$*$-G-FRAMES IN HILBERT MODULES OVER PRO-C^*-ALGEBRAS

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In this paper, the $*$-g-frames with algebraic bounds in Hilbert pro-C^*-modules are introduced. Some properties of $*$-g-frames in Hilbert pro-C^*-modules are studied. Moreover, the dual $*$-g-frames in Hilbert pro-C^*-modules are presented. As a result, a reconstruction formula of the elements of Hilbert pro-C^*-modules is provided.

Keywords: Hilbert pro-C^*-modules, $*$-g-frame, $*$-g-frame operator.

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1. Introduction

Frames that are a generalization of bases in Hilbert space, were introduced by [7] in 1952, to deal with some problems in the non-harmonic Fourier series. In 1986, Daubechies et al. [6] reintroduced them and characterized function spaces. In other words, they replaced the sequence of bounded linear operators instead of the sequence of element in Hilbert space. Frames have many applications, such as: study and characterization of function spaces, signal and image processing, wireless communications, transceiver design, data compression and so on; we refer to [2], [4], [5], [12] and [23] for an introduction to the frame theory and its applications. Diverse applications of frame theory in sciences and engineering, led to the theory, should be extended to different forms. Many generalizations of frames were presented; for instance, the fusion frames by Casazza et al. [3] and g-frames by Sun [22]. In 2000, Frank and Larson introduced and studied the concept of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces; for the details see [8], [9]. More results for the generalizations of frames in Hilbert C^*-modules are available in [11] and [14].

Pro-C^*-algebras which are the generalizations of C^*-algebras such that the topology of a pro-C^*-algebra is given by a directed family of C^*-seminorms instead of a single C^*-norm. In the literature, pro-C^*-algebras have been given by different names such as b^*-algebras (C. Apostol), LMC^*-algebras (G. Lessner, K. Schmudgen) or locally C^*-algebras (A. Inoue, M. Fragoulopoulou, A. Mallios, etc). Hilbert modules over pro-C^*-algebras were considered independently by Mallios [19] and Phillips [20]. In fact, Phillips showed that the Kasparov stabilisation theorem is valid for countably generated Hilbert modules over metrizable pro-C^*-algebras and Joita showed that this theorem is true for countably generated Hilbert modules over arbitrary pro-C^*-algebras [16]. Later, Zhuraev and Sharipov [24] considered pro-C^*- algebras and introduced Hilbert module over pro-C^*-algebras. Furthermore, Raeburn and Thompson [21] showed that every Hilbert C^*-module countably generated in the multiplier module admits a frame of multipliers. In 2008, Joita [17] reconsidered ideas Raeburn and Thompson in Hilbert modules over pro-C^*-algebras and proposed frames of multipliers in Hilbert pro-C*-modules.

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In this paper, we generalize the concept of $*$-g-frame into a general space which is called, Hilbert module over a Pro-C*-algebra. We also introduce the $*$-g-frame transforms and study their properties. Finally, by the canonical dual $*$-g-frames, we provide a reconstruction formula of the elements of such spaces.

2. **Definitions and Preliminaries**

In this section, we recall some basic definitions and properties of pro-C^*-algebras and Hilbert modules over pro-C^*-algebras.

Definition 2.1. A *pro-C^*-algebra* is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net (a_λ) converges to 0 if and only if $\rho(a_\lambda) \to 0$ for any continuous C^*-seminorm ρ on A. For any C^*-seminorm ρ on A and each $a, b \in A$, we have

(i) $\rho(ab) \leq \rho(a)\rho(b)$;
(ii) $\rho(a^*a) = \rho(a)^2$.

For each pro-C^*-algebra A, the set of all positive elements in A is denoted by A^+. Moreover, $a \geq 0$ denotes $a \in A^+$ and $a \leq b$ means that $b - a \geq 0$. We recall that every C^*-algebra is a pro-C^*-algebra.

The set of all continuous C^*-seminorms on A is denoted by $S(A)$. An element $a \in A$ is bounded if $\|a\|_\infty = \sup\{\rho(a) : \rho \in S(A)\} < \infty$. The set of all bounded elements in A is denoted by $b(A)$. Let A be a unitary pro-C^*-algebra and $a \in A$. Then, a non-zero element $a \in A$ is called *strictly non-zero* if zero does not belong to $\sigma(a)$. Here, we remember the following elementary result from [13].

Proposition 2.1. *Let A be a unital pro- C^*-algebra with the identity 1_A and $\rho \in S(A)$. Then*

(1) $\rho(a) = \rho(a^*)$ *for all* $a \in A$;
(2) $\rho(1_A) = 1$;
(3) *if* $a, b \in A^+$ *and* $a \leq b$, *then* $\rho(a) \leq \rho(b)$;
(4) *if* $1_A \leq b$, *then* b *is invertible and* $b^{-1} \leq 1_A$;
(5) *if* $a, b \in A^+$ *are invertible and* $0 \leq a \leq b$, *then* $0 \leq b^{-1} \leq a^{-1}$;
(6) *if* $a, b, c \in A$ *and* $a \leq b$, *then* $c^*ac \leq c^*bc$;
(7) *if* $a, b \in A^+$ *and* $a^2 \leq b^2$, *then* $0 \leq a \leq b$.

