AN APPROXIMATION SCHEME FOR SOLVING QUASI-VARIATIONAL INCLUSIONS AND QUASIMONOTONE VARIATIONAL INEQUALITIES

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In this article, we discuss iterative schemes for solving quasi-variational inclusions and quasimonotone variational inequalities in Hilbert spaces. An iterative scheme is presented which consists of resolvent technique, extragradient-type algorithm and self-adaptive linear search rule. Under several appropriate conditions, we prove the convergence of the investigated scheme.

Keywords: quasi-variational inclusions, quasimonotone variational inequalities, extragradient-type method, linear search rule.

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1. Introduction

In this article, we are concerned with iterative schemes for the quasi-variational inclusion

$$\text{Find a point } s^* \in 2^{\mathcal{H}} \text{ such that } 0 \in U(s^*) + G(s^*), \tag{1}$$

where $\mathcal{D}$ is a nonempty, convex and closed subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $U \colon \mathcal{D} \to \mathcal{H}$ and $G \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be two nonlinear operators. The solution set of (1) is expressed by $(U + G)^{-1}(0)$.

A great variety of problems originating from natural science, engineering applications and management services can be converted to this kind of quasi-variational inclusion problems, see, [18, 19]. Iterative algorithms for seeking $s^* \in (U + G)^{-1}(0)$ are studied in ([6, 19, 32, 54, 57]). These algorithms define a procedure through the following way

$$t_0 \in \mathcal{D},\ t_{n+1} = \mathrm{Res}_{\alpha}^{G}(t_n - \alpha U(t_n)),\ n \geq 0, \tag{2}$$

where $\alpha > 0$ is any constant and the operator $\mathrm{Res}_{\alpha}^{G} := (I + \alpha U)^{-1}$ is the resolvent of G. This way is so-called the resolvent method.

Let $F : \mathcal{D} \to \mathcal{H}$ be a nonlinear operator. Now, we focus on the following variational inequality

$$\text{Find } s^* \in \mathcal{D} \text{ such that } \langle F(s^*), s - s^* \rangle \geq 0,\ \forall s \in \mathcal{D}, \tag{3}$$

The prologue of variational inequality occurred in 1964 when Stampacchia ([24]) utilized it as a technique for solving partial differential equations. As we know, variational inequality theory has become a central concept in numerous problems such as optimization ([2, 7, 11, 14, 21, 30, 40, 42, 48, 56]), equilibrium problems ([33, 39, 54]), fixed point problems ([8, 20, 22, 26–28, 31, 36, 38, 41]) and so on. Various numerical approaches have been put forward for computing a solution of variational inequality. For some related valuable methods and techniques, please refer to [1, 4, 17, 25, 34, 37, 43–46, 52, 53, 55].

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Now, we all know that finding a solution s^* of (3) can be translated into solving a fixed point problem, i.e.,

$$s^* \in \mathrm{VI}(F, \mathcal{D}) \Leftrightarrow s^* = proj_{\mathcal{D}}(s^* - \alpha F(s^*)), \tag{4}$$

where $\mathrm{VI}(F, \mathcal{D})$ denotes the set of solution of (3), $proj_{\mathcal{D}}$ means the metric projection from $\mathcal{H}$ onto $\mathcal{D}$ and α is a positive constant. Based on the equivalent relation (4), we can compute a solution of (3) by utilizing the following projection algorithm ([16, 23, 35])

$$t_0 \in \mathcal{D}, t_{n+1} = proj_{\mathcal{D}}(t_n - \alpha F(t_n)),\ \forall n \geq 0. \tag{5}$$

In fact, if F is strongly monotone or inverse strongly monotone ([3]), we can guarantee that $proj_{\mathcal{D}}(I - \alpha F)$ is a contractive operator provided $\alpha \in (0, 1/\nu)$ with ν being the Lipschitz constant of F. In this sense, the projection algorithm (5) is a popular method that can be easily implemented. In 1976, Korpelevich ([15]) designed an iterative algorithm below for computing a solution of (3)

$$\begin{cases} t_0 \in \mathcal{D}, \\ s_n = proj_{\mathcal{D}}(t_n - \alpha F(t_n)), \\ t_{n+1} = proj_{\mathcal{D}}(t_n - \alpha F(s_n)),\ n \geq 0. \end{cases} \tag{6}$$

This novel algorithm (6) named "extragradient algorithm" is remarkable because it can be performed to compute a solution of pseudomonotone variational inequalities but the projection algorithm (5) fails. As a matter of fact, the projection algorithm (5) may be divergent if F is pseudomonotone ([13, 29]). In this meaning, the extragradient algorithm (6) is easier to perform and it has been extensively used and generalized, see, e.g., [5, 10, 12, 47, 50, 51, 58]. On the other hand, in order to ensure the convergence of the sequence $\{t_n\}$, an additional condition

$$\mathrm{VI}(F, \mathcal{D}) \subset \mathrm{DVI}(F, \mathcal{D}) \tag{7}$$

was employed where $\mathrm{DVI}(F, \mathcal{D})$ is the solution set of the following variational inequality

$$\text{Find } s^* \in \mathcal{D} \text{ such that } \langle F(s), s - s^* \rangle \geq 0,\ \forall s \in \mathcal{D}. \tag{8}$$

The variational inequality (8) is so-called the dual variational inequality of (3). From (8), it is easy to see that $\mathrm{DVI}(F, \mathcal{D})$ is a closed convex set. It is known that if $\mathcal{D}$ is convex and F is continuous, then $\mathrm{DVI}(F, \mathcal{D}) \subset \mathrm{VI}(F, \mathcal{D})$. Note that the assumption (7) holds when F is pseudomonotone however it may not hold when F is quasimonotone.

