SOME TRIDIAGONAL MATRICES AND DETERMINANTS OF SCHUR-COHN CRITERION FOR TRINOMIALS

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Matrices of two types, which have three diagonals with nonzero elements, are investigated. Formulas for determinants of these tridiagonal matrices are derived. These formulas are tested on large-order matrices, where the calculation is shown to be truly efficient. Then the formulas are used for Schur-Cohn criterion for trinomials.

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1. Introduction

The original aim of this paper was to give explicit formulas for determinants of tridiagonal matrices of a certain type. See e. g. [3] as an overview in the topic of tridiagonal matrices. The motivation was the study of Schur-Cohn determinants (for the case of trinomials) which are used for the purpose of locating roots of polynomials. However, results are also of the autonomous value. Derived formulas are easy to use and effective even for very large matrices. We can recommend the paper [5] of Qi, Čerňanová and Semenov as a suitable reference to compare our results with results already known.

In Section 2, main results are formulated and proved. The Section 3 is focused on Schur-Cohn determinants, in particular, for the case of trinomials. The Schur-Cohn criterion for the location of polynomial roots is very well explained in the classical Marden's monograph Geometry of Polynomials ([4]).

The use of effective calculating of these determinants is then demonstrated on specific examples in the Section 4. The examples we study here follow the research [2] of one of the authors.

2. Determinants of tridiagonal matrices

We consider complex square matrices of order n

$$H_n^{(1)}(c,s,v)=\begin{pmatrix} v & s & 0 & 0 & \dots & 0 & 0 & 0\\ s & c & s & 0 & \dots & 0 & 0 & 0\\ 0 & s & c & s & \dots & 0 & 0 & 0\\ 0 & 0 & s & c & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & c & s & 0\\ 0 & 0 & 0 & 0 & \dots & s & c & s\\ 0 & 0 & 0 & 0 & \dots & 0 & s & c \end{pmatrix}$$

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and

$$H_n^{(2)}(c,s,v) = \begin{pmatrix} v & 0 & s & 0 & \dots & 0 & 0 & 0 \\ 0 & v & 0 & s & \dots & 0 & 0 & 0 \\ s & 0 & c & 0 & \dots & 0 & 0 & 0 \\ 0 & s & 0 & c & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c & 0 & s \\ 0 & 0 & 0 & 0 & \dots & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \dots & s & 0 & c \end{pmatrix},$$

where $c, s, v \in \mathbb{C}$. In this section, we demonstrate how to evaluate the determinants of these matrices for any $n \in \mathbb{N}$ explicitly. We set $\varepsilon_n = \frac{1-(-1)^n}{2}$ for expressions that will follow below. It is obvious that $\varepsilon_{n+1} = 1 - \varepsilon_n$ and $\varepsilon_m = \varepsilon_n$ if and only if $m \equiv n(mod2)$.

2.1. Results

Theorem 2.1. *The determinant of the matrix $H_n^{(1)}(c,s,v)$ can be given by the formula*

$$\begin{aligned} \det H_n^{(1)}(c,s,v) \quad &= \quad \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2-2t} \binom{n-2-t}{t} \left(vc\frac{n-1-t}{n-1-2t} - s^2 \right) \\ &\quad + \varepsilon_n (-1)^{\frac{n-1}{2}} v s^{n-1}. \end{aligned}$$

Theorem 2.2. *The determinant of the matrix $H_n^{(2)}(c,s,v)$ can be given by the formula*

$$\det H_n^{(2)}(c,s,v) = \det H^{(1)}_{\frac{n+\varepsilon_n}{2}}(c,s,v) \det H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s,v).$$

2.2. Proofs

We will consider the matrix

$$B_n(c,r,s) = \begin{pmatrix} c & s & 0 & 0 & \dots & 0 & 0 & 0 \\ r & c & s & 0 & \dots & 0 & 0 & 0 \\ 0 & r & c & s & \dots & 0 & 0 & 0 \\ 0 & 0 & r & c & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c & s & 0 \\ 0 & 0 & 0 & 0 & \dots & r & c & s \\ 0 & 0 & 0 & 0 & \dots & 0 & r & c \end{pmatrix}$$

and write shortly $|\ \ |$ for det.

Lemma 2.1. *Let us set $|B_{-1}(c,r,s)| = 0$ and $|B_0(c,r,s)| = 1$. Then the recurrence relation*

$$|B_n(c,r,s)| = c|B_{n-1}(c,r,s)| - rs|B_{n-2}(c,r,s)| \tag{1}$$

holds for every $n \in \mathbb{N}$.

Proof. We obtain the formula (1) directly for $n = 1$ and $n = 2$. For $n \geq 3$ we expand the determinant $|B_n(c,r,s)|$ according to the first row and the first column. □

Lemma 2.2. *The determinant of the matrix $B_n(c,r,s)$ can be given by the formula*

$$|B_n(c,r,s)| = \sum_{t=0}^{\frac{n-\varepsilon_n}{2}} (-1)^t r^t s^t c^{n-2t} \binom{n-t}{t}. \tag{2}$$

Proof. This lemma can be verified by induction for $n \geq 1$. It is easy to verify the formula (2) for $n = 1$ and $n = 2$. Suppose the formula (2) is valid for $n \geq 2$, and we will prove its validity for $n+1$. So we have to prove:

$$|B_{n+1}(c,r,s)| = \sum_{t=0}^{\frac{n+1-\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t} \binom{n+1-t}{t}.$$

