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WEAK AND F-WEAK MULTIPLICATION BCK-MODULES

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In this paper by considering the notion of BCK-module X, we have introduced weak and F-weak multiplication BCK-modules and we have obtained some related results. As a result we have shown that, every homomorphic image of a weak multiplication BCK-module is a weak multiplication BCK-module.

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1. Introduction

In 1966, Imai and Iseki [10, 13] introduced BCK-algebras. This notion was originated from two different ways: (1) set theory, and (2) classical and no classical propositional calculi. Certain algebraic structures, for example Boolean-algebra, MV-algebras,… are introduced as BCK-algebras [12]. Boolean-algebras are certain type of BCK-algebras. Today BCK-algebras have been applied to many branches of mathematics, including group theory, functional analysis, probability theory, topology, fuzzy set theory. Investigation of the mathematical foundations of quantum mechanics showed that the Kolmogorov model of probability theory, holding for classical mechanics, fails in the case of quantum mechanics. Birkhoff and Von Neumann [3] introduced quantum logics, as algebraic systems describing the kinds of propositional logics that can be associated to Hilbert spaces. Quantum logics are more general than Boolean-algebras; the most important example is given by the quantum logics £(H) consisting of all closed subspaces of a real, complex, or quaternion Hilbert space H. For more details see [7].

Every module is an action of ring on certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. BCK-module is an action of BCK-algebra on commutative group. In 1994, the notion of BCK-module was introduced by M. Aslam, H. A. S. Abujabal and A. B. Thaheem [2]. They established isomorphism theorems and studied some

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properties of BCK-modules. The theory of BCK-modules was further developed by Z. Perveen and M. Aslam [16]. Also in 2003, the notion of MV-module was introduced by A. Di Nola, P. Flondor and I. Leustean as an action of MV-algebra on pmv-algebra [6].

Multiplication modules have been recently considered by many authors, either over a commutative ring [8] or over a non commutative ring [18].

Now, in this note, we have defined the concept of weak and F-weak multiplication BCK- modules and we have obtained some results as mentioned in the abstract.

2. Preliminaries

Let us to begin this section with the definition of a BCK-algebra.

Definition 2.1 [14] Let X be a set with a binary operation * and a constant 0. Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following axioms:

(BCK1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCK2) $((x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = y * x = 0$ imply that $x = y$, for all $x, y, z \in X$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If there is an element 1 of a BCK- algebra X, satisfying $x * 1 = 0$, for all $x \in X$, the element 1 is called unit of X. A BCK-algebra with unit is called to be bounded.

Definition 2.2 [14] Let $(X, *, 0)$ be a BCK- algebra and X_0 be a non-empty subset of X. Then X_0 is said to be a sub-algebra of X, if for any $x, y \in X_0$, $x * y \in X_0$ i.e., X_0 is closed under the binary operation * of X.

Definition 2.3 [14] A BCK-algebra $(X, *, 0)$ is said to be commutative, if it satisfies, $x * (x * y) = y * (y * x)$, for all x, y in X.

Definition 2.4 [14] A BCK-algebra $(X, *, 0)$ is called implicative, if $x = x * (y * x)$, for all x, y in X.

Definition 2.5 [1] Let $(X, *, 0)$ be a BCK-algebra, M be an abelian group under + and let $(x, m) \rightarrow x \cdot$ m be a mapping of $X \times M \rightarrow M$ such that

(i) $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(iii) $0 \cdot m = 0$,

For all $x, y \in X, m_1, m_2 \in M$, where $x \wedge y = y * (y * x)$. Then M is called a left X-module.

If X is bounded, then the following additional condition holds:

(iv) $1 \cdot m = m$.

A right X-module can be defined similarly.

A non-empty subset N of M is called a X-sub module of M, if $(N, +)$ is a X-module.

In the sequel X is a BCK-algebra.

Definition 2.6 [1] Let M be a left BCK-module over X and N be a BCK-sub module of M. Then we define $(N: M) = \{x \in X \mid x \cdot M \subseteq N\}$.