The aim of this article is to introduce an iterative procedure for solving the quasi-variational inclusion problem (1) and the quasimonotone variational inequality (3) in Hilbert spaces. Our procedure consists of resolvent technique, extragradient-type algorithm and self-adaptive linear search rule. Under several appropriate conditions, we prove the convergence of the investigated scheme.

2. **Preliminaries**

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In the sequel, we use the symbols " $\rightharpoonup$ " and " $\to$ " to indicate weak convergence and strong convergence, respectively. Let $G\colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a multi-valued operator with domain $\mathrm{D}(G) = \{s^* \in \mathcal{H} : G(s^*) \neq \emptyset\}$. Set $G^{-1}(0) := \{s^* \in \mathcal{H} : 0 \in G(s^*)\}$. G is said to be monotone if and only if $\langle p - q, s - t \rangle \geq 0$, $\forall p, q \in \mathrm{D}(G)$ where $s \in G(p)$ and $t \in G(q)$. An operator $G\colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is said to be maximal monotone ([19]) if and only if G is monotone and its graph is not strictly contained in the graph of any other monotone operator.

For a maximal monotone operator $G\colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$, we can define its resolvent Res_α^G : $\mathcal{H} \to \mathrm{D}(G)$ by $\mathrm{Res}_\alpha^G := (I + \alpha G)^{-1}$ where $\alpha > 0$ is any constant.

Next, we list several important conclusions on Res_α^G:

(c1): $\mathrm{Res}_{\alpha}^{G}$ is a single-valued operator.
(c2): For any $\alpha > 0$, $G^{-1}(0) = \mathrm{Fix}(\mathrm{Res}_{\alpha}^{G}) := \{t \in \mathcal{H} : \mathrm{Res}_{\alpha}^{G}(t) = t\}$.
(c3): $\mathrm{Res}_{\alpha}^{G}$ fulfils the following inequality, for all $s, t \in \mathcal{H}$,

$$\|\mathrm{Res}_{\alpha}^{G}(s) - \mathrm{Res}_{\alpha}^{G}(t)\|^2 \leq \langle \mathrm{Res}_{\alpha}^{G}(s) - \mathrm{Res}_{\alpha}^{G}(t), s - t\rangle.$$

(c4): For all $\alpha, \tau > 0$, we have, for all $s \in \mathcal{H}$,

$$\|\mathrm{Res}_{\alpha}^{G}(s) - \mathrm{Res}_{\tau}^{G}(s)\|^2 \leq \frac{\alpha - \tau}{\tau}\langle \mathrm{Res}_{\alpha}^{G}(s) - \mathrm{Res}_{\tau}^{G}(s), \mathrm{Res}_{\alpha}^{G}(s) - s\rangle.$$

Let $\mathcal{D}$ be a nonempty convex and closed subset of a real Hilbert space $\mathcal{H}$. Let $F : \mathcal{D} \to \mathcal{H}$ be a single-valued mapping. Recall that F is said to be
(f1): λ-inverse strongly monotone if $\langle F(s) - F(t), s - t\rangle \geq \lambda\|F(s) - F(t)\|^2$ for all $s, t \in \mathcal{D}$.
(f2): monotone if $\langle F(s) - F(t), s - t\rangle \geq 0$ for all $s, t \in \mathcal{D}$.
(f3): pseudomonotone if $\langle F(s), t - s\rangle \geq 0$ implies that $\langle F(t), t - s\rangle \geq 0$ for all $s, t \in \mathcal{D}$.
(f4): quasimonotone if $\langle F(s), t - s\rangle > 0$ implies that $\langle F(t), t - s\rangle \geq 0$ for all $s, t \in \mathcal{D}$.
Recall that an operator $F : \mathcal{D} \to \mathcal{H}$ is said to be ν-Lipschitz continuous if $\|F(s) - F(t)\| \leq \nu\|s - t\|$, $\forall s, t \in \mathcal{D}$, where $\nu > 0$ is a constant. If $\nu = 1$, F is nonexpansive.

Let $U \colon \mathcal{D} \to \mathcal{H}$ be a λ-inverse strongly monotone operator. For the composition operator $\mathrm{Res}_{\alpha}^{G}(I - \alpha U) : \mathcal{D} \to \mathcal{D}$, we have the following conclusions:
(c5): $\mathrm{Res}_{\alpha}^{G}(I - \alpha U)$ is an averaged operator provided $\alpha \in (0, 2\lambda)$.
(c6): $s^* \in (U + G)^{-1}(0) \Leftrightarrow s^* = \mathrm{Res}_{\alpha}^{G}(I - \alpha U)(s^*)$ for all $\alpha \in (0, 2\lambda)$.
Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D} \subset \mathcal{H}$ be a closed convex set. A metric projection from $\mathcal{H}$ onto $\mathcal{D}$, denoted by $proj_{\mathcal{D}}$ satisfies, for $z \in \mathcal{H}$, $\|z - proj_{\mathcal{D}}(z)\| \leq \|\hat{z} - z\|$, $\forall \hat{z} \in \mathcal{D}$. It is well known that $proj_{\mathcal{D}}$ has the following characteristic inequality ([49])

$$z \in \mathcal{H},\ \langle z - proj_{\mathcal{D}}(z), \hat{z} - proj_{\mathcal{D}}(z)\rangle \leq 0,\ \forall \hat{z} \in \mathcal{D}. \tag{9}$$

