \|\phi\|^2 \sum_{i\in I} \sum_{j\in\mathbb{Z}} \int_{c_i a^j}^{d_i a^j} |f(x)|^2 dx \\
&= |I| \|\phi\|^2 \|f\|^2.
\end{aligned}$$

□

Corollary 2.2. *If $\phi \in L_2(0,\infty)$ is compactly supported, then $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$.*

The following example shows that the converse of Proposition 2.1 is not correct.

Example 2.1. *Let $a > 1$ and let ϕ be a function on $(0,\infty)$ defined by*

$$\phi(x) = \begin{cases} \sqrt{\frac{2^{-j}}{a^{j+1}-a^j}} & \text{if } x \in [a^j, a^{j+1}), j \geq 0, \\ 0 & \text{if } x \in (0,1) \end{cases},$$

then $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$ with Bessel bound $\frac{2}{3-2\sqrt{2}}$. Indeed, for all $f \in L_2(0,\infty)$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\langle f, D_{a^j}\phi\rangle|^2 &= \sum_{j\in\mathbb{Z}} a^j \Big| \sum_{\ell\in\mathbb{Z}} \int_{a^\ell}^{a^{\ell+1}} f(a^j x)\overline{\phi(x)}dx\Big|^2 \\
&= \sum_{j\in\mathbb{Z}} a^j \Big| \sum_{\ell=0}^{\infty} \sqrt[4]{2^{-\ell}} \int_{a^\ell}^{a^{\ell+1}} f(a^j x)\frac{\sqrt[4]{2^{-\ell}}}{\sqrt{a^{\ell+1}-a^\ell}}dx\Big|^2 \\
&\leq \sum_{j\in\mathbb{Z}} a^j \sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}} \sum_{\ell=0}^{\infty} \Big| \int_{a^\ell}^{a^{\ell+1}} f(a^j x)\frac{\sqrt[4]{2^{-\ell}}}{\sqrt{a^{\ell+1}-a^\ell}}dx\Big|^2 \\
&\leq \sum_{j\in\mathbb{Z}} a^j \sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}} \sum_{\ell=0}^{\infty} \int_{a^\ell}^{a^{\ell+1}} |f(a^j x)|^2 dx \int_{a^\ell}^{a^{\ell+1}} \frac{\sqrt{2^{-\ell}}}{a^{\ell+1}-a^\ell}dx \\
&= \sum_{j\in\mathbb{Z}} a^j \sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}} \sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}} \int_{a^\ell}^{a^{\ell+1}} |f(a^j x)|^2 dx \\
&= \sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}} \sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}} \sum_{j\in\mathbb{Z}} \int_{a^{\ell+j}}^{a^{\ell+j+1}} |f(x)|^2 dx
\end{aligned}$$

$$= (\sum_{\ell=0}^{\infty} \sqrt{2^{-\ell}})^2 \|f\|^2$$
$$= \frac{2}{3-2\sqrt{2}} \|f\|^2$$

For any $1 \leq p < \infty$ and $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}} \in \ell_p(\mathbb{Z})$ with $\gamma_j \neq 0$, for all $j \in \mathbb{Z}$, we define $L_p^\gamma(0,\infty)$ by

$$L_p^\gamma(0,\infty) := \{f : (0,\infty) \longrightarrow \mathbb{C};\ \sum_{j\in\mathbb{Z}} \|\frac{1}{\gamma_j}\phi\chi_{[a^j,a^{j+1})}\|_p^p < \infty\}.$$

Also for $\phi \in L_p^\gamma(0,\infty)$, we define $\|\phi\|_\gamma$ by

$$\|\phi\|_\gamma := \|\gamma\|(\sum_{j\in\mathbb{Z}} \|\frac{1}{\gamma_j}\phi\chi_{[a^j,a^{j+1})}\|_p^p)^{\frac{1}{p}}.$$

For all $j \in \mathbb{Z}$ we have

$$|\gamma_j|^p \leq \sum_{j\in\mathbb{Z}} |\gamma_j|^p = \|\gamma\|^p$$

and hence for all $\phi \in L_p^\gamma(0,\infty)$

$$\|\phi\|_\gamma^p \geq \|\gamma\|^p \frac{1}{\|\gamma\|^p}(\sum_{j\in\mathbb{Z}} \|\phi\chi_{[a^j,a^{j+1})}\|_p^p) = \|\phi\|^p.$$

Thus $\|\phi\| \leq \|\phi\|_\gamma$ for any $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ with $\gamma_j \neq 0$, for all $j \in \mathbb{Z}$ and hence $L_p^\gamma(0,\infty)$ is a sub space of $L_p(0,\infty)$.