Definition 2.7 [5] Let $(X, *, 0)$ be a BCK-algebra, M be an abelian group under $+$ and let $(x, m) \to x \cdot$ m be a mapping of $X \times M \to M$ such that

i. $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,
ii. $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
iii. $0 \cdot m = 0$,
iv. $(x \vee y).m = x.m + (y * x).m$

Then M is called extended BCK-module.

Definition 2.8 [14] A non-empty subset A of BCK-algebra $(X, *, 0)$ is called a BCK-ideal of X if it satisfies the following conditions:

(i) $0 \in A$,
(ii) $(\forall x \in X)(\forall y \in A)(x * y \in A \Rightarrow x \in A)$.

Lemma 2.9 [2] Let X be a bounded implicative BCK-algebra and $x + y = (x * y) * (y * x)$, for all $x, y \in X$. Then we have:

(i) $(X, +)$ forms a commutative group,
(ii) Any ideal I of X consisting of two elements forms a X- module.

Definition 2.10 [11] Let X be a lower BCK-semi lattice and A be a proper ideal of X. Then A is said to be prime if $a \wedge b = b * (b * a) \in A$ implies that $a \in A$ or $b \in A$, for any a,b in X.

Lemma 2.11 [14] In a lower BCK-semi lattice $(X, *, 0)$ the following are equivalent:

(i) I is a prime ideal,
(ii) I is an ideal and satisfies that for any $A,B \in I(X)$, $A \subseteq I$ *or* $B \subseteq I$, whenever $A \cap B \subseteq I$.

Definition 2.12 [9] Let M be a X-module. Then M is called of type 2 if for all m in M and x_1,x_2 in X, $(x_1+x_2).m=x_1.m+x_2.m$.

Theorem 2.13 Let M be a X- module of type 2, $m \in M$. Then $X \cdot m$ is a X- sub module.

Proof: For all $m \in M$, it is obvious $0 \cdot m = 0$. Also for all m_1, m_2 in M, and $x_1,x_2 \in X$, $x_1.m-x_2.m=(x_1-x_2).m$. So $X \cdot m$ is a X- sub module of M.

Lemma 2.14 [1] Every bounded implicative BCK-algebra forms module over itself.

Example 2.15 [1] Let A be a non-empty set and $X = P(A)$ be the power set of A. Then X is a bounded commutative BCK-algebra with $x \wedge y = x \cap y$, for all $x,y \in X$. Define $x+y = (x \cup y) \cap (x \cap y)'$, the symmetric difference. Then $M = (X, +)$ is an abelian group with empty set as an identity element and $x + x = \emptyset$. Define $x \cdot m = x \cap m$, for any $x , m \in X$. Then simple calculations show that:

(i) $(x \wedge y) \cdot m = (x \cap y) \cap m = x \cap (y \cap m) = x \cdot (y \cdot m)$,

(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,

(iii) $0 \cdot m = \emptyset \cap m = \emptyset = 0$,

(iv) $1 \cdot m = A \cap m = m$. Thus X itself is an X-module.

Definition 2.16 [1] Let M_1 , M_2 be X-modules. A mapping $f : M_1 \rightarrow M_2$ is called a BCK- homomorphism, if for any $m_1 , m_2 \in M_1$, we have :

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$,

(ii) $f(x \cdot m_1) = x \cdot f(m_1)$, for all $x \in X$.

Ker (f) and *Image* (f) have usual meaning.

Definition 2.17 [14] In a lower BCK-semi lattice X, a non -empty subset S of X is said to be $\wedge$-closed if $x \wedge y \in S$ whenever $x, y \in S$.

Theorem 2.18 [9] Let X be bounded implicative and M be an X-module. If S is a $\wedge$-closed subset of X, then the sub modules of M_s are on the form N_s where $N = \{n \in M : \frac{n}{1} \in N_s\}$.