In any Hilbert space $\mathcal{H}$, we have the following equality

$$\|tu + (1 - t)u^{\dagger}\|^2 = t\|u\|^2 + (1 - t)\|u^{\dagger}\|^2 - t(1 - t)\|u - u^{\dagger}\|^2, \tag{10}$$

$\forall u, u^{\dagger} \in \mathcal{H}$ and $\forall t \in \mathbb{R}$.

Lemma 2.1 ([9]). *Assume that $\mathcal{H}$ is a Hilbert space and $\mathcal{D} \subset \mathcal{H}$ is a nonempty, convex and closed set. Suppose that T is a nonexpansive self-mapping of $\mathcal{D}$. For a given sequence $\{t_n\} \subset \mathcal{D}$, $t_n \rightharpoonup z \in \mathcal{D}$ and $t_n - T(t_n) \to 0$ imply that $z \in \mathrm{Fix}(T)$.*

3. Main results

In this section, we demonstrate an iterative algorithm and a relevant convergence theorem for solving (1) and (3). Let $\mathcal{D}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Assume that the following assumptions are fulfilled:
(a1): $G \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is a maximal monotone operator fulfilling $\mathrm{D}(G) \subset \mathcal{D}$.
(a2): $U \colon \mathcal{D} \to \mathcal{H}$ is a λ-inverse strongly-monotone operator.
(a3): F is a quasimonotone operator on $\mathcal{H}$ and F is ν-Lipschitz on $\mathcal{D}$.
(a4): If $\{t_n\} \subset \mathcal{H}$ is a sequence satisfying $t_n \rightharpoonup t^{\dagger}$ and $\underline{\lim}_{n\to+\infty}\|F(t_n)\| = 0$, then $F(t^{\dagger}) = 0$.
Set $\Delta := (U + G)^{-1}(0) \cap \mathrm{DVI}(F, \mathcal{D})$. Suppose that $\Delta \neq \emptyset$ and the set $\{p \in \mathcal{D} : F(p) = 0\} \setminus \mathrm{DVI}(F, \mathcal{D})$ is finite. Let ϑ, ϱ and κ be three constants in $(0, 1)$. Let $\{\gamma_n\} \subset (0, 2)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\tau_n\} \subset (0, 1)$ be three real number sequences satisfying $0 < \underline{\lim}_{n\to\infty}\gamma_n \leq \overline{\lim}_{n\to\infty}\gamma_n < 2$, $0 < \underline{\lim}_{n\to\infty}\tau_n \leq \overline{\lim}_{n\to\infty}\tau_n < 1$ and $0 < \underline{\alpha} < \alpha_n < \overline{\alpha} < 2\lambda$ for all $n \geq 0$.

Now, we first present an iterative procedure for finding a point in Δ.

Algorithm 3.1. *Select an initial guess $t_0 \in \mathcal{D}$. Let $n = 0$.*

Step 1. Assume that t_n is obtained. Compute

$$s_n = (1 - \tau_n)t_n + \tau_n\mathrm{Res}_{\alpha_n}^{G}(I - \alpha_n U)t_n, \tag{11}$$

and

$$r_n = proj_{\mathcal{D}}[s_n - \vartheta \varrho_n F(s_n)], \tag{12}$$

where $\varrho_n = \max\{1, \varrho, \varrho^2, \cdots\}$ *such that*

$$\vartheta \varrho_n \|F(r_n) - F(s_n)\| \leq (1-\kappa)\|r_n - s_n\|. \tag{13}$$

Step 2. (i) If $r_n = s_n$*, then set* $t_{n+1} = s_n$ *and skip to step 1. (ii) If* $r_n \neq s_n$*, then compute*

$$t_{n+1} = proj_{\mathcal{D}}(s_n - \kappa \gamma_n q_n), \tag{14}$$

where

$$q_n = \left(\frac{\|r_n - s_n\|}{\|p_n\|}\right)^2 p_n, \tag{15}$$

and

$$p_n = s_n - r_n + \vartheta \varrho_n F(r_n). \tag{16}$$

Set $n := n+1$ *and return to Step 1.*

Proposition 3.1. *There exists* ϱ_n *satisfying* (13)*. Meanwhile,* $\frac{(1-\kappa)\varrho}{\vartheta\nu} < \varrho_n \leq \frac{1-\kappa}{\vartheta\nu}$*.*