By the formula (1) we have:

$$|B_{n+1}(c,r,s)| = c|B_n(c,r,s)| - rs|B_{n-1}(c,r,s)|$$

As the formula (2) is valid for n and $n-1$ we can write:

$$\begin{aligned} |B_{n+1}(c,r,s)| &= c\sum_{t=0}^{\frac{n-\varepsilon_n}{2}} (-1)^t r^t s^t c^{n-2t} \binom{n-t}{t} \\ &\qquad -rs \sum_{t=0}^{\frac{n-1-\varepsilon_{n-1}}{2}} (-1)^t r^t s^t c^{n-1-2t} \binom{n-1-t}{t} \end{aligned}$$

But $\varepsilon_n = 1 - \varepsilon_{n+1}$ and $\varepsilon_{n-1} = \varepsilon_{n+1}$, so:

$$\begin{aligned} |B_{n+1}(c,r,s)| &= \sum_{t=0}^{\frac{n-1+\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t} \binom{n-t}{t} \\ &\qquad + \sum_{t=0}^{\frac{n-1-\varepsilon_{n+1}}{2}} (-1)^{t+1} r^{t+1} s^{t+1} c^{n+1-2(t+1)} \binom{n-(t+1)}{t} \end{aligned}$$

Substituting each t by $t-1$ in the second sum of the last equation, we get:

$$\begin{aligned} |B_{n+1}(c,r,s)| &= \sum_{t=0}^{\frac{n-1+\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t} \binom{n-t}{t} \\ &\qquad + \sum_{t=1}^{\frac{n+1-\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t} \binom{n-t}{t-1} \end{aligned}$$

We note that the upper bounds of these two sums satisfy:

$$\frac{n+1-\varepsilon_{n+1}}{2} = \frac{n-1+\varepsilon_{n+1}}{2} + 1 - \varepsilon_{n+1}$$

where $\varepsilon_{n+1} = \frac{1-(-1)^{n+1}}{2} = 0$ or 1, so by calculating the term corresponding to $t = 0$ in the first sum and the term corresponding to $t = \frac{n+1-\varepsilon_{n+1}}{2}$ in the second sum, we find:

$$\begin{aligned} |B_{n+1}(c,r,s)| &= (-1)^0 r^0 s^0 c^{n+1}\binom{n}{0} + \sum_{t=1}^{\frac{n-1+\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t} \binom{n-t}{t} \\ &\qquad + \sum_{t=1}^{\frac{n-1+\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t} \binom{n-t}{t-1} \\ &\qquad +(1-\varepsilon_{n+1})(-1)^p r^p s^p c^{n+1-2p} \binom{n-p}{p-1} \end{aligned}$$

where $p = \frac{n+1-\varepsilon_{n+1}}{2}$, then:

$$\begin{aligned}
|B_{n+1}(c,r,s)| &= (-1)^0 r^0 s^0 c^{n+1}\binom{n}{0} \\
&\quad + \sum_{t=1}^{\frac{n-1+\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t}\left[\binom{n-t}{t}+\binom{n-t}{t-1}\right] \\
&\quad +(1-\varepsilon_{n+1})(-1)^p r^p s^p c^{n+1-2p}\binom{n-p}{p-1}
\end{aligned}$$

Since $\binom{n}{0} = \binom{n+1}{0}$, $\binom{n-t}{t}+\binom{n-t}{t-1} = \binom{n+1-t}{t}$ and $\binom{n-p}{p-1} = \binom{n+1-p}{p}$ when $\varepsilon_{n+1} = 0$ (when $\varepsilon_{n+1} = 1$, the last term in the statement of $|B_{n+1}(c,r,s)|$ is not found) we get:

$$\begin{aligned}
|B_{n+1}(c,r,s)| &= (-1)^0 r^0 s^0 c^{n+1}\binom{n+1}{0} \\
&\quad + \sum_{t=1}^{\frac{n-1+\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t}\binom{n+1-t}{t} \\
&\quad +(1-\varepsilon_{n+1})(-1)^p r^p s^p c^{n+1-2p}\binom{n+1-p}{p} \\
&= \sum_{t=0}^{\frac{n+1-\varepsilon_{n+1}}{2}} (-1)^t r^t s^t c^{n+1-2t}\binom{n+1-t}{t}
\end{aligned}$$

□

If we denote $B_n(c,s)$ the matrix $B_n(c,r,s)$ when $r = s$, then immediately we get

$$|B_n(c,s)| = \sum_{t=0}^{\frac{n-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2t}\binom{n-t}{t} \tag{3}$$

Remark 2.1. We note that $B_n(c,1)$ is the matrix studied by Qi, Čerňanová and Semenov in [5]. They denoted this matrix by $M_n(c)$. Our formula for its determinant is

$$|B_n(c,1)| = \sum_{t=0}^{\frac{n-\varepsilon_n}{2}} (-1)^t c^{n-2t}\binom{n-t}{t}.$$

It corresponds with the result in [5], however, our sum includes only half of terms, as we do not use such binomial coefficients $\binom{p}{q}$ in which $q > p$.

Now, we turn our attention to the matrices $H_n^{(1)}(c,s,v)$.