Theorem 2.19[11] Let M be a X-module and the operation $o : X_s \times M_s \rightarrow M_s$ be defined by $\frac{x}{s} \, o \, \frac{m}{t} = \frac{(x.m)}{(s \wedge t)}$. Then M_s is a X_s-module.

Theorem 2.20 Let X be an X-module and N be a X-sub module of M. Then

a) $\frac{n}{1}$ is a X -module by $x. (m + N) = (x \cdot m) + N$, for all $x \in X$ and $m + N \in \frac{M}{N}$.

b) $\varphi : M \rightarrow \frac{M}{N}$ such that $\varphi (m) = m + N$, for all $m \in M$, is an epiorphism.

Proof: It is obvious.

Definition 2.21 Let M be a left BCK- module over X and N be a BCK-sub module of M. Then we define $Ann_X(M) = \{x \in X \mid x \cdot m = 0$, for all $m \in M\}$. M is called faithful if $Ann_X(M) = 0$.

Theorem 2.22 [2] Any ideal consisting of two elements in a bounded commutative BCK- algebra X forms a X- module under the binary operations $x \cdot m = x \wedge m$.

Example 2.23 [5] Let X be a non -empty set. Then $(P(X),-)$ is a bounded BCK-algebras, Z (integer set) with the followings operations is a $P(X)$-module, $x_0 \in X$ and $\cdot : P(X) \times Z \rightarrow Z$ such that

$$A.n = \begin{cases} n & if\ x_0 \in A \\ 0 & if\ x_0 \notin A \end{cases}$$

Definition 2.24 [14] In a lower BCK-semi lattice X, a non -empty subset S of X is said to be ∧-closed if $x \wedge y \in S$ whenever $x, y \in S$.

Definition 2.25 [15] Let X be a BCK-algebra and M be a X-module. Then a proper sub module N of M is said to be prime if for every $x \in X$, $m \in M$; $x \cdot m \in N$ implies that $m \in N$ or $x \in (N : M)$. If X be positive implicative and M be extended module, and $P = (N : M)$ is a prime ideal of X , N is said to be P-prime sub module of M.

Example 2.26 [15] Let $X = P\ (A = \{1, 2, ..., n\})$, $B = \{1, 2, ..., n\} - \{i\}$, for $i \in \{1, 2, ..., n\}$. Then P (B) is a prime BCK- sub module of $P\ (A)$.

Theorem 2.27 [15] Let M_1 and M_2 be left BCK- modules over X and φ be a BCK-homomorphism from M_1 to M_2. Also N be a prime BCK-sub module of M_2. Then $\varphi^{-1}(N)$ is a prime BCK-sub module of M_1.

3. Weak and F-Weak Multiplication BCK- module

In 1994, the notion of BCK-module was introduced by M. Aslam, H. A. S. Abujabal and A. B. Thaheem [2]. In this section we have defined weak and F-weak multiplication BCK-modules and we have obtained some theorems.

Definition 3.1 Let X be a BCK-algebra and M be a X-module. Then M is called a weak multiplication BCK-module, if $spec(M) = \emptyset$ or for each prime BCK-sub module N of M, there exists an ideal I of X, such that $N = I \cdot M$.

Example 3.2 Assume $A = \{1, 2\}$ and $X = P(A)$. Simple calculations and Example 2.15 show that all of X-sub modules of $P(A)$ and all of BCK-ideals of $P(A)$ are $\{\emptyset\}$, $\{\emptyset, \{1\}\}$, $\{\emptyset, \{2\}\}$, $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. By Example 2.26 we have the prime BCK-sub modules of $P(A)$ in this example are $\{\emptyset\}$, $\{\emptyset, \{1\}\}$,$\{\emptyset, \{2\}\}$. By some simple calculations we get that X is a weak multiplication X-module.

Theorem 3.3 Let X be a positive implicative BCK-algebra and M be an extended weak multiplication BCK-module. If N be a P-prime BCK-sub module of M, then $N = P \cdot M$.