Definition 2.2. Let A be a pro-C^*-algebra. A *pre-Hilbert A-module* is a complex vector space X which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : X \times X \longrightarrow A$ which is $\mathbb{C}$ and A-linear in second variable and satisfies the following conditions:

(i) $\langle x, y \rangle^* = \langle y, x \rangle$;
(ii) $\langle x, x \rangle \geq 0$;
(iii) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

We say that X is a *Hilbert A-module* or *Hilbert pro-C^*-module over A* if X is complete with respect to the topology determined by the family of seminorms

$$\bar{\rho}_X(x) = \sqrt{\rho(\langle x, x \rangle)}, \quad x \in X_\rho, \ \rho \in S(A).$$

Let X be a pre-Hilbert A-module. For every $\rho \in S(A)$ and for each $x, y \in X$, the following Cauchy-Schwartz inequality holds

$$\rho(\langle x, y \rangle)^2 \leq \rho(\langle x, x \rangle)\rho(\langle y, y \rangle).$$

Consequently, $\bar{\rho}_X(ax) \leq \rho(a)\bar{\rho}_X(x)$ for all $a \in A, x \in X$.

Next, we bring two examples of Hilbert modules over pro-C^*-algebras.

Example 2.1. (i) Let $l^2(A)$ be the set of all sequences $(a_n)_{n\in\mathbb{N}}$ of elements of a pro-C^*-algebra A such that the series $\sum_{i\in I} a_i a_i^*$ is convergent in A. Then, $l^2(A)$ is a Hilbert module over A with respect to the pointwise operations and inner product defined by

$$\langle (a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}\rangle := \sum_{i\in\mathbb{N}} a_i b_i^*.$$

(ii) [20] Suppose that A is a pro-C^*-algebra and X_i for $i\in\mathbb{N}$ is a Hilbert A-module with the topology induced by the family of continuous seminorms $\{\bar{\rho}_i\}_{\rho\in S(A)}$ defined through

$$\bar{\rho}_i(x) := \sqrt{\rho(\langle x, x\rangle)} \quad x\in X_i.$$

Then, the direct sum of $\{X_i\}_{i\in\mathbb{N}}$ is considered as

$$\oplus_{i\in\mathbb{N}} X_i = \{(x_i)_{i\in\mathbb{N}} : x_i\in X_i, \sum_{i\in\mathbb{N}}\langle x_i, x_i\rangle \text{ is convergent in } A\}.$$

It has been shown in [18, Example 3.2] that the direct sum $\oplus_{i\in\mathbb{N}} X_i$ is a Hilbert A-module with A-valued inner product $\langle x, y\rangle = \sum_i \langle x_i, y_i\rangle$, where $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ are in $\oplus_{i\in\mathbb{N}} X_i$, pointwise operations and a topology determined by the family of seminorms

$$\bar{\rho}(x) = \sqrt{\rho(\langle x, x\rangle)} \;\; x\in\oplus_{i\in\mathbb{N}} X_i, \quad \rho\in S(A).$$

The direct sum of a countable copies of a Hilbert module X is denoted by $l^2(X)$.

Let A be a pro-C^*-algebra and X be a pre-Hilbert A-module. We recall that an element x in E is bounded if $\|x\|_\infty = \sup\{\bar{\rho}_X(x) : \rho\in S(A)\} < \infty$. We denote by $b(X)$, the set of all bounded elements in X. It is well-known that $b(A)$ is a C^*-algebra in the C^*-norm $\|.\|_\infty$ and $b(X)$ is a Hilbert $b(A)$-module (see [20, Proposition 1.11] and [24, Theorem 2.1] for more details).

Let A be a pro-C^*-algebra and X,Y be two Hilbert A-modules. An A-module map $T : X \longrightarrow Y$ is said to *bounded* if for each $\rho\in S(A)$, there is $C_\rho > 0$ such that $\bar{\rho}_Y(T(x)) \le C_\rho\bar{\rho}_X(x)$ for all $x\in X$, where $\bar{\rho}_X$,and $\bar{\rho}_Y$ are continuous seminorms on X and Y, respectively. A bounded A-module map from X to Y is called an operator from X to Y. We denote the set of all operators from X to Y by $\mathrm{Hom}_A(X,Y)$, and set $\mathrm{End}_A(X) = \mathrm{Hom}_A(X,X)$. Let $T \in \mathrm{Hom}_A(X,Y)$. We say T is *adjointable* if there exists an operator $T^* \in \mathrm{Hom}_A(X,Y)$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x\in X$ and $y\in Y$. We denote by $\mathrm{Hom}_A^*(X,Y)$, the set of all adjointable operators from X to Y and $\mathrm{End}_A^*(X) = \mathrm{Hom}_A^*(X,X)$. By a little modification in the proof of Lemma 3.2 from [24], we have the next result.