Proof. Since F is ν-Lipschitz, $\|F(r_n) - F(t_n)\| \leq \nu\|r_n - t_n\|$. Then, there exists ϱ_n such that $\vartheta\varrho_n\nu\|r_n - t_n\| \leq (1-\kappa)\|r_n - t_n\|$ which implies that $\varrho_n \leq \frac{1-\kappa}{\vartheta\nu}$. By the definition of ϱ_n, ϱ_n/ϱ does not satisfy (13), that is, $\vartheta\frac{\varrho_n}{\varrho}\nu\|r_n - t_n\| \geq \vartheta\frac{\varrho_n}{\varrho}\|F(r_n) - F(t_n)\| > (1-\kappa)\|r_n - t_n\|$. It follows that $\frac{(1-\kappa)\varrho}{\vartheta\nu} < \varrho_n$. Therefore, $\frac{(1-\kappa)\varrho}{\vartheta\nu} < \varrho_n \leq \frac{1-\kappa}{\vartheta\nu}$. □

Proposition 3.2. *(a) If* $r_n = s_n$*, then* $s_n \in \mathrm{VI}(F, \mathcal{D})$*. (b) If* $r_n \neq s_n$*, then* $p_n \neq 0$*.*

Proof. (a) If $r_n = proj_{\mathcal{D}}[s_n - \vartheta\varrho_n F(s_n)] = s_n$, utilizing (9) we attain that $\langle s_n - [s_n - \vartheta\varrho_n F(s_n)], p - s_n\rangle \geq 0, \forall p \in \mathcal{D}$,, i.e., $\vartheta\varrho_n\langle F(s_n), p - s_n\rangle \geq 0, \forall p \in \mathcal{D}$. By Proposition 3.1, $\vartheta\varrho_n > \frac{(1-\kappa)\varrho}{\nu} > 0$. This together with the last inequality implies that $s_n \in \mathrm{VI}(F, \mathcal{D})$. (b) Let $q^* \in \mathrm{DVI}(F, \mathcal{D})$. Since $s_n \in \mathcal{D}$ and $r_n \in \mathcal{D}$, from (8), we obtain that $\langle F(s_n), s_n - q^*\rangle \geq 0$ and $\langle F(r_n), r_n - q^*\rangle \geq 0$. Thanks to (16), we gain

$$\begin{aligned}
\langle p_n, s_n - q^*\rangle &= \langle s_n - r_n + \vartheta\varrho_n F(r_n), s_n - q^*\rangle \\
&= \langle s_n - r_n - \vartheta\varrho_n F(s_n), s_n - q^*\rangle + \vartheta\varrho_n\langle F(r_n), s_n - r_n\rangle \\
&\quad + \vartheta\varrho_n\langle F(s_n), s_n - q^*\rangle + \vartheta\varrho_n\langle F(r_n), r_n - q^*\rangle \\
&\geq \langle s_n - r_n - \vartheta\varrho_n F(s_n), s_n - q^*\rangle + \vartheta\varrho_n\langle F(r_n), s_n - r_n\rangle \\
&= \langle s_n - r_n - \vartheta\varrho_n(F(s_n) - F(r_n)), s_n - r_n\rangle \\
&\quad + \langle s_n - r_n - \vartheta\varrho_n F(s_n), r_n - q^*\rangle \\
&= \|s_n - r_n\|^2 - \vartheta\varrho_n\langle F(s_n) - F(r_n), s_n - r_n\rangle \\
&\quad + \langle s_n - r_n - \vartheta\varrho_n F(s_n), r_n - q^*\rangle.
\end{aligned} \tag{17}$$

Taking into account (13), we have

$$\langle F(s_n) - F(r_n), s_n - r_n\rangle \leq \|F(s_n) - F(r_n)\|\|s_n - r_n\| \leq \frac{1-\kappa}{\vartheta\varrho_n}\|r_n - s_n\|^2. \tag{18}$$

Since $q^* \in \mathcal{D}$, from (12) and (9), we deduce

$$\langle s_n - r_n - \vartheta\varrho_n F(s_n), r_n - q^*\rangle \geq 0. \tag{19}$$

According to (17)-(19), we acquire

$$\langle p_n, s_n - q^*\rangle \geq \kappa\|r_n - s_n\|^2. \tag{20}$$

If $r_n \neq s_n$, then from (20) we get $\langle p_n, s_n - q^*\rangle > 0$ which results in that $p_n \neq 0$. □

In the sequel, we prove a convergence result.

Theorem 3.1. *The sequence $\{t_n\}$ defined by Algorithm 3.1 converges weakly to a point in $(U+G)^{-1}(0)\cap \mathrm{VI}(F,\mathcal{D})$.*

Proof. Let $q^*\in\Delta$. Take into account of (14) and (15), we have

$$\begin{aligned}\|t_{n+1}-q^*\|^2 &= \|proj_{\mathcal{D}}(s_n-\kappa\gamma_n q_n)-proj_{\mathcal{D}}(q^*)\|^2\\ &\le \|s_n-q^*-\kappa\gamma_n q_n\|^2\\ &= \|s_n-q^*\|^2-2\kappa\gamma_n\langle q_n, s_n-q^*\rangle+(\kappa\gamma_n)^2\|q_n\|^2\\ &= \|s_n-q^*\|^2-2\kappa\gamma_n\frac{\|r_n-s_n\|^2}{\|p_n\|^2}\langle p_n, s_n-q^*\rangle+(\kappa\gamma_n)^2\frac{\|r_n-s_n\|^4}{\|p_n\|^2}.\end{aligned}\tag{21}$$

In view of (20) and (21), we receive

$$\|t_{n+1}-q^*\|^2\le\|s_n-q^*\|^2-\kappa^2(2-\gamma_n)\gamma_n\frac{\|r_n-s_n\|^4}{\|p_n\|^2}.\tag{22}$$