Proof: Since N is a P-prime sub module of M, then for all p in P, $p \cdot M \subseteq N$, so $P \cdot M \subseteq N$. Since M is a weak multiplication X-module, there exists an ideal I of X, such that $N = I.M$, hence $I.M \subseteq N$ i.e. $I \subseteq (N : M) = P$. So $I.M = N \subseteq P \cdot M$. Therefore the proof is complete.

Lemma 3.4 Let X be a bounded and implicative BCK-algebra and M be an extended X-module. Then M is a weak multiplication X-module if and only if for each prime sub module N of M, $N = (N : M) \cdot M$.

Proof: **Necessity.** Let M be a weak multiplication BCK-module. Then, if spec $(M) = \emptyset$, the proof is complete. Otherwise for each prime sub module N of M, there exists an ideal I of X, such that $N = I.M$, hence $I \subseteq (N : M)$ and sequence $N = I \cdot M \subseteq (N : M) \cdot M \subseteq N$, so $N = (N : M) \cdot M$.
Sufficiency. It is obvious.

Theorem 3.5 Let X be a bounded and implicative BCK-algebra. If M is an extended cyclic X-module, then it is a weak multiplication X-module.
Proof: Let M be a cyclic X-module and spec $(M) \neq \emptyset$. Then there exists $m \in M$ such that $M = X \cdot m$. Now let N be a prime X-sub module of M and $n \in N$. Hence there exists $x \in X$, such that $n = x \cdot m$. So, $x \cdot M = x \cdot (X \cdot m) = (x \wedge X) \cdot m = X \cdot (x \cdot m) = X \cdot n \subseteq N$ and so $x \in (N : M)$.
Therefore $N \subseteq (N : M) \cdot M$. On the other hand, since $(N : M) \cdot M \subseteq N$, thus $N = (N : M) \cdot M$, so by Theorem 3.4, M is a weak multiplication X-module.

Theorem 3.6 Let X be a BCK-algebra, M be a weak multiplication X -module and N be a prime X-sub module of M. Then $N = (ann(\frac{M}{N})) \cdot M$.

Proof: Since M is a X-module, then $\frac{M}{N}$ is a X-module by $(x , m + N) \rightarrow x \cdot m + N$. First we show that $N \subseteq ann(\frac{M}{N}) \cdot M$. We have M is a weak multiplication X-module, so $spec(M) = \emptyset$ or for prime X-sub module N of M, there exists an ideal I of X such that $N = I \cdot M$. If $spec(M) = \emptyset$, the proof is complete. Otherwise for $m \in M$, $x \in I$, we get that $x \cdot m \in N$, so $I \subseteq ann(\frac{M}{N})$ and sequence $N = I \cdot M \subseteq (ann(\frac{M}{N})) \cdot M$. Now we show $(ann(\frac{M}{N})) \cdot M \subseteq N$. *Let* $x \in (ann(\frac{M}{N})) \cdot M$. Then for $r_i \in ann(\frac{M}{N})$ and $m_i \in M$, $x = \sum_{i=1}^{k} r_i . m_i$, since $r_i \in ann(\frac{M}{N})$, So $r_i \cdot \frac{M}{N} = N$.
Hence $r_i . m_i \subseteq N$. So $x \in N$, and the proof is complete.

Theorem 3.7 Let X be a BCK-algebra, M be a weak multiplication X –module and $\varphi: M \rightarrow M_1$ be a X-module epimorphism. Then $\varphi(M)$ is a weak multiplication module.
Proof: Let $M_1 = \varphi(M)$. Then if spec $(M_1) = \emptyset$, then the proof is complete. Suppose N' be a prime sub module of $\varphi(M)$. Since $\varphi : M \rightarrow M_1$ is onto, then by Theorem 2.27 $\varphi^{-1}(N')$ is a prime sub module of M. Now since M is a weak multiplication X-module, then there exists an ideal I in X such that $\varphi^{-1}(N') = I \cdot M$. Since φ is a X-module epimorphism. So $N' = I \cdot \varphi(M)$, namely $\varphi(M)$ is a weak multiplication X-module.