Theorem 2.1. *Let $\phi \in L_2(0,\infty)$ and $B > 0$.*

1) *If $\|\phi\|_\gamma \leq \sqrt{B}$, for some $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ with $\gamma_j \neq 0$, for all $j \in \mathbb{Z}$, then $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$ with Bessel bound B.*
2) *If $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence in $L_2(0,\infty)$ with Bessel bound B, then $\|\phi\| \leq \sqrt{B}$.*
3) *If $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a frame in $L_2(0,\infty)$ with frame bounds A and B, then $\sqrt{A} \leq \|\phi\| \leq \sqrt{B}$.*

Proof.

1) For all $f \in L_2(0,\infty)$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\langle f, D_{a^j}\phi\rangle|^2 &= \sum_{j\in\mathbb{Z}} a^j |\sum_{\ell\in\mathbb{Z}} |\gamma_\ell| \frac{1}{|\gamma_\ell|} \int_{a^\ell}^{a^{\ell+1}} f(a^j x)\overline{\phi(x)}dx|^2 \\
&\leq \sum_{j\in\mathbb{Z}} a^j \sum_{\ell\in\mathbb{Z}} |\gamma_\ell|^2 \sum_{\ell\in\mathbb{Z}} \int_{a^\ell}^{a^{\ell+1}} |f(a^j x)|^2 dx \int_{a^\ell}^{a^{\ell+1}} \frac{1}{|\gamma_\ell|^2}|\phi(x)|^2 dx \\
&\leq \|\gamma\|^2 \sum_{\ell\in\mathbb{Z}} \|\frac{1}{\gamma_\ell}\phi\chi_{[a^\ell,a^{\ell+1})}\|_2^2 \sum_{j\in\mathbb{Z}} a^j \int_{a^\ell}^{a^{\ell+1}} |f(a^j x)|^2 dx \\
&\leq \|\phi\|_\gamma^2 \|f\|^2 \\
&\leq B\|f\|^2
\end{aligned}$$

2) Let $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ be a Bessel sequence in $L_2(0,\infty)$ with Bessel bound B. Then $T:\ell_2(\mathbb{Z})\longrightarrow L_2(0,\infty)$, with $T(\{c_j\}_{j\in\mathbb{Z}})=\sum_{j\in\mathbb{Z}}c_jD_{a^j}\phi$ is well defined bounded linear operator with $\|T\|\le\sqrt{B}$. The sequence $\{d_j\}_{j\in\mathbb{Z}}$ defined by

$$d_j=\left\{\begin{array}{ll}1 & \text{if } j=0,\\ 0 & \text{if } j\neq 0\end{array}\right.,$$

has $\|\{d_j\}_{j\in\mathbb{Z}}\|=1$. Thus

$$B\ge\|T\|^2=\sup_{\|\{c_j\}_{j\in\mathbb{Z}}\|=1}\|T(\{c_j\}_{j\in\mathbb{Z}})\|^2\ge\|T(\{d_j\}_{j\in\mathbb{Z}})\|^2=\|\phi\|^2.$$

3) We have $A\|f\|^2\le\sum_{j\in\mathbb{Z}}|\langle f,D_{a^j}\phi\rangle|^2$, for all $f\in L_2(0,\infty)$. If $f:=\chi_{[1,a)}$, then

$$\begin{aligned}A(a-1)&\le\sum_{j\in\mathbb{Z}}|\left\langle\chi_{[1,a)},D_{a^j}\phi\right\rangle|^2\\&=\sum_{j\in\mathbb{Z}}a^{-j}|\int_1^a\overline{\phi(a^{-j}x)}dx|^2\\&\le(a-1)\sum_{j\in\mathbb{Z}}a^{-j}\int_1^a|\phi(a^{-j}x)|^2dx\\&=(a-1)\int_1^a\sum_{j\in\mathbb{Z}}a^{-j}|\phi(a^{-j}x)|^2dx\\&=(a-1)\int_1^a\Phi(x)dx\\&=(a-1)\|\phi\|^2\end{aligned}$$

and hence $\sqrt{A}\le\|\phi\|$. □

Open problems:

1) Which conditions on ϕ will imply that the sequence $\{D_{a^j}\phi\}_{j\in\mathbb{Z}}$ is a Bessel sequence, frame sequence, Riesz sequence or orthonormal sequence in $L_2(0,\infty)$?
2) Can $\{D_{\lambda_j}\phi\}_{j\in\mathbb{Z}}$ to be a frame, Riesz basis or orthonormal basis for $L_2(0,\infty)$?

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