By (c6), we obtain $q^*=\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)(q^*),\forall n\ge 0$. Furthermore, $\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)$ is nonexpansive. By (10) and (11), we have

$$\begin{aligned}\|s_n-q^*\|^2 &= \|(1-\tau_n)(t_n-q^*)+\tau_n(\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n-q^*)\|^2\\ &= (1-\tau_n)\|t_n-q^*\|^2-\tau_n(1-\tau_n)\|t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n\|^2\\ &\quad+\tau_n\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)q^*\|^2\\ &\le (1-\tau_n)\|t_n-q^*\|^2+\tau_n\|t_n-q^*\|^2\\ &\quad-\tau_n(1-\tau_n)\|t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n\|^2\\ &= \|t_n-q^*\|^2-\tau_n(1-\tau_n)\|t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n\|^2.\end{aligned}\tag{23}$$

Combining (22) and (23), we get

$$\begin{aligned}\|t_{n+1}-q^*\|^2\le\|t_n-q^*\|^2&-\tau_n(1-\tau_n)\|t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n\|^2\\ &-\kappa^2(2-\gamma_n)\gamma_n\frac{\|r_n-s_n\|^4}{\|p_n\|^2}.\end{aligned}\tag{24}$$

By (24), we have $\|t_{n+1}-q^*\|\le\|t_n-q^*\|$ which leads to that $\lim_{n\to\infty}\|t_n-q^*\|$ exists. Thus, we have the following results:

(b1): the sequence $\{t_n\}$ is bounded and so is $\{F(t_n)\}$ due to the Lipschitz continuity of F.
(b2): the sequence $\{s_n\}$ is bounded by (23) and so is $\{F(s_n)\}$.
(b3): the sequence $\{r_n\}$ is bounded because of $\|r_n\|\le\|s_n\|+\vartheta\varrho_n\|F(s_n)\|$ and so is $\{F(r_n)\}$.
(b4): the sequence $\{p_n\}$ is bounded by (16).

Further, by (24), we deduce

$$\begin{aligned}\tau_n(1-\tau_n)\|t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n\|^2&+\kappa^2(2-\gamma_n)\gamma_n\frac{\|r_n-s_n\|^4}{\|p_n\|^2}\\ &\le\|t_n-q^*\|^2-\|t_{n+1}-q^*\|^2\to 0.\end{aligned}\tag{25}$$

By the assumptions, we have $\underline{\lim}_{n\to\infty}\tau_n(1-\tau_n)>0$ and $\underline{\lim}_{n\to\infty}(2-\gamma_n)\gamma_n>0$. This together with (25) implies that

$$\lim_{n\to\infty}\|t_n-\mathrm{Res}^G_{\alpha_n}(I-\alpha_nU)t_n\|=0\tag{26}$$

and

$$\lim_{n\to\infty}\frac{\|r_n-s_n\|^2}{\|p_n\|}=0.\tag{27}$$

Let $\tau^\dagger > 0$ be any constant. Then, we have

$$\begin{aligned}
&\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\tau^\dagger U)t_n\| \\
&\qquad \le \|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n U)t_n\| \\
&\qquad\quad + \|\mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\tau^\dagger U)t_n\| \\
&\qquad \le \|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n H)t_n\| + |\alpha_n - \tau^\dagger|\|U(t_n)\|.
\end{aligned} \tag{28}$$

Using (c4), we attain

$$\begin{aligned}
&\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n U)t_n\|^2 \\
&\qquad \le \frac{\alpha_n-\tau^\dagger}{\tau^\dagger}\langle \mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n U)t_n, \\
&\qquad\quad \mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - (I-\alpha_n U)t_n\rangle \\
&\qquad \le \frac{|\alpha_n-\tau^\dagger|}{\tau^\dagger}\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n U)t_n\| \\
&\qquad\quad \times \|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - (I-\alpha_n U)t_n\|,
\end{aligned}$$

which results in that

$$\begin{aligned}
&\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\alpha_n U)t_n\| \\
&\qquad \le \frac{|\alpha_n-\tau^\dagger|}{\tau^\dagger}\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - (I-\alpha_n U)t_n\|.
\end{aligned} \tag{29}$$

Based on (28) and (29), we receive

$$\begin{aligned}
\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - \mathrm{Res}^G_{\tau^\dagger}(I-\tau^\dagger U)t_n\| &\le |\alpha_n-\tau^\dagger|\|U(t_n)\| \\
&+ \frac{|\alpha_n-\tau^\dagger|}{\tau^\dagger}\|\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - (I-\alpha_n U)t_n\|.
\end{aligned} \tag{30}$$

Since U is λ-inverse strongly-monotone, U is $1/\lambda$-Lipschitz continuous. By the boundedness of $\{t_n\}$, we deduce $\{U(t_n)\}$ is bounded. Hence, $\{\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n\}$ is bounded.

Next, we show that $\omega_w(t_n) \subset \Delta$. First, we prove $\omega_w(t_n) \subset (U+G)^{-1}(0)$. Pick up any $t^\dagger \in \omega_w(t_n)$.