Corollary 3.8 Let X be a BCK-algebra, M be a weak multiplication X -module and N, be a prime X-sub module of M. Then $\frac{M}{N}$ is a weak multiplication X-module.

Theorem 3.9 Let X be bounded implicative, M be a X-module and S be a ∧-closed subset of X. Then if J is a prime sub module of M_s, then there exists a prime sub module N of M such that $J = N_s$.
Proof: By Theorem 2.18, we get that $J = N_s$ is a sub module of M_s. Now we show that if $J = N_s$ be a prime sub module of M_s, then N is a prime sub module of M. Since $N_s \neq M_s$, it is obvious that $N \neq M$. Let $x \in X$ and $m \in M$, such that $x.m \in N$. Then we must to show $m \in N$ or $x \cdot M \subseteq N$. If $x \cdot m \in N$, then $\frac{x \cdot m}{1} = \frac{x}{1} \cdot \frac{m}{1} = \frac{x \cdot m}{1 \wedge 1} \in J = N_s$. By primitivity $J = N_s$, $\frac{m}{1} \in J$ or $\frac{x}{1} \cdot M_s \subseteq J$. If $\frac{m}{1} \in J$, then $m \in N$ and the proof is complete. If $\frac{x}{1} \cdot M_s \subseteq J$, then for all $m \in M$ and $s \in S$, $\frac{x}{1} \cdot \frac{m}{s} = \frac{x \cdot m}{1 \wedge s} = \frac{x \cdot m}{s} \in J$. Let $s = 1$. Then for all $m \in M$, $x \cdot m_1 \in J$, so $x \cdot m \in N$. i.e. $x \cdot M \subseteq N$, and the proof is complete.

Theorem 3.10 Let X be a bounded implicative BCK-algebra. If S is a ∧-closed subset of X and M is a weak multiplication X-module, then M_s is a weak multiplication X_s -module.
Proof: If spec $(M_s) = \emptyset$, the proof is complete. If T is a prime sub module of M_s, then by Theorem 3.9 there exists a prime sub module N of M such that $T = N_s$. There exists an ideal I of X such that $N = I \cdot M$. So $Ns = I_s \cdot M_s$, hence M_s is a weak multiplication X_s -module.

Corollary 3.11 Let X be a bounded implicative BCK-algebra. Then, if M be a weak multiplication X-module and P be a prime ideal of X, then M_P is a weak multiplication as a X_P -module.
Definition 3.12 An X-module M is said to be F-weak multiplication, if it satisfies the following conditions:
1) M is a weak multiplication.
2) For every $P \in spec(X)$, $P \cdot M$ is a prime sub module of M and $(P \cdot M : M) = P$.

Theorem 3.13 Let M be a F-weak multiplication X-module and let $I \subseteq X$ and $P \in spec(X)$. If $I.M \subseteq P.M$, then $I \subseteq P$.
Proof: If $I.M \subseteq P \cdot M$, then $(I \cdot M : M) \subseteq (P \cdot M : M) = P$ and clearly $I \subseteq (I \cdot M : M)$. So $I \subseteq P$.

Corollary 3.14 Let M be a F-weak multiplication X-module, I be an ideal of X and $P \in spec(X)$. Then, if $I \cdot M \subseteq P \cdot M$, then $I \subseteq P$.

Definition 3.15 A BCK-module M is called a prime cancellation BCK-module or a P-cancellation BCK-module, if for every $P, Q \in spec(X)$, $P \cdot M = Q \cdot M$ implies that $P = Q$.

Theorem 3.16 Let M be a F-weak multiplication X-module. Then M is a P-cancellation X-module.

Proof: It is a particular case of theorem 3.13.

Theorem 3.17 Let M be a F-weak multiplication X-module, M_1 be a X -module and $\varphi : M \rightarrow M_1$ be an epimorphism such that ker φ is contained in every prime sub modules of M. Then M_1 is a F-weak multiplication X -module.