Proof of the Theorem 2.1. By expanding the determinant of the matrix $H_n^{(1)}(c,s,v)$ according to the first row and the first column we obtain the recurrence relation

$$|H_n^{(1)}(c,s,v)| = v|B_{n-1}(c,s)| - s^2|B_{n-2}(c,s)| \tag{4}$$

which holds for every $n \geq 1$. By the formula (3) and the fact $\varepsilon_{n-2} = \varepsilon_n$ and $\varepsilon_{n-1} = 1 - \varepsilon_n$ we find

$$\begin{aligned}
|B_{n-2}(c,s)| &= \sum_{t=0}^{\frac{n-2-\varepsilon_{n-2}}{2}} (-1)^t s^{2t} c^{n-2-2t} \binom{n-2-t}{t} \\
&= \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2-2t} \binom{n-2-t}{t}, \\
|B_{n-1}(c,s)| &= \sum_{t=0}^{\frac{n-1-\varepsilon_{n-1}}{2}} (-1)^t s^{2t} c^{n-1-2t} \binom{n-1-t}{t} \\
&= \sum_{t=0}^{\frac{n-2+\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-1-2t} \binom{n-1-t}{t}.
\end{aligned}$$

But:

$$\frac{n-2+\varepsilon_n}{2} = \frac{n-2-\varepsilon_n}{2} + \varepsilon_n$$

where $\varepsilon_n = \frac{1-(-1)^n}{2} = 0$ or 1, Thus:

$$\begin{aligned}
|B_{n-1}(c,s)| &= \sum_{t=0}^{\frac{n-2+\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-1-2t} \binom{n-1-t}{t} \\
&= \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-1-2t} \binom{n-1-t}{t} \\
&\quad + \varepsilon_n . (-1)^{\frac{n-2+\varepsilon_n}{2}} s^{2\frac{n-2+\varepsilon_n}{2}} c^{n-1-2\frac{n-2+\varepsilon_n}{2}} \binom{n-1-\frac{n-2+\varepsilon_n}{2}}{\frac{n-2+\varepsilon_n}{2}} \\
&= \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-1-2t} \binom{n-1-t}{t} \\
&\quad + \varepsilon_n (-1)^{\frac{n-1}{2}} s^{2\frac{n-1}{2}} c^{n-1-2(\frac{n-1}{2})} \binom{n-1-\frac{n-1}{2}}{\frac{n-1}{2}} \\
&= \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-1-2t} \binom{n-1-t}{t} + \varepsilon_n (-1)^{\frac{n-1}{2}} s^{n-1}
\end{aligned}$$

Thus:

$$\begin{aligned}
\det H_n^{(1)}(c,s,v) &= v.|B_{n-1}(c,s)| - s^2.|B_{n-2}(c,s)| \\
&= v\sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-1-2t}\binom{n-1-t}{t} + \varepsilon_n(-1)^{\frac{n-1}{2}} v s^{n-1} \\
&\quad -s^2 \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2-2t}\binom{n-2-t}{t} \\
&= vc\sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2-2t}\frac{n-1-t}{n-1-2t}\binom{n-2-t}{t} \\
&\quad -s^2 \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2-2t}\binom{n-2-t}{t} \\
&\quad +\varepsilon_n(-1)^{\frac{n-1}{2}} v s^{n-1}
\end{aligned}$$

So we can write:

$$\begin{aligned}
\det H_n^{(1)}(c,s,v) &= \sum_{t=0}^{\frac{n-2-\varepsilon_n}{2}} (-1)^t s^{2t} c^{n-2-2t}\binom{n-2-t}{t}\left(vc\frac{n-1-t}{n-1-2t} - s^2\right) \\
&\quad +\varepsilon_n(-1)^{\frac{n-1}{2}} v s^{n-1}
\end{aligned}$$

□

Furthermore, for $n \in \mathbb{N}$, let us consider the matrix

$$E_n(c,s) = \begin{pmatrix} c & 0 & s & 0 & \dots & 0 & 0 & 0 \\ 0 & c & 0 & s & \dots & 0 & 0 & 0 \\ s & 0 & c & 0 & \dots & 0 & 0 & 0 \\ 0 & s & 0 & c & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c & 0 & s \\ 0 & 0 & 0 & 0 & \dots & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \dots & s & 0 & c \end{pmatrix}$$

and the matrix

$$I_n(c,s) = \begin{pmatrix} c & s & 0 & 0 & \dots & 0 & 0 & 0 \\ s & c & 0 & s & \dots & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \dots & 0 & 0 & 0 \\ 0 & s & 0 & c & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c & 0 & s \\ 0 & 0 & 0 & 0 & \dots & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \dots & s & 0 & c \end{pmatrix}$$

(for clarification, we specify that $I_n(c,s)$ differs from $E_n(c,s)$ only in the first row and in the first column). For $n = 0$ we put $|E_0(c,s)| = 1$ and $|I_0(c,s)| = 0$.

Lemma 2.3. *For $n \geq 2$, the recurrence relation*

$$|I_n(c,s)| = c|E_{n-1}(c,s)| - s^2|E_{n-2}(c,s)| \tag{5}$$

holds.

Proof. We can obtain the formula (5) by expanding the determinant $|I_n(c,s)|$ according to the first row then according to the first column. $\square$

Lemma 2.4. *For $n \geq 4$, the recurrence relation*

$$|E_n(c,s)| = c|E_{n-1}(c,s)| - cs^2|E_{n-3}(c,s)| + s^4|E_{n-4}(c,s)| \tag{6}$$

holds.