Since $\{t_n\}$ and $\{\alpha_n\}$ are bounded, there exists a common subsequence $\{n_i\} \subset \{n\}$ fulfilling $t_{n_i} \rightharpoonup t^\dagger \in \mathcal{D}$ and $\alpha_{n_i} \to \alpha^\dagger \in (0, 2\lambda)$ as $i \to \infty$.

Selecting $\tau^\dagger = \alpha^\dagger$, from (30), we obtain

$$\begin{aligned}
\|\mathrm{Res}^G_{\alpha_{n_i}}(I-\alpha_{n_i} U)t_{n_i} - \mathrm{Res}^G_{\alpha^\dagger}(I-\alpha^\dagger U)t_{n_i}\| &\le |\alpha_{n_i}-\alpha^\dagger|\|U(t_{n_i})\| \\
&+ \frac{|\alpha_{n_i}-\alpha^\dagger|}{\alpha^\dagger}\|\mathrm{Res}^G_{\alpha_{n_i}}(I-\alpha_{n_i} U)t_{n_i} - (I-\alpha_{n_i} U)t_{n_i}\| \to 0.
\end{aligned}$$

This together with (26) implies that

$$\lim_{i\to\infty}\|t_{n_i} - \mathrm{Res}^G_{\alpha^\dagger}(I-\alpha^\dagger U)t_{n_i}\| = 0. \tag{31}$$

Now, we have the facts that $t_{n_i} \rightharpoonup t^\dagger \in \mathcal{D}$ and $\mathrm{Res}^G_{\alpha^\dagger}(I-\alpha^\dagger U)$ is nonexpansive. Applying Lemma 2.1 to (31), we derive that $t^\dagger \in \mathrm{Fix}(\mathrm{Res}^G_{\alpha^\dagger}(I-\alpha^\dagger U)) = (U+G)^{-1}(0)$. Hence, $\omega_w(t_n) \subset (U+G)^{-1}(0)$. Next, we show $t^\dagger \in \mathrm{VI}(F, \mathcal{D})$. According to (11), we get $s_n - t_n = \tau_n(\mathrm{Res}^G_{\alpha_n}(I-\alpha_n U)t_n - t_n)$. Then, by (26), we have

$$\lim_{n\to\infty}\|t_n - s_n\| = 0. \tag{32}$$

With the help of (14) and (15), we have

$$\|t_{n+1} - s_n\| = \|proj_{\mathcal{D}}(s_n - \kappa\gamma_n q_n) - proj_{\mathcal{D}}(s_n)\| \le \kappa\gamma_n\|q_n\| \le \kappa\gamma_n\frac{\|r_n - s_n\|^2}{\|p_n\|}. \tag{33}$$

It follows from (27) and (33) that $\lim_{n\to\infty}\|s_n - t_{n+1}\| = 0$. Hence, from (32), we receive

$$\lim_{n\to\infty}\|t_{n+1} - t_n\| = 0. \tag{34}$$

By the boundedness of $p_n = t_n - r_n + \vartheta\varrho_n F(r_n)$, thanks to (27), we attain

$$\lim_{n\to\infty}\|r_n - s_n\| = 0 \tag{35}$$

which together with the Lipschitz continuity of F results in that

$$\lim_{n\to\infty}\|F(r_n) - F(s_n)\| = 0. \tag{36}$$

By (9) and (11), we have

$$\langle s_n - \vartheta\varrho_n F(s_n) - r_n, z - r_n\rangle \le 0,\ \forall z \in \mathcal{D}.$$

Thus,

$$\langle F(s_n), z - s_n\rangle \ge \frac{1}{\vartheta\varrho_n}\langle s_n - r_n, z - r_n\rangle + \langle F(s_n), r_n - s_n\rangle,\ \forall z \in \mathcal{D}. \tag{37}$$

Observing that $\{s_{n_i}\}$, $\{r_{n_i}\}$ and $\{F(s_{n_i})\}$ are bounded, from (35) and (37), we acquire

$$\underline{\lim}_{i\to+\infty}\langle F(s_{n_i}), z - s_{n_i}\rangle \ge 0,\ \forall z \in \mathcal{D}. \tag{38}$$

Next, we divide two cases to prove that $t^\dagger \in \mathrm{VI}(F,\mathcal{D})$: (a) $\underline{\lim}_{i\to+\infty}\|F(s_{n_i})\| = 0$ and (b) $\underline{\lim}_{i\to+\infty}\|F(s_{n_i})\| > 0$.

For Case (a), since $t_{n_i} \rightharpoonup t^\dagger$, from (32), we have $s_{n_i} \rightharpoonup t^\dagger$. This together with condition (a4) implies that $F(t^\dagger) = 0$.