Proposition 2.2. *Let $T : X \longrightarrow Y$ and $T^* : Y \longrightarrow X$ be two maps such that the equality*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

holds for all $x\in X$ and $y\in Y$. Then, $T\in \mathrm{Hom}_A^*(X,Y)$.

It is easy to see that for any $\rho\in S(A)$, the map defined via

$$\hat{\rho}_{X,Y}(T) = \sup\{\bar{\rho}_Y(T(x)) : x\in X, \bar{\rho}_X(x)\le 1\}, \quad T\in\mathrm{Hom}_A(X,Y),$$

is a seminorm on $\mathrm{Hom}_A(X,Y)$. Moreover, $\mathrm{Hom}_A(X,Y)$ with the topology determined by the family of seminorms $\{\hat{\rho}_{X,Y}\}_{\rho\in S(A)}$ is a complete locally convex space [15, Proposition 3.1]. Furthermore, by using [24, Lemma 2.2], for each $y\in Y$ and $\rho\in S(A)$, we get

$$\begin{aligned}\bar{\rho}_X(T^*(x)) &= \sup\{\rho\langle T^*(y), x\rangle : \bar{\rho}_X(x)\le 1\}\\ &= \sup\{\rho\langle T(x), y\rangle : \bar{\rho}_X(x)\le 1\}\\ &\le \sup\{\bar{\rho}_Y(T(x)) : \bar{\rho}_X(x)\le 1\}\bar{\rho}_Y(y)\\ &= \bar{\rho}(T)\bar{\rho}_Y(y).\end{aligned}$$

Thus, for each $\rho \in S(A)$, we have $\hat{\rho}_{Y,X}(T^*) \leq \hat{\rho}_{X,Y}(T)$. Since $T^{**} = T$, by replacing T with T^*, for each $\rho \in S(A)$, we obtain $\hat{\rho}_{Y,X}(T^*) = \hat{\rho}_{X,Y}(T)$. It follows from [20, Proposition 4.7] that $\mathrm{End}_A^*(X)$ is a pro-C^*-algebra for any Hilbert A-module X such that its topology is taken from $\{\hat{\rho}_X\}_{\rho \in S(A)}$ [24]. By [24, Proposition 3.2], T is a positive element of $\mathrm{End}_A^*(X)$ if and only if $\langle T(x), x\rangle \geq 0$ for all $x \in X$.

Definition 2.3. Let X and Y be two Hilbert modules over pro-C^*-algebra A. Then, the operator $T : X \longrightarrow Y$ is called uniformly bounded (below), if there exists $C > 0$ such that $\bar{\rho}_Y(T(x)) \leq C\bar{\rho}_X(x)$ $(C\bar{\rho}_X(x) \leq \bar{\rho}_Y(T(x)))$ for all $\rho \in S(A)$ and $x \in X$. The number C is called an upper bound for T and hence we set

$$\|T\|_\infty = \inf\{C : C \textit{ is an upper bound for } T\}.$$

Clearly, in this case we have $\hat{\rho}(T) \leq \|T\|_\infty$, for all $\rho \in S(A)$.

Suppose that T is an invertible element in $\mathrm{End}_A^*(X)$ which is uniformly bounded. By [1, Proposition 3.2], we find

$$\|T^{-1}\|_\infty^{-2}\langle x, x\rangle \leq \langle T(x), T(x)\rangle \leq \|T\|_\infty^2\langle x, x\rangle, \tag{1}$$

for all $x \in X$.

3. Main results

Throughout this section, A is a pro-C^*-algebra, X and Y are two Hilbert A-modules. Moreover, $\{Y_i\}_{i\in I}$ is a countable sequence of closed submodules of Y.

Definition 3.1. Let X be a Hilbert pro-C^*-module. A sequence $\{x_i\}_{i\in I}$ in X is said to be the *standard $*$-frame* for X if for each $x \in X$, the series $\sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle$ is convergent in A and there exist two strictly non-zero elements C and D in A such that

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle D^*$$

for all $x \in X$. The elements C and D are called *$*$-frame bounds* for $\{x_i\}_{i\in I}$. The $*$-frame is called *tight* if $C = D$ and called *Parseval* if $C = D = 1$. In the above relation, if we only have the upper bound, then $\{x_i\}_{i\in I}$ is called a *$*$-Bessel sequence.*

Similar to Definition 3.1, we have the incoming definition for the sequences in the operator setting.

Definition 3.2. A sequence $\Lambda = \{\Lambda_i \in \mathrm{Hom}_A^*(X, Y_i)\}_{i\in I}$ is called a *$*$-g-frame* for X with respect to $\{Y_i\}_{i\in I}$ if for each $x \in X$, the series $\sum_{i\in I}\langle\Lambda_i(x), \Lambda_i(x)\rangle$ is convergent in A and there exist two strictly non-zero elements C and D in A such that for every $x \in X$,

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle\Lambda_i(x), \Lambda_i(x)\rangle \leq D\langle x, x\rangle D^*.$$

The elements C and D are called *$*$-g-frame bounds* for Λ. The $*$-g-frame is called *tight* if $C = D$ and called *Parseval* if $C = D = 1$. If in the above we only need to have the upper bound, then Λ is called a $*$-g-Bessel sequence. Since the sequence $\sum_{i\in I}\langle\Lambda_i(x), \Lambda_i(x)\rangle$ is convergent in A, the $*$-g-frame can be called *standard* but we use this definition without the word standard if there is no risk of ambiguity. Besides, if for each $i \in I$, $Y_i = Y$, we call it a $*$-g-frame for X with respect to Y.