Proof: First, let L_1 be an arbitrary prime sub module of M. Then there exists a prime sub module L of M such that $\varphi(L) = L_1$. Since M is a F-weak multiplication module, there exists an ideal $P \in spec(X)$ such that $P \cdot M = L$. Hence $L = P \cdot M = \varphi^{-1}(L_1)$ implies that $\varphi(P \cdot M) = L_1$, that is, $P \cdot \varphi(M) = L_1$ which means $P \cdot M_1 = L_1$. Therefore M_1 is a weak multiplication X-module. Second, let $P \in spec(X)$ be an arbitrary prime ideal. Then we must prove that $P \cdot M_1 \in spec(M_1)$ and $(P \cdot M_1 : M_1) = P$. But $P \cdot M_1 = P \cdot \varphi(M) = \varphi(P \cdot M) \subseteq M_1$. Since M is F-weak multiplication, then $P.M \in spec(M)$ and so $P \cdot M_1 = \varphi(P \cdot M) \in spec(M_1)$. Now we prove that $(P \cdot M_1 : M_1) = P$. Obviously, $P \subseteq (P \cdot M_1 : M_1)$. We show that $(P \cdot M_1 : M_1) \subseteq P$. *But* $(P \cdot M_1 : M_1) = (P \cdot \varphi(M) : \varphi(M)) = (\varphi(P \cdot M) : \varphi(M))$. Let $x \in (P \cdot M_1 : M_1) = (\varphi(P \cdot M) : \varphi(M))$. Then $x \cdot \varphi(M) \subseteq \varphi(P \cdot M)$ that is, $\varphi(x \cdot M) \subseteq \varphi(P \cdot M)$. Since $x \cdot M \subseteq \varphi^{-1}(\varphi(x \cdot M)) = \varphi^{-1}(P \cdot M_1)$, we know that $\varphi^{-1}(P \cdot M_1)$ is a prime sub module of M, so there exists an ideal I of X such that $\varphi^{-1}(P \cdot M_1) = I \cdot M$, since M is a weak multiplication X-module, hence $\varphi(\varphi^{-1}(P \cdot M_1)) = \varphi(I \cdot M)$, namely $P \cdot M_1 = I \cdot \varphi(M) = I \cdot M_1$, therefore $(I - P) \cdot \varphi(M) = 0$, *so* $(I - P) \cdot M \in ker\ \varphi \subseteq P \cdot M$, by Theorem 3.13 we get $I - P \subseteq P$, so $I = P$, namely $\varphi^{-1}(P \cdot M_1) = P.M$, then $x \cdot M \subseteq P \cdot M$ and so $x \in (P \cdot M : M) = P$. Therefore $(P \cdot M_1 : M_1) \subseteq P$. Hence $(P \cdot M_1 : M_1) = P$ and so M_1 is a F-weak multiplication X-module.

Corollary 3.18 Let M be a F-weak multiplication X-module and N be a sub module of M such that N is contained in every prime sub module of M. Then $\frac{M}{N}$ is a F-weak multiplication X-module.

Corollary 3.19 Let M_i, $1 \leq i \leq n$, be a collection of X-modules. Then, if $M = \bigoplus_{i=1}^{n}(M_i)$ is a weak multiplication X-module, then for every $1 \leq i \leq n$, M_i is a weak multiplication X-module.

Proof: We define the map φ_i as follows:
$\varphi_i : M = \bigoplus_{i=1}^{n}(M_i) \rightarrow M_i$, $i = 1, 2, ..., n$ by $\varphi_i(m_1, m_2, ..., m_n) = m_i$, for $(m_1, m_2, ..., m_n) \in \bigoplus_{i=1}^{n}(M_i)$.
Since φ_i is an epimorphism, the result follows by the first part of the proof of Theorem 3.17.

4. Conclusion

The authors have defined a weak and F-weak multiplication BCK-module and have obtained some theorems. As a result they have proved every cyclic BCK-module is a weak multiplication BCK-module.

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