Now, we study Case (b). We can find an integer $k > 0$ such that $\|F(s_{n_i})\| > 0$ for all $i \ge k$. In terms of (38), we assert

$$\underline{\lim}_{i\to+\infty}\langle F(s_{n_i})/\|F(s_{n_i})\|, z - s_{n_i}\rangle \ge 0,\ \forall z \in \mathcal{D}. \tag{39}$$

Let $\{\beta_j\}$ be a positive real number sequence satisfying $\beta_{j+1} < \beta_j$ and $\beta_j \to 0$ as $j \to +\infty$. With the help of (39), there is $\{n_{i_j}\} \subset \{n_i\}$ fulfilling $n_{i_j} \ge k, \forall j \ge 0$ and

$$\forall z \in \mathcal{D},\ \langle F(s_{n_{i_j}})/\|F(s_{n_{i_j}})\|, z - s_{n_{i_j}}\rangle + \beta_j > 0,\ \forall j \ge 0,$$

which indicates that

$$\forall z \in \mathcal{D},\ \langle F(s_{n_{i_j}}), z - s_{n_{i_j}}\rangle + \beta_j\|F(s_{n_{i_j}})\| > 0,\ \forall j \ge 0. \tag{40}$$

Set $\hat\beta_j = F(s_{n_{i_j}})/\|F(s_{n_{i_j}})\|^2$ for all $j \ge 0$ which implies that $\langle F(s_{n_{i_j}}), \hat\beta_j\rangle = 1$ for all $j \ge 0$. Thanks to (40), we obtain

$$\forall z \in \mathcal{D},\ \langle F(s_{n_{i_j}}), \beta_j\hat\beta_j\|F(s_{n_{i_j}})\| + z - s_{n_{i_j}}\rangle > 0,\ \forall j \ge 0. \tag{41}$$

Based on (41) and the quasimonotonicity of F, we assert

$$\forall z \in \mathcal{D},\ \langle F(z + \beta_j\hat\beta_j\|F(s_{n_{i_j}})\|), \beta_j\hat\beta_j\|F(s_{n_{i_j}})\| + z - s_{n_{i_j}}\rangle \ge 0,\ \forall j \ge 0. \tag{42}$$

Notice that $\lim_{j\to+\infty}\beta_j\|\hat\beta_j\|\|F(s_{n_{i_j}})\| = \lim_{j\to+\infty}\beta_j = 0$. Using the Lipschitz continuity of F, we have $F(z + \beta_j\hat\beta_j\|F(s_{n_{i_j}})\|) \to F(z)$ as $j \to +\infty$. Letting $j \to +\infty$ in (42), we deduce $\langle F(z), z - t^\dagger\rangle \ge 0,\ \forall z \in \mathcal{D}$, which implies that $t^\dagger \in \mathrm{DVI}(F,\mathcal{D})$. According to Case (a) and Case (b), we conclude that $\omega_w(t_n) \subset \big(\{x \in \mathcal{D} : F(x) = 0\} \cup \mathrm{DVI}(F,\mathcal{D})\big) \subset \mathrm{VI}(F,\mathcal{D})$. Therefore, $\omega_w(t_n) \subset (U+G)^{-1}(0) \cap \mathrm{VI}(F,\mathcal{D})$.

Next, we prove $\{t_n\}$ has at most one weak cluster point in $\mathrm{DVI}(F,\mathcal{D})$. Assume $\tilde t$ and $\hat t$ are two weak cluster points of $\{t_n\}$ in $\mathrm{DVI}(F,\mathcal{D})$. Then, there are $\{t_{n_i}\} \subset \{t_n\}$ and $\{t_{n_j}\} \subset \{t_n\}$ satisfying $t_{n_i} \rightharpoonup \tilde t(i \to \infty)$ and $t_{n_j} \rightharpoonup \hat t(j \to \infty)$. It is obviously that

$$2\langle t_n, \tilde t - \hat t\rangle = \|t_n - \hat t\|^2 - \|t_n - \tilde t\|^2 + \|\tilde t\|^2 - \|\hat t\|^2,\ \forall n \ge 0,$$

which results in that $\lim_{n\to+\infty}\langle t_n, \tilde{t}-\hat{t}\rangle$ exists. Therefore,

$$\lim_{i\to+\infty}\langle t_{n_i}, \tilde{t}-\hat{t}\rangle = \lim_{j\to+\infty}\langle t_{n_j}, \tilde{t}-\hat{t}\rangle \tag{43}$$

Taking into account $t_{n_i} \rightharpoonup \tilde{t}$ and $t_{n_j} \rightharpoonup \hat{t}$, by (44), we receive

$$\langle \tilde{t}, \tilde{t}-\hat{t}\rangle = \langle \hat{t}, \tilde{t}-\hat{t}\rangle$$

which yields that $\tilde{t} = \hat{t}$. So, $\{t_n\}$ has no more than one weak cluster point in $\mathrm{DVI}(F, \mathcal{D})$. Since the set $\{x \in \mathcal{D} : F(x) = 0\} \setminus \mathrm{DVI}(F, \mathcal{D})$ is finite, we conclude that $\{t_n\}$ has only finite weak cluster points in $(U + G)^{-1}(0) \cap \mathrm{VI}(F, \mathcal{D})$. Suppose that $t^{(1)}, t^{(2)}, \cdots, t^{(m)}$ are weak cluster points of $\{t_n\}$ in $(U + G)^{-1}(0) \cap \mathrm{VI}(F, \mathcal{D})$. Set $I = \{1, 2, \cdots, m\}$ and

$$d = \min\{\|t^{(r)} - t^{(s)}\|/3, r, s \in I, r \neq s\}. \tag{44}$$

For $t^{(r)}, r \in I$, there is $\{t^r_{n_i}\} \subset \{t_n\}$ satisfying $t^r_{n_i} \rightharpoonup t^{(r)}$ as $i \to +\infty$. Then,

$$\lim_{i\to+\infty}\langle t^r_{n_i}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle = \langle t^{(r)}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle,\ \forall s \in I. \tag{45}$$