An example regarding to an $*$-g-frame for a Hilbert A-module X is indicated as follows.

Example 3.1. Let $\{x_i\}_{i\in I}$ be a $*$- frame for X with bounds C and D. For $i \in I$, consider the operator Λ_{x_i} defined via

$$\Lambda_{x_i} : X \longrightarrow A; \quad \Lambda_{x_i}(x) = \langle x, x_i\rangle.$$

It is obvious that Λ_{x_i} is a bounded operator in $\mathrm{Hom}_A(X, A)$ which its adjoint is

$$\Lambda^*_{x_i} : A \to X \quad \Lambda^*_{x_i}(a) = ax_i.$$

Hence, $\Lambda_{x_i} \in \mathrm{Hom}^*_A(X, A)$, $i \in I$. Moreover, by assumption, for each $x \in X$

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle = \sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle \leq D\langle x, x\rangle D^*.$$

Therefore, $\Lambda = \{\Lambda_{x_i}\}_{i\in I}$ is a $*$-g-frame for X with respect to A.

The following result will be used in this section.

Proposition 3.1. *Let T be an uniformly bounded below operator in* $\mathrm{Hom}_A(X, Y)$. *Then, T is closed and injective.*

Proof. Refer to [10, Proposition 2.3]. □

Lemma 3.1. [1, Lemma 3.1] *Let X be a Hilbert module over C^*-algebra A, $S \in \mathrm{End}^*_A(X)$ and $S \geq 0$, i.e., this element is positive in C^*-algebra* $\mathrm{End}^*_A(X)$. *Then, for each $x \in X$, $\langle S(x), x\rangle \leq \|S\|\langle x, x\rangle$.*

Let $\Lambda = \{\Lambda_i \in \mathrm{Hom}^*_A(X, Y_i)\}_{i\in I}$ be a $*$-g-frame for X with respect to $\{Y_i\}_{i\in I}$ and bounds C and D in A. We define the corresponding $*$-g-frame transform

$$T_\Lambda : X \to \oplus_{i\in I}Y_i \quad T_\Lambda(x) = \{\Lambda_i x\}_{i\in I}.$$

Since Λ is a $*$-g-frame, we have

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle \leq D\langle x, x\rangle D^*,$$

for all $x \in X$. Thus, T_Λ is well-defined. Besides, for any $\rho \in S(A)$ and $x \in X$, we obtain

$$(\rho(C))\bar{\rho}_X(x) \leq \bar{\rho}_{\oplus_{i\in I}Y_i}(T_\Lambda(x)) \leq (\rho(D))\bar{\rho}_X(x).$$

From the above, it concludes that the $*$-g-frame transform is an uniformly bounded below operator in $\mathrm{Hom}_A(X, \oplus_{i\in I}Y_i)$. Thus, by Proposition 3.1 , T_Λ is closed and injective. Here, we define the synthesis operator $T^*_\Lambda : \oplus_{i\in I}Y_i \longrightarrow X$ for $*$-g-frame Λ through

$$T^*_\Lambda(\{y_i\}_i) := \sum_{i\in I}\Lambda^*_i(y_i), \tag{2}$$

where Λ^*_i is the adjoint operator of Λ_i.

Proposition 3.2. *The synthesis operator defined by* (2) *is well-defined, uniformly bounded and adjoint of the transform operator.*

Proof. Since $\Lambda = \{\Lambda_i : i \in I\}$ is a $*$-g-frame for X with respect to $\{Y_i : i \in I\}$, there exist two strictly nonzero elements C and D in A such that for every $x \in X$,

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle \leq D\langle x, x\rangle D^*.$$

Let J be an arbitrary finite subset of I. Using Cauchy-Bunyakovskii inequality and [24, Lemma 2.2], for any $\rho \in S(A)$ and $(y_i) \in \oplus_{i\in I} Y_i$ we have

$$\begin{aligned}
\left(\bar{\rho}_X\left(\sum_{i\in J}\Lambda_i^*(y_i)\right)\right)^2 &= \left\{\sup\left(\rho\langle\sum_{i\in J}\Lambda_i^*(y_i),x\rangle\right) : x\in X, \bar{\rho}_X(x)\le 1\right\}^2 \\
&= \left\{\sup\left(\rho\left(\sum_{i\in J}\langle y_i,\Lambda_i(x)\rangle\right)\right) : x\in X, \bar{\rho}_X(x)\le 1\right\}^2 \\
&\le \sup_{\bar{\rho}_X(x)\le 1}\left(\rho\left(\sum_{i\in J}\langle y_i,y_i\rangle\right)\right)\left(\rho\left(\sum_{i\in J}\langle\Lambda_i(x),\Lambda_i(x)\rangle\right)\right) \\
&\le \sup_{\bar{\rho}_X(x)\le 1}(\rho(D))^2(\bar{\rho}_X(x))^2\rho\left(\sum_{i\in J}\langle y_i,y_i\rangle\right) \\
&\le \sup_{\bar{\rho}_X(x)\le 1}(\rho(D))^2\rho\left(\sum_{i\in J}\langle y_i,y_i\rangle\right).
\end{aligned}$$