Proof. By expanding the determinant $|E_n(c,s)|$ according to the first row then to the first column when $n \geq 4$ we obtain:

$$|E_n(c,s)| = c|E_{n-1}(c,s)| - s^2|I_{n-2}(c,s)|$$

By the formula (5) we find:

$$\begin{aligned} |E_n(c,s)| &= c|E_{n-1}(c,s)| - s^2\left[c|E_{n-3}(c,s)| - s^2|E_{n-4}(c,s)|\right] \\ &= c|E_{n-1}(c,s)| - cs^2|E_{n-3}(c,s)| + s^4|E_{n-4}(c,s)| \end{aligned}$$

$\square$

Lemma 2.5. *For $n \geq 1$, the determinant of the matrix $E_n(c,s)$ can be given by the formula:*

$$|E_n(c,s)| = |B_{\frac{n+\varepsilon_n}{2}}(c,s)||B_{\frac{n-\varepsilon_n}{2}}(c,s)|. \tag{7}$$

Proof. This lemma can be verified by induction for $n \geq 1$. It is easy to verify the formula (7) for $n = 1$, 2, 3 and 4.
Suppose the assertion holds for $n \geq 4$, and we will prove that for $n+1$. So we have to prove:

$$|E_{n+1}(c,s)| = |B_{\frac{n+1+\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)|.$$

By Lemma 2.4 we have

$$|E_{n+1}(c,s)| = c|E_n(c,s)| - cs^2|E_{n-2}(c,s)| + s^4|E_{n-3}(c,s)|.$$

As the formula (7) is valid for n , $n-2$ and $n-3$, we have

$$\begin{aligned} |E_{n+1}(c,s)| &= c|B_{\frac{n+\varepsilon_n}{2}}(c,s)||B_{\frac{n-\varepsilon_n}{2}}(c,s)| \\ &\quad -cs^2|B_{\frac{n-2+\varepsilon_{n-2}}{2}}(c,s)||B_{\frac{n-2-\varepsilon_{n-2}}{2}}(c,s)| \\ &\quad +s^4|B_{\frac{n-3+\varepsilon_{n-3}}{2}}(c,s)||B_{\frac{n-3-\varepsilon_{n-3}}{2}}(c,s)| \end{aligned}$$

But $\varepsilon_n = \varepsilon_{n-2} = 1 - \varepsilon_{n+1}$ and $\varepsilon_{n-3} = \varepsilon_{n+1}$, so:

$$\begin{aligned} |E_{n+1}(c,s)| &= c|B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n-1+\varepsilon_{n+1}}{2}}(c,s)| \\ &\quad -cs^2|B_{\frac{n-1-\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n-3+\varepsilon_{n+1}}{2}}(c,s)| \\ &\quad +s^4|B_{\frac{n-3+\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n-3-\varepsilon_{n+1}}{2}}(c,s)| \end{aligned}$$

By Lemma 2.1 we have

$$|B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)| = c|B_{\frac{n-1-\varepsilon_{n+1}}{2}}(c,s)| - s^2|B_{\frac{n-3-\varepsilon_{n+1}}{2}}(c,s)|.$$

So

$$\begin{aligned} |E_{n+1}(c,s)| &= c|B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n-1+\varepsilon_{n+1}}{2}}(c,s)| \\ &\quad -s^2|B_{\frac{n-3+\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)| \end{aligned}$$

Again, we can write

$$\begin{aligned}|E_{n+1}(c,s)| &= |B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)|\left[c|B_{\frac{n-1+\varepsilon_{n+1}}{2}}(c,s)| - s^2|B_{\frac{n-3+\varepsilon_{n+1}}{2}}(c,s)|\right] \\ &= |B_{\frac{n+1-\varepsilon_{n+1}}{2}}(c,s)||B_{\frac{n+1+\varepsilon_{n+1}}{2}}(c,s)|.\end{aligned}$$

The proof is complete. □

Corollary 2.1. *For $n \geq 2$, the determinant of the matrix $I_n(c,s)$ can be given by the formula*

$$|I_n(c,s)| = |B_{\frac{n-2+\varepsilon_n}{2}}(c,s)||B_{\frac{n+2-\varepsilon_n}{2}}(c,s)|. \tag{8}$$

Proof. By Lemma 2.3, for $n \geq 2$ we have

$$|I_n(c,s)| = c|E_{n-1}(c,s)| - s^2|E_{n-2}(c,s)|.$$

By Lemma 2.5 and the fact $\varepsilon_{n-1} = 1 - \varepsilon_n$ and $\varepsilon_{n-2} = \varepsilon_n$ we can write

$$\begin{aligned}|I_n(c,s)| &= c|B_{\frac{n-1+\varepsilon_{n-1}}{2}}(c,s)||B_{\frac{n-1-\varepsilon_{n-1}}{2}}(c,s)| \\ &\quad -s^2|B_{\frac{n-2+\varepsilon_{n-2}}{2}}(c,s)||B_{\frac{n-2-\varepsilon_{n-2}}{2}}(c,s)| \\ &= c|B_{\frac{n-\varepsilon_n}{2}}(c,s)||B_{\frac{n-2+\varepsilon_n}{2}}(c,s)| - s^2|B_{\frac{n-2+\varepsilon_n}{2}}(c,s)||B_{\frac{n-2-\varepsilon_n}{2}}(c,s)| \\ &= |B_{\frac{n-2+\varepsilon_n}{2}}(c,s)|\left[c|B_{\frac{n-\varepsilon_n}{2}}(c,s)| - s^2|B_{\frac{n-2-\varepsilon_n}{2}}(c,s)|\right].\end{aligned}$$