Note that

$$\begin{aligned}\langle t^{(r)}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle &= \frac{\|t^{(r)} - t^{(s)}\|}{2} + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|} \\ &> d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|},\ \forall s \neq r.\end{aligned} \tag{46}$$

Combining (45) and (46), there is a positive integer $\mathbb{N}^r_i > 0$ fulfilling for all $i \geq \mathbb{N}^r_i$,

$$t^r_{n_i} \in \left\{\hat{t} : \langle \hat{t}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|}\right\},\ s \in I, s \neq r. \tag{47}$$

Write

$$\Theta_r = \bigcap_{s=1, s\neq r}^{m} \left\{\hat{t} : \langle \hat{t}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|}\right\}. \tag{48}$$

Combining (47) with (48), we deduce that $t^r_{n_i} \in \Theta_r$ for all $i \geq \max\{\mathbb{N}^r_i, r \in I\}$.

Next we show that $t_n \in \bigcup_{r=1}^m \Theta_r$ for large enough n. Suppose that there is $\{t_{n_j}\} \subset \{t_n\}$ verifying $t_{n_j} \notin \bigcup_{r=1}^m \Theta_r$. According to the boundedness of $\{t_{n_j}\}$, there exists a subsequence of $\{t_{n_j}\}$, without loss of generality, still denoted by $\{t_{n_j}\}$, which converges weakly to $\hat{v}$. Hence $t_{n_j} \notin \Theta_r$ for any $r \in I$. Then, there is a subsequence $\{t_{n_{j_s}}\}$ of $\{t_{n_j}\}$ such that $\forall s \geq 0$,

$$t_{n_{j_s}} \notin \left\{\hat{t} : \langle \hat{t}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|}\right\}, s \in I, s \neq r. \tag{49}$$

So,

$$\hat{v} \notin \left\{\hat{t} : \langle \hat{t}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|}\right\}, s \in I, s \neq r,$$

which results in that $\hat{v} \neq t^{(r)} (\forall r \in I)$. It is impossible. Thus, there is an integer N_1 large enough such that $t_n \in \bigcup_{r=1}^m \Theta_r$ for all $n \geq N_1$.

Finally, we demonstrate that $\{t_n\}$ has the unique weak cluster point in $(U+G)^{-1}(0) \cap \mathrm{VI}(F, \mathcal{D})$. Assume that $m \geq 2$. From (34), there exists a positive integer $N_2 \geq N_1$ such that $\|t_{n+1} - t_n\| < d$ for all $n \geq N_2$. Then, there exists $p \geq N_2$ such that $t_p \in \Theta_r$ and $t_{p+1} \in \Theta_s$, where $r, s \in I$ and $m \geq 2$, that is,

$$t_p \in \Theta_r = \bigcap_{s=1, s\neq r}^{m} \left\{\hat{t} : \langle \hat{t}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|}\right\}$$

and

$$t_{p+1} \in \Theta_s = \bigcap_{r=1, r\neq s}^{m} \Big\{\hat{t} : \langle \hat{t}, \frac{t^{(s)} - t^{(r)}}{\|t^{(s)} - t^{(r)}\|}\rangle > d + \frac{\|t^{(s)}\|^2 - \|t^{(r)}\|^2}{2\|t^{(s)} - t^{(r)}\|}\Big\}.$$

Thus, we have

$$\langle t_p, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > d + \frac{\|t^{(r)}\|^2 - \|t^{(s)}\|^2}{2\|t^{(r)} - t^{(s)}\|} \tag{50}$$

and

$$\langle t_{p+1}, \frac{t^{(s)} - t^{(r)}}{\|t^{(s)} - t^{(r)}\|}\rangle > d + \frac{\|t^{(s)}\|^2 - \|t^{(r)}\|^2}{2\|t^{(s)} - t^{(r)}\|}. \tag{51}$$

Thanks to (50) and (51), we acquire

$$\langle t_p - t_{p+1}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle > 2d. \tag{52}$$

Note that

$$\|t_{p+1} - t_p\| < d. \tag{53}$$

By virtue of (52) and (53), we receive

$$2d < \langle t_p - t_{p+1}, \frac{t^{(r)} - t^{(s)}}{\|t^{(r)} - t^{(s)}\|}\rangle \leq \|t_p - t_{p+1}\| < d.$$

It leads to a contradiction. Thus, $\{t_n\}$ has a unique weak cluster point in $(U+G)^{-1}(0) \cap \mathrm{VI}(F, \mathcal{D})$. Therefore, $\{t_n\}$ converges weakly to a point in $(U+G)^{-1}(0) \cap \mathrm{VI}(F, \mathcal{D})$. □

4. **Concluding remarks**

In this paper, we investigate iterative schemes for finding a common solution of quasi-variational inclusion (1) and quasimonotone variational inequality (3) in a Hilbert space $\mathcal{H}$. We present an iterative scheme $\{t_n\}$ generated by Algorithm 3.1 which consists of resolvent operator step (11), extragradient step (12)-(14) and self-adaptive search step (13). Under several appropriate conditions, we show that the proposed scheme $\{t_n\}$ defined by Algorithm 3.1 converges weakly to some common solution of quasi-variational inclusion (1) and quasimonotone variational inequality (3).

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