Due to the convergence of series $\sum_{i\in I}\langle y_i, y_i\rangle$ in A, the above relation shows that $\sum_{i\in I}\Lambda_i^*(y_i)$ is convergent. Hence, T_Λ^* is well-defined. On the other hand, for any $x \in X$ and $(y_i) \in \oplus_{i\in I} Y_i$ we get

$$\begin{aligned}
\langle T_\Lambda(x),(y_i)_i\rangle &= \langle(\Lambda_i x)_i,(y_i)_i\rangle = \sum_{i\in I}\langle\Lambda_i(x),y_i\rangle \\
&= \sum_{i\in I}\langle x,\Lambda_i^*(y_i)\rangle = \langle x,\sum_{i\in I}\Lambda_i^*(y_i)\rangle = \langle x,T_\Lambda^*(\{y_i\}_i)\rangle.
\end{aligned}$$

Thus, Proposition 2.2 implies that the synthesis operator is adjoint of the transform operator. Furthermore, for any $\rho \in S(A)$ we reach

$$\bar{\rho}_X(T_\Lambda^*(y)) \le \rho(D)\bar{\rho}_{\oplus_{i\in I}Y_i}(y), \quad y=(y_i)_i\in\oplus_{i\in I}Y_i.$$

Therefore, the synthesis operator is uniformly bounded. □

Let $\Lambda = \{\Lambda_i : i \in I\}$ be a $*$-g-frame for X with respect to $\{Y_i : i \in I\}$. Define the corresponding $*$-g-frame operator $S_\Lambda = T_\Lambda^* T_\Lambda : X \longrightarrow X$ via $S_\Lambda(x) = \sum_{i\in I}\Lambda_i^*\Lambda_i x$. Then, S_Λ is a combination of two bounded operators and so it is a bounded operator.

Theorem 3.1. *Let $\Lambda = \{\Lambda_i : i \in I\}$ be a $*$-g-frame for X with respect to $\{Y_i : i \in I\}$ with frame bounds C and D. Then, S_Λ is an invertible positive operator. Moreover, it is a self-adjoint operator such that*

$$C^*CI_X \le S_\Lambda \le D^*DI_X \tag{3}$$

and

$$D^{-1}(D^*)^{-1}I_X \le S_\Lambda^{-1} \le C^{-1}(C^*)^{-1}I_X, \tag{4}$$

where I_X is the identity function on X, and also we have

$$(\rho(C^{-1})^{-2} \le \bar{\rho_X}(S_\Lambda) \le (\rho(D))^2.$$

Proof. According to the definition of the transform operator, for any $x \in X$ we can write

$$\langle T_\Lambda(x),T_\Lambda(x)\rangle = \langle\{\Lambda_i(x)\}_{i\in I},\{\Lambda_i(x)\}_{i\in I}\rangle = \sum_{i\in I}\langle\Lambda_i(x),\Lambda_i(x)\rangle.$$

By hypotheses, we get

$$C\langle x,x\rangle C^* \le \langle T_\Lambda(x),T_\Lambda(x)\rangle \le D\langle x,x\rangle D^*.$$

On the other hand,

$$\langle S_\Lambda(x), x\rangle = \langle T_\Lambda^* T_\Lambda(x), x\rangle = \langle T_\Lambda(x), T_\Lambda(x)\rangle = \langle x, T_\Lambda^* T_\Lambda(x)\rangle = \langle x, S_\Lambda(x)\rangle.$$

Consequently, S_Λ is a self-adjoint operator. For any $x \in X$, we find

$$C\langle x, x\rangle C^* \leq \langle S_\Lambda(x), x\rangle \leq D\langle x, x\rangle D^*. \tag{5}$$

From (5), it follows that the $*$-g-frame operator is positive and (3) is obtained as well. Now, suppose that $S_\Lambda(x) = 0$ for any $x \in X$. By (5), we observe that $\langle x, x\rangle = 0$, which implies S_Λ is invertible. For $x \in X$, we have

$$C\langle S_\Lambda^{-1}x, S_\Lambda^{-1}x\rangle C^* \leq \sum_{i\in I}\langle \Lambda_i S_\Lambda^{-1}x, \Lambda_i S_\Lambda^{-1}x\rangle = \langle S_\Lambda^{-1}x, x\rangle,$$

and

$$\langle x, S_\Lambda^{-1}(x)\rangle = \sum_{i\in I}\langle \Lambda_i S_\Lambda^{-1}(x), \Lambda_i S_\Lambda^{-1}(x)\rangle \leq D\langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle D^*.$$