By Lemma 2.1 we have $|B_{\frac{n+2-\varepsilon_n}{2}}(c,s)| = c|B_{\frac{n-\varepsilon_n}{2}}(c,s)| - s^2|B_{\frac{n-2-\varepsilon_n}{2}}(c,s)|$. Then

$$|I_n(c,s)| = |B_{\frac{n-2+\varepsilon_n}{2}}(c,s)||B_{\frac{n+2-\varepsilon_n}{2}}(c,s)|.$$

□

Lemma 2.6. *Let us consider the matrix*

$$G_n(c,s,v) = \begin{pmatrix} v & 0 & s & 0 & \ldots & 0 & 0 & 0 \\ 0 & c & 0 & s & \ldots & 0 & 0 & 0 \\ s & 0 & c & 0 & \ldots & 0 & 0 & 0 \\ 0 & s & 0 & c & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & c & 0 & s \\ 0 & 0 & 0 & 0 & \ldots & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \ldots & s & 0 & c \end{pmatrix}.$$

Then for $n \in \mathbb{N}$

$$|G_n(c,s,v)| = |B_{\frac{n-\varepsilon_n}{2}}(c,s)||H^{(1)}_{\frac{n+\varepsilon_n}{2}}(c,s,v)|.$$

holds.

Proof. It can be verified easily that this lemma is valid for $n = 1$, 2 and 3. For $n \geq 4$ we can write

$$|G_n(c,s,v)| = v|E_{n-1}(c,s)| - s^2|I_{n-2}(c,s)|.$$

By Lemma 2.5 and Corollary 2.1, we have

$$\begin{aligned}
|G_n(c,s,v)| &= v|B_{\frac{n-1+\varepsilon_{n-1}}{2}}(c,s)||B_{\frac{n-1-\varepsilon_{n-1}}{2}}(c,s)| \\
&\quad -s^2|B_{\frac{n-2-2+\varepsilon_{n-2}}{2}}(c,s)||B_{\frac{n-2+2-\varepsilon_{n-2}}{2}}(c,s)| \\
&= v|B_{\frac{n-1+(1-\varepsilon_n)}{2}}(c,s)||B_{\frac{n-1-(1-\varepsilon_n)}{2}}(c,s)| \\
&\quad -s^2|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)||B_{\frac{n-\varepsilon_n}{2}}(c,s)| \\
&= v|B_{\frac{n-\varepsilon_n}{2}}(c,s)||B_{\frac{n-2+\varepsilon_n}{2}}(c,s)| \\
&\quad -s^2|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)||B_{\frac{n-\varepsilon_n}{2}}(c,s)| \\
&= |B_{\frac{n-\varepsilon_n}{2}}(c,s)|\left[v|B_{\frac{n-2+\varepsilon_n}{2}}(c,s)| - s^2|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)|\right].
\end{aligned}$$

But $\det H^{(1)}_{\frac{n+\varepsilon_n}{2}}(c,s,v) = v|B_{\frac{n-2+\varepsilon_n}{2}}(c,s)| - s^2|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)|$ by the formula (4). So

$$|G_n(c,s,v)| = |B_{\frac{n-\varepsilon_n}{2}}(c,s)||H^{(1)}_{\frac{n+\varepsilon_n}{2}}(c,s,v)|.$$

□

Proof of the Theorem 2.2. The Theorem 2.2 can be easily proven for $n = 1$, 2, 3 and 4. By expanding the determinant $\det H^{(2)}_n(c,s,v)$ according to the first two rows and first two columns when $n \geq 5$, we obtain

$$|H^{(2)}_n(c,s,v)| = v|G_{n-1}(c,s)| - vs^2|E_{n-3}(c,s)| + s^4|E_{n-4}(c,s)|.$$

Then by lemma 2.5 and lemma 2.6 we have:

$$\begin{aligned}
\det H^{(2)}_n(c,s,v) &= v|B_{\frac{n-1-\varepsilon_{n-1}}{2}}(c,s)||H^{(1)}_{\frac{n-1+\varepsilon_{n-1}}{2}}(c,s,v)| \\
&\quad -vs^2|B_{\frac{n-3+\varepsilon_{n-3}}{2}}(c,s)||B_{\frac{n-3-\varepsilon_{n-3}}{2}}(c,s)| \\
&\quad +s^4|B_{\frac{n-4+\varepsilon_{n-4}}{2}}(c,s)||B_{\frac{n-4-\varepsilon_{n-4}}{2}}(c,s)|
\end{aligned}$$

Since $\varepsilon_{n-1} = \varepsilon_{n-3} = 1-\varepsilon_n$ and $\varepsilon_{n-4} = \varepsilon_n$ we find:

$$\begin{aligned}
\det H^{(2)}_n(c,s,v) &= v|B_{\frac{n-2+\varepsilon_n}{2}}(c,s)||H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s,v)| \\
&\quad -vs^2|B_{\frac{n-2-\varepsilon_n}{2}}(c,s)||B_{\frac{n-4+\varepsilon_n}{2}}(c,s)| \\
&\quad +s^4|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)||B_{\frac{n-4-\varepsilon_n}{2}}(c,s)|
\end{aligned}$$