The last relations necessitate that for all $x \in X$

$$D^{-1}\langle S_\Lambda^{-1}(x), x\rangle (D^*)^{-1} \leq \langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle \leq C^{-1}\langle S_\Lambda^{-1}(x), x\rangle (C^*)^{-1}$$

and so

$$D^{-1}(D^*)^{-1}S_\Lambda^{-1} \leq (S_\Lambda^{-1})^2 \leq C^{-1}(C^*)^{-1}S_\Lambda^{-1}.$$

Since S is a positive operator, $D^{-1}(D^*)^{-1}I_X \leq S_\Lambda^{-1} \leq C^{-1}(C^*)^{-1}I_X$. Applying the Cauchy-Bunyakovskii inequality and [24, Lemma 2.2], we have

$$\begin{aligned}
\left(\bar\rho_X\left(\sum_{i\in J}\Lambda_i^*\Lambda_i(x)\right)\right)^2 &= \left\{\sup\rho\left(\langle\sum_{i\in J}\Lambda_i^*\Lambda_i(x), y\rangle\right) : y \in X, \bar\rho_X(y) \leq 1\right\}^2 \\
&= \left\{\sup\rho\left(\sum_{i\in J}\langle\Lambda_i(x), \Lambda_i(y)\rangle\right) : y \in X, \bar\rho_X(y) \leq 1\right\}^2 \\
&\leq \sup_{\bar\rho_X(y)\leq 1}\left(\rho\left(\sum_{i\in J}\langle\Lambda_i(x), \Lambda_i(x)\rangle\right)\right)\left(\rho\left(\sum_{i\in J}\langle\Lambda_i(y), \Lambda_i(y)\rangle\right)\right) \\
&\leq \sup_{\bar\rho_X(y)\leq 1}(\rho(D))^2(\bar\rho_X(y))^2(\rho(D))^2(\bar\rho_X(x))^2 \\
&\leq (\rho(D))^4(\bar\rho_X(x))^2.
\end{aligned}$$

for all $\rho \in S(A)$ and $x, y \in X$. Hence,

$$\begin{aligned}
(\bar\rho_X(S_\Lambda(x)))^2 &= \left(\bar\rho_X\left(\sum_{i\in J}\Lambda_i^*\Lambda_i(x)\right)\right)^2 \\
&= \left\{\sup\rho\left(\langle\sum_{i\in J}\Lambda_i^*\Lambda_i(x), y\rangle\right) : y \in X, \bar\rho_X(y) \leq 1\right\}^2 \leq (\rho(D))^4(\bar\rho_X(x))^2.
\end{aligned}$$

Furthermore, $\bar\rho_X(S_\Lambda(x))(\rho(C^{-1}))^{-2} \leq \bar\rho_X(S_\Lambda(x))$. Therefore

$$(\rho(C^{-1}))^{-2} \leq \bar\rho_X(S_\Lambda(x)) \leq (\rho(D))^2.$$

This finishes the proof. □

Proposition 3.3. *Let $\Lambda = \{\Lambda_i : i \in I\}$ be a standard $*$-g-frame for X with respect to $\{Y_i : i \in I\}$, with $*$-frame bound in $b(A)$. Then, $\Lambda = \{\Lambda_i : i \in I\}$ is a standard $*$-g-frame for X with respect to $\{Y_i : i \in I\}$, with lower and upper frame bounds $\|S_\Lambda^{\frac{-1}{2}}\|_\infty^{-2}$ and $\|S_\Lambda^{\frac{1}{2}}\|_\infty^2$, respectively.*

Proof. By our assumptions, $\sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle$ is convergent in A. Theorem 3.1 implies that $S_\Lambda^{\frac{1}{2}}$ is invertible and positive and there are C, D in $b(A)$ such that

$$C\langle x,x\rangle C^* \leq \langle S_\Lambda^{\frac{1}{2}}(x), S_\Lambda^{\frac{1}{2}}(x)\rangle \leq D\langle x,x\rangle D^*.$$

The last relation shows that $\bar{\rho}_X(S_\Lambda^{\frac{1}{2}}) \leq \rho(D)\bar{\rho}_X(x)$ and $\bar{\rho}_X(S_\Lambda^{\frac{1}{2}}) \leq \rho(D)$ for all $x \in X$. Since $D \in b(A)$, $S_\Lambda^{\frac{1}{2}} \in b(\mathrm{End}_A^*(X))$. According [22], we have

$$\|S_\Lambda^{\frac{-1}{2}}\|_\infty^{-2}\langle x,x\rangle \leq \textstyle\sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle \leq \|S_\Lambda^{\frac{1}{2}}\|_\infty^2\langle x,x\rangle. \qquad \square$$

In the upcoming result, some relations between Parseval $*$-frame and Parseval $*$-g-frame for Hilbert modules and also for operators are presented.