By the formula (4) we find

$$\begin{aligned}
\det H^{(2)}_n(c,s,v) &= v|B_{\frac{n-2+\varepsilon_n}{2}}(c,s)||H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s,v)| \\
&\quad -s^2.|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)||H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s)| \\
&= |H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s)|\left[v|B_{\frac{n-2+\varepsilon_n}{2}}(c,s)| - s^2|B_{\frac{n-4+\varepsilon_n}{2}}(c,s)|\right] \\
&= |H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s,v)||H^{(1)}_{\frac{n+\varepsilon_n}{2}}(c,s,v)|
\end{aligned}$$

So

$$\det H^{(2)}_n(c,s,v) = \det H^{(1)}_{\frac{n-\varepsilon_n}{2}}(c,s,v)\det H^{(1)}_{\frac{n+\varepsilon_n}{2}}(c,s,v).$$

and the proof is complete. □

Remark 2.2. We remark that the calculation is truly efficient. For instance, the computer demonstrated really fast work during the evaluation of

$$\det H^{(1)}_{100511}(3,5,7) = -2279029\dots672932 \qquad \text{(70255 digits)}.$$

3. Schur-Cohn determinants Δ_i

Theorem 3.1. (Schur-Cohn criterion)
If for the polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_k z^k, \quad a_k \neq 0, k \in \mathbb{N}$$

all the determinants Δ_i of matrices

$$S_i = \begin{pmatrix} a_0 & 0 & \dots & 0 & a_k & a_{k-1} & \dots & a_{k-i+1} \\ a_1 & a_0 & \dots & 0 & 0 & a_k & \dots & a_{k-i+2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i-1} & a_{i-2} & \dots & a_0 & 0 & 0 & \dots & a_k \\ \bar{a}_k & 0 & \dots & 0 & \bar{a}_0 & \bar{a}_1 & \dots & \bar{a}_{i-1} \\ \bar{a}_{k-1} & \bar{a}_k & \dots & 0 & 0 & \bar{a}_0 & \dots & \bar{a}_{i-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \bar{a}_{k-i+1} & \bar{a}_{k-i+2} & \dots & \bar{a}_k & 0 & 0 & \dots & \bar{a}_0 \end{pmatrix}, \quad i = 1, \dots, k,$$

are different from zero, the $f(z)$ has no zero on the circle $|z| = 1$ and p zeros in this circle, p being the number of variations of sign in the sequence $1, \Delta_1, \Delta_2, \dots, \Delta_k$.

Proof. See e.g. [4]. □

We observe that S_i are block matrices

$$S_i = \begin{pmatrix} J_i & K_i \\ L_i & M_i \end{pmatrix}$$

and hence

$$\Delta_i = \det(J_i) \det\left(M_i - L_i J_i^{-1} K_i\right).$$

Lemma 3.1. $\det J_i = a_0^i$.

Proof. The matrix J_i is a lower triadgonal matrix of the order i. □

Further, we apply the criterion to the trinomial

$$z^k - a z^{k-m} - b$$

where $k > m$ are coprime positive integers and a and b nonzero real coefficients.

Let us denote $H_i = M_i - L_i J_i^{-1} K_i$. It follows:

Lemma 3.2. *Let $i \leq k - m$. Then for $m = 1$*

$$H_i = \begin{pmatrix} \frac{1}{b} - b & -\frac{a}{b} & 0 & \dots & 0 & 0 \\ -\frac{a}{b} & \frac{a^2+1}{b} - b & -\frac{a}{b} & \dots & 0 & 0 \\ 0 & -\frac{a}{b} & \frac{a^2+1}{b} - b & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{a^2+1}{b} - b & -\frac{a}{b} \\ 0 & 0 & 0 & \dots & -\frac{a}{b} & \frac{a^2+1}{b} - b \end{pmatrix},$$

for $m = 2$

$$H_i = \begin{pmatrix} \frac{1}{b} - b & 0 & -\frac{a}{b} & \dots & 0 & 0 \\ 0 & \frac{1}{b} - b & 0 & \dots & 0 & 0 \\ -\frac{a}{b} & 0 & \frac{a^2+1}{b} - b & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{a^2+1}{b} - b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{a^2+1}{b} - b \end{pmatrix},$$

etc.

Proof. Let $m = 1$, $i \leq k - 1$. Then

$$L_i = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -a & 1 & 0 & \dots & 0 & 0 \\ 0 & -a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -a & 1 \end{pmatrix}, K_i = \begin{pmatrix} 1 & -a & 0 & \dots & 0 & 0 \\ 0 & 1 & -a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -a \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$J_i = \begin{pmatrix} -b & 0 & 0 & \dots & 0 & 0 \\ 0 & -b & 0 & \dots & 0 & 0 \\ 0 & 0 & -b & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -b & 0 \\ 0 & 0 & 0 & \dots & 0 & -b \end{pmatrix}, J_i^{-1} = \begin{pmatrix} -\frac{1}{b} & 0 & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{b} & 0 & \dots & 0 & 0 \\ 0 & 0 & -\frac{1}{b} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\frac{1}{b} & 0 \\ 0 & 0 & 0 & \dots & 0 & -\frac{1}{b} \end{pmatrix}$$

$$\text{and } M_i = \begin{pmatrix} -b & 0 & 0 & \dots & 0 & 0 \\ 0 & -b & 0 & \dots & 0 & 0 \\ 0 & 0 & -b & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -b & 0 \\ 0 & 0 & 0 & \dots & 0 & -b \end{pmatrix}.$$

Then H_i is obtained directly in the form stated in the statement above.