Theorem 3.2. *For each $i \in I$, let $\Lambda = \{\Lambda_i \in Hom_A^*(X, Y_i)\}_{i\in I}$ and $\{x_{ij} : j \in J_i\}$ be a Parseval $*$-frame for Y_i. Then, the following assertions hold.*

(i) *$\{\Lambda_i : i \in I\}$ is a Parseval $*$-g-frame for X if and only if $\{\Lambda_i^* x_{ij} : j \in J_i, i \in I\}$ is a Parseval $*$-frame for X.*

(ii) *The $*$-g-frame operator of $\{\Lambda_i : i \in I\}$ is the $*$-frame operator of $\Gamma = \{\Lambda_i^* x_{ij} : j \in J_i, i \in I\}$.*

Proof. (i) It follows from the assumptions that

$$\langle \Lambda_i(x), \Lambda_i(x)\rangle = \sum_{j\in J_i}\langle \Lambda_i(x), x_{ij}\rangle\langle x_{ij}, \Lambda_i(x)\rangle.$$

Therefore

$$\sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle = \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i(x), x_{ij}\rangle\langle x_{ij}, \Lambda_i(x)\rangle.$$

for all $x \in X$. Since for every i , Λ_i is adjointable and so the above equality can be summarized as follow:

$$\sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle = \sum_{i\in I}\sum_{j\in J_i}\langle x, \Lambda_i^*(x_{ij})\rangle\langle \Lambda_i^*(x_{ij}), x\rangle,$$

which shows that $\{\Lambda_i : i \in I\}$ is a Parseval $*$-g-frame for X if and only if $\{\Lambda_i^*(x_{ij}) : j \in J_i, i \in I\}$ is a Parseval $*$-frame for X.

(ii) Let S_Λ and S_Γ be the $*$-frame operators for Λ and Γ, respectively. Then

$$S_\Gamma(x) = \sum_{i\in I}\sum_{j\in J_i}\langle x, \Lambda_i^*(x_{ij})\rangle\Lambda_i^*(x_{ij}), \quad S_\Lambda(x) = \sum_{i\in I}\Lambda_i^*\Lambda_i(x)$$

for all $x \in X$. On the other hand, for every $i \in I$ and $x \in X$, $\Lambda_i(x) = \sum_{j\in J_i}\langle \Lambda_i(x), x_{ij}\rangle x_{ij}$. Since $\Lambda_i(x) \in Y_i$ and the last equality is the reconstruction formula for $\Lambda_i x$ with respect to Parseval $*$-frame $\{x_{ij} : j \in J_i\}$, we get

$$\begin{aligned} S_\Gamma(x) &= \sum_{i\in I}\sum_{j\in J_i}\langle x, \Lambda_i^*(x_{ij})\rangle\Lambda_i^*(x_{ij}) = \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i(x), x_{ij}\rangle\Lambda_i^*(x_{ij}) \\ &= \sum_{i\in I}\Lambda_i^*(\sum_{j\in J_i}\langle \Lambda_i(x), x_{ij}\rangle x_{ij}) = \sum_{i\in I}\Lambda_i^*\Lambda_i(x) = S_\Lambda(x). \end{aligned}$$

for all $x \in X$. The proof of part (ii) is now complete. $\square$

Proposition 3.4. *Let $\Lambda = \{\Lambda_i \in \mathrm{Hom}_A^*(X, Y_i)\}_{i\in I}$ be a $*$-g-frame for X whit bounds C and D and $*$-g-frame operator S_Λ. If $T \in \mathrm{End}_A^*(X)$ is an invertible operator such that both are uniformly bounded, then $\{\Lambda_i T : i \in I\}$ is a $*$-g-frame for X with respect to $\{Y_i\}_{i\in I}$ and with $*$-g-frame operator $T^* S_\Lambda T$.*

Proof. Note that $\Lambda_i T \in \mathrm{Hom}_A^*(X, Y_i)$. In addition, by (1), for each $x \in X$ we have

$$\begin{aligned} C\|T^{-1}\|_\infty^{-1}\langle x,x\rangle\|T^{-1}\|_\infty^{-1}C^* &= C\|T^{-1}\|_\infty^{-2}\langle x,x\rangle C^* \\ &\le C\langle T(x),T(x)\rangle C^* \\ &\le \sum_{i\in I}\langle \Lambda_i T(x), \Lambda_i T(x)\rangle \\ &\le D\langle T(x),T(x)\rangle D^* \\ &\le D\|T\|_\infty^2\langle x,x\rangle D^* \\ &= D\|T\|_\infty\langle x,x\rangle\|T\|_\infty D^*. \end{aligned}$$

Therefore, the sequence $\{\Lambda_i T : i \in I\}$ is a $*$-g-frame for X whit respect to $\{Y_i\}_{i\in I}$ and bounds $C\|T^{-1}\|_\infty^{-1}, D\|T\|_\infty$. Besides, for any $x \in X$ we obtain

$$T^*S_\Lambda T(x) = T^*\sum_{i\in I}\Lambda_i^*\Lambda_i T(x) = \sum_{i\in I}T^*\Lambda_i^*\Lambda_i T(x) = \sum_{i\in I}(\Lambda_i T)^*(\Lambda_i T)(x).$$

This means that $T^*S_\Lambda T$ is the $*$-g-frame operator for $\Lambda_i T \in \mathrm{Hom}_A^*(X, Y_i)$. □

As a main result in this section, we present a reconstruction formula for elements of a Hilbert pro-C*-module as follows.