Let $m = 2$, $i \leq k - 2$. We proceed analogously with the only difference that

$$L_i = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ -a & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, K_i = \begin{pmatrix} 1 & 0 & -a & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

□

We observe that matrices H_i are nothing but $H_i^{(1)}\left(\frac{a^2+1}{b} - b, -\frac{a}{b}, \frac{1}{b} - b\right)$ for $m = 1$ and $H_i^{(2)}\left(\frac{a^2+1}{b} - b, -\frac{a}{b}, \frac{1}{b} - b\right)$ for $m = 2$ which are studied in the previuous section.

We found:

Theorem 3.2. *Let $k > m$ be coprime positive integers and a and b nonzero real numbers. Then the Schur-Cohn determinants for trinomials*

$$z^k - az^{k-m} - b,$$

can be expressed as

$$\Delta_i = (-b)^i \, H_i^{(1)}\left(\frac{a^2+1}{b} - b, -\frac{a}{b}, \frac{1}{b} - b\right) \qquad \textit{for } m = 1 \tag{9}$$

and

$$\Delta_i = (-b)^i \, H_i^{(2)} \left(\frac{a^2+1}{b} - b, -\frac{a}{b}, \frac{1}{b} - b \right) \qquad \textit{for } m = 2. \tag{10}$$

$i = 1, \ldots, k$.

Proof. The Theorem follows from Lemma 3.1 ($a_0 = -b$) and Lemma 3.2. □

4. **Two septic trinomials**

First, we note that an interesting question is an existence of real trinomials which are irreducible over rationals but solvable by radicals. Such trinomials are known for degrees 5, 6, 8 and other, but there is not known any such trinomial of degree 7 up to now. For research in this area, see e. g. [1]. This can also be a motivation for the study of septic trinomials from different points of view.

Let us consider a particular example of septic trinomials in the form

$$P_a^n(\lambda) = \lambda^7 - a\lambda^n - \left(a - \frac{1}{2} \right),$$

which was investigated as to counts of their interior and exterior zeros for cases $n = 1$ and $n = 2$ in [2]. Now, we will investigate counts of interior and exterior zeros of these trinomials for cases $n = 6$ and $n = 5$, i.e. we consider

$$P_a^6(\lambda) = \lambda^7 - a\lambda^6 - \left(a - \frac{1}{2} \right), \qquad P_a^2(\lambda) = \lambda^7 - a\lambda^5 - \left(a - \frac{1}{2} \right)$$

and, by method described in [2], we present their dependence on a varying real parameter a. Then we compare the results with the direct use of the Schur-Cohn criterion in which we apply our explicit formulas for determinants.

First we consider $P_a^6(\lambda)$. Then the critical values of parameter a (when the studied zero configuration is changing) are solutions of nonlinear equations

$$7 \arccos \frac{3+4a}{8|a|} + \arccos \frac{5-4a}{8\left|a - \frac{1}{2}\right|} = z\pi, \quad z = 1, \ldots, 6. \tag{11}$$

For all $z = 1, \ldots, 4$, (11) has a unique negative solution (we denote it a_{Nz}^6); for both $z = 1, 2$, (11) has a unique positive solution (we denote it a_{Pz}^6). Other solutions do not exist.

Similarly, we consider the trinomial $P_a^5(\lambda)$ and corresponding nonlinear equations

$$7 \arccos \frac{3+4a}{8|a|} + 2 \arccos \frac{5-4a}{8\left|a - \frac{1}{2}\right|} = z\pi, \quad z = 1, \ldots, 6. \tag{12}$$

For all $z = 1, \ldots, 5$, (12) has a unique negative solution (we denote it a_{Nz}^5); for all $z = 1, \ldots, 3$, (12) has a unique positive solution (we denote it a_{Pz}^5). Other solutions do not exist.

Because of monotony of the left-hand sides of (11) and (12) on appropriate intervals, it is easy to check that $a_{Nz}^6, a_{Nz}^5 < -\frac{1}{4}$ and $a_{Pz}^6, a_{Pz}^5 > \frac{3}{4}$.

Results of [2] yield the classification scheme described in Table 1. The approximate values of solutions $a_{Nz}^6, a_{Nz}^5, a_{Pz}^6, a_{Pz}^5$, involved in Table 1, are

$$a_{N2}^6 \approx -0.3242, \quad a_{N4}^6 \approx -0.4750, \quad a_{P2}^6 \approx 1.3142$$

and

$$a_{N1}^5 \approx -0.2646, \quad a_{N3}^5 \approx -0.4426, \quad a_{N5}^5 \approx -3.5560, \quad a_{P2}^5 \approx 1.0132.$$

Using the formula (9), we obtain

$\Delta_1 = a^2 - a - \frac{3}{4}$,

$\Delta_2 = -a^3 - \frac{3a^2}{4} + \frac{3a}{2} + \frac{9}{16}$,

TABLE 1. The zero distribution for $P_a^6(\lambda)$ (the left table) and $P_a^5(\lambda)$ (the right table).

An interval for a	Zeros of $P_a^6(\lambda)$		
	r_{in}	r_{on}	r_{out}
$(-\infty, a_{N4}^6)$	2	0	5
a_{N4}^6	2	2	3
(a_{N4}^6, a_{N2}^6)	4	0	3
a_{N2}^6	4	2	1
$(a_{N2}^6, -\frac{1}{4})$	6	0	1
$-\frac{1}{4}$	6	1	0
$(-\frac{1}{4}, \frac{3}{4})$	7	0	0
$\frac{3}{4}$	6	1	0
$(\frac{3}{4}, a_{P2}^6)$	6	0	1
a_{P2}^6	4	2	1
(a_{P2}^6, ∞)	4	0	3

An interval for a	Zeros of $P_a^5(\lambda)$		
	r_{in}	r_{on}	r_{out}
$(-\infty, a_{N5}^5)$	1	0	6
a_{N5}^5	1	2	4
(a_{N5}^5, a_{N3}^5)	3	0	4
a_{N3}^5	3	2	2
(a_{N3}^5, a_{N1}^5)	5	0	2
a_{N1}^5	5	2	0
$(a_{N1}^5, \frac{3}{4})$	7	0	0
$\frac{3}{4}$	6	1	0
$(\frac{3}{4}, a_{P2}^5)$	6	0	1
a_{P2}^5	4	2	1
(a_{P2}^5, ∞)	4	0	3

$$\Delta_3 = \frac{5a^3}{2} - \frac{3a^2}{16} - \frac{27a}{16} - \frac{27}{64},$$
$$\Delta_4 = a^5 - \frac{7a^4}{4} - \frac{51a^3}{16} + \frac{81a^2}{64} + \frac{27a}{16} + \frac{81}{256},$$
$$\Delta_5 = -a^6 - \frac{3a^5}{2} + \frac{75a^4}{16} + \frac{45a^3}{16} - \frac{567a^2}{256} - \frac{405a}{256} - \frac{243}{1024},$$
$$\Delta_6 = 4a^6 - \frac{3a^5}{8} - \frac{243a^4}{32} - \frac{405a^3}{256} + \frac{2997a^2}{1024} + \frac{729a}{512} + \frac{729}{4096}.$$

In particular, approximate values of roots of Δ_6 are -0.9900, -0.4750, -0.3242, -0.2660, 0.8347 and 1.3142. We can observe in Figure 1, how the signs of Δ_i's alternate. We remark, however, that the polynomial Δ_7, which is of a different type, is not included.

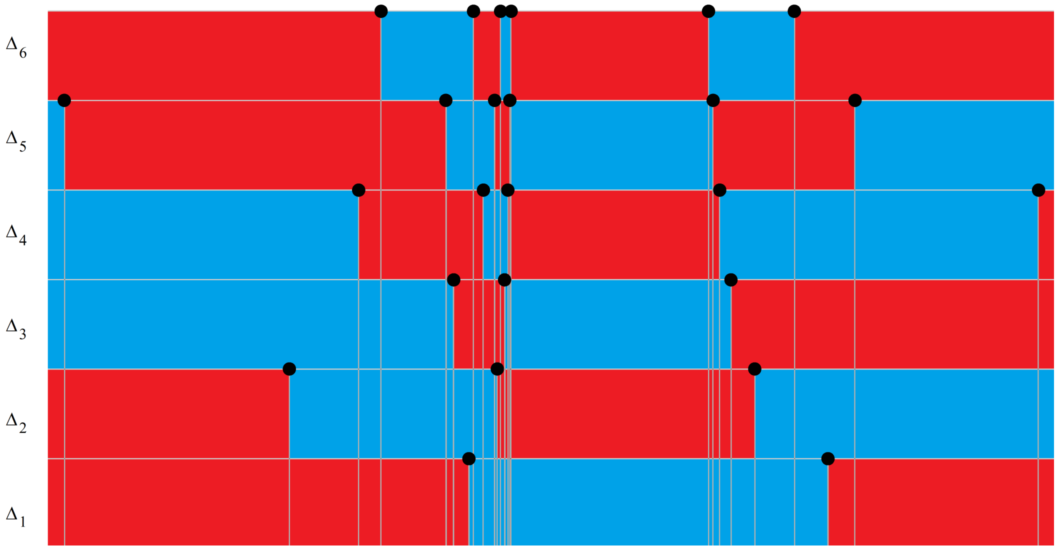

FIGURE 1. *We see the roots of the polynomial Δ_1 as black points in the lowest strip and it is marked in red, where this polynomial is positive and in blue, where it is negative. Analogically, for other strips.*

Similar analysis of the signs of Schur-Cohn determinantes can be realized on the basis of the formula (10) for $P_a^5(\lambda)$. For brevity, however, we will omit it.

5. **Conclusions**

Theorems 2.1 and 2.2 provide new, author-derived formulas for calculating the determinants of tridiagonal matrices of two types. The usefulness of these formulas is then shown for use in Schur Cohn criterion. We have demonstrated this criterion to locate the roots of trinomials. We dealt in more detail with examples of special trinomials depending on one real parameter.

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REFERENCES

[1] *N. Bruin and N. D. Elkies*, Trinomials $ax^7 + bx + c$ and $ax^8 + bx + c$ with Galois groups of order 168 and $8 \cdot 168$, Lecture Notes in Computer Science **2369** (2002), 172-188.

[2] *J. Čermák, L. Fedorková. and M. Kureš*, Complete classification scheme for distribution of trinomial zeros with respect to their moduli, Publ. Math. Debrecen **101**(2022), No. 1-2, 119-146.

[3] *C.M. da Fonseca, V. Kowalenko, and L. Losonczi*, Ninety years of k-tridiagonal matrices, Studia Scientiarum Mathematicarum Hungarica **57** (2020), No. 3, 298-311.

[4] *M. Marden*, Geometry of Polynomials, American Mathematical Society, Providence, RI, 1966.

[5] *F. Qi, V. Čerňanová and Y. S. Semenov*, Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials, Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **81**(2019), No. 1, 123-136.