Theorem 3.3. *Let $\Lambda = \{\Lambda_i \in \mathrm{Hom}_A^*(X, Y_i)\}_{i\in I}$ be a $*$-g-frame for X with bounds C and D and $*$-g-frame operator S_Λ . For each $i \in I$, set $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$. Then, $\tilde{\Lambda} = \{\tilde{\Lambda}_i : i \in I\}$ is a $*$-g-frame for X whit respect to $\{Y_i\}_{i\in I}$ and bounds $CD^{-1}(D^*)^{-1}, DC^{-1}(C^*)^{-1}$ and $*$-g-frame operator S_Λ^{-1}. Furthermore, for each $x \in X$ we have the following reconstruction formula $x = \sum_{i\in I}(\tilde{\Lambda}_i)^*\Lambda_i(x) = \sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i(x)$. $\tilde{\Lambda}$ is called the canonical dual *-g-frame of Λ.*

Proof. In Proposition 3.4, put $T = S_\Lambda^{-1}$. Then, we conclude $\{\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1} : i \in I\}$ is a $*$-g-frame for X with respect to $\{Y_i\}_{i\in I}$ and $*$-g-frame operator such that $T^*S_\Lambda T = S_\Lambda^{-1}S_\Lambda S_\Lambda^{-1} = S_\Lambda^{-1}$.
Moreover, by Theorem 3.1, for $x \in X$ we have

$$C\langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle C^* \le \sum_{i\in I}\langle \Lambda_i S_\Lambda^{-1}(x), \Lambda_i S_\Lambda^{-1}(x)\rangle = \langle S_\Lambda^{-1}(x), x\rangle$$

and $\langle x, S_\Lambda^{-1}(x)\rangle = \sum_{i\in I}\langle \Lambda_i S_\Lambda^{-1}(x), \Lambda_i S_\Lambda^{-1}(x)\rangle \le D\langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle D^*$. The two last relations show that $D^{-1}\langle S_\Lambda^{-1}(x), x\rangle(D^*)^{-1} \le \langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle \le C^{-1}\langle S_\Lambda^{-1}(x), x\rangle(C^*)^{-1}$ for all $x \in X$. Hence $D^{-1}(D^*)^{-1}S_\Lambda^{-1} \le (S_\Lambda^{-1})^2 \le C^{-1}(C^*)^{-1}S_\Lambda^{-1}$. Since S is positive operator, we arrive at $D^{-1}(D^*)^{-1}I_X \le S_\Lambda^{-1} \le C^{-1}(C^*)^{-1}I_X$. According to this and that Λ is a $*$-g-frame, we get

$$\begin{aligned} &\sum_{i\in I}\langle \tilde{\Lambda}_i(x), \tilde{\Lambda}_i(x)\rangle = \sum_{i\in I}\langle \Lambda_i S_\Lambda^{-1}(x), \Lambda_i S_\Lambda^{-1}(x)\rangle \\ &\le D\langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle D^* \le DC^{-1}\langle S_\Lambda^{-1}(x), x\rangle(C^*)^{-1}D^* \\ &\le DC^{-1}(C^*)^{-1}C^{-1}\langle x,x\rangle(C^*)^{-1}D^* \le DC^{-1}(C^*)^{-1}\langle x,x\rangle C^{-1}(C^*)^{-1}D^*, \end{aligned}$$

for all $x \in X$. Similarly, we find

$$\begin{aligned} &\sum_{i\in I}\langle \tilde{\Lambda}_i(x), \tilde{\Lambda}_i(x)\rangle = \sum_{i\in I}\langle \Lambda_i S_\Lambda^{-1}(x), \Lambda_i S_\Lambda^{-1}(x)\rangle \\ &\ge C\langle S_\Lambda^{-1}(x), S_\Lambda^{-1}(x)\rangle C^* \ge CD^{-1}\langle S_\Lambda^{-1}(x), x\rangle(D^*)^{-1}C^* \ge CD^{-1}(D^*)^{-1}D^{-1}\langle x,x\rangle(D^*)^{-1}D^* \\ &\ge CD^{-1}(D^*)^{-1}\langle x,x\rangle D^{-1}(D^*)^{-1}C^*, \end{aligned}$$

for all $x \in X$. In addition

$$x = S_\Lambda^{-1} S_\Lambda(x) = S_\Lambda^{-1} \sum_{i\in I} \Lambda_i^* \Lambda_i(x) = \sum_{i\in I} S_\Lambda^{-1} \Lambda_i^* \Lambda_i(x) = \sum_{i\in I} (\tilde{\Lambda}_i)^* \Lambda_i(x),$$

for all $x \in X$. Once more, $x = S_\Lambda S_\Lambda^{-1}(x) = \sum_{i\in I} \Lambda_i^* \Lambda_i(S_\Lambda^{-1}(x)) = \sum_{i\in I} \Lambda_i^* \tilde{\Lambda}_i(x)$. This completes the proof. □

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