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MINIMIZATION THEOREMS IN GENERATING SPACES OF QUASI G-METRIC FAMILY

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In this paper the concept of $\{\Omega_\alpha : \alpha \in (0,1]\}$ *that is a family* Ω_α*-quasi distances on generating spaces of quasi* G*-metric family* $(X, G_\alpha : \alpha \in (0,1])$ *is considered and non-convex minimization theorem on such spaces is proved.*[3] *Then as its application we prove the Caristi type fixed point theorem and Ekeland's* ε*-variational principle. Our results generalized and improve some recent results in the literature.*

Keywords: generating space of quasi G-metric space; Ω_α-distance; fixed point; non-convex minimization theorems.

AMS Subject Classification 47H10; 54H25

1. Introduction

In 1994, Chang, Cho and Kang ([5],[6]) gave definition of generating spaces of quasi-metric family, which included fuzzy metric spaces ([8]) and Meneger probabilistic metric spaces ([20]) as special cases. They proved some fixed point theorems and Takahashi-type minimization theorems in complete generating spaces of quasi-metric family. In 1996, Kada, Suzuki and Takahashi ([9]) introduced the concept ω-distance in metric spaces and proved the Caristi's fixed point theorem ([4]), Ekeland's ε-variational principle ([7]) and Takahashi type nonconvex minimization theorems ([23]) in complete metric spaces by using the ω-distance. Later, in 1997, Chang, Jung, Lee ([10]) by following the approaches of Kada et al. defined a family of weak quasi metric in generating spaces of quasi metric family and proved minimization theorems. In 2006, Mustafa and Sims ([14]) introduced the concept of G-metric. Some authors ([1], [2], [3], [11], [12], [13], [15], [19]) have proved some fixed point theorems in these spaces. Recently, Saadati, Vaezpour, Vetro and Rhoades ([21]), using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of ω-distance.

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In this paper, inspire of above conceptes we introduce the concept of generating spaces of quasi G-metric family that included Q-fuzzy metric spaces [22] and Ω_α -quasi distances. Then, we prove Takahashi type nonconvex minimization theorems, Caristi type fixed point theorem and ε-variational principle in these spaces.

At first we recall some definitions and lemmas in the G-metric space. For more information see ([2], [3], [13], [14], [21]).

2. *G*-metric space

Definition 2.1 ([14]) Let X be a non-empty set. A function $G: X \times X \times X \to [0, \infty)$ is called a G- metric if the following conditions are satisfied :

1. $G(x, y, z) = 0$ if $x = y = z$ (coincidence),
2. $G(x, x, y)>0$ for all $x, y \in X$, where $x \neq y$,
3. $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
4. $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.2([14]) Let (X, G) be a G-metric space,

1. a sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$.
2. a sequence $\{x_n\}$ in X is said to be G-convergent to a point $x \in X$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

Definition 2.3 ([21]) Let (X, G) be a G-metric space. Then a function $\Omega: X \times X \times X \to [0, \infty)$ is called an Ω-distance on X if the following conditions are satisfied :

1. $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$,
2. for any $x, y \in X$, $\Omega(x, y, .)$, $\Omega(x, ., y)$: $X \to [0, \infty)$ are lower semi-continuous,
3. for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \varepsilon$.

Example 2.4 ([21]) Let (X, d) be a metric space and $G: X^3 \to [0, \infty)$ defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},$$

for all $x, y, z \in X$. Then $\Omega = G$ is an Ω-distance on X.

For more examples see ([21]).

Lemma 2.5 ([21]) Let X be a metric space with metric G and Ω be an Ω-distance on X. Let $\{x_n\}, \{y_n\}$ be sequences in X, $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following :

1. If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \boldsymbol{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$.
2. If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$ then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$.
3. If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \boldsymbol{N}$ with $n \leq m \leq l$, then $\{x_n\}$ is a G-Cauchy sequence.
4. If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \boldsymbol{N}$, then $\{x_n\}$ is a G-Cauchy sequence.

Definition 2.6 ([21]) A G-metric space X is said to be Ω-bounded if there is a constant $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.

3. Generating space of quasi G-metric family

In this section we present the definition of a generating space of quasi G-metric family.

Definition 3.1 Let X be a nonempty set and $\{G_\alpha: \alpha \in (0,1]\}$ be a family of mappings $G_\alpha: X \times X \times X \to [0, \infty)$. Then $(X, G_\alpha: \alpha \in (0,1])$ called a generating space of quasi G-metric family if the following conditions are satisfied :

1. $G_\alpha(x, y, z) = 0$ for all $\alpha \in (0,1]$ if and only if $x = y = z$,
2. $G_\alpha(x, x, y)>0$ for all $x, y \in X$, where $x \neq y$ and $\alpha \in (0,1]$,
3. $G_\alpha(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$ and $\alpha \in (0,1]$,
4. $G_\alpha(x, y, z) = G_\alpha(p\{x, y, z\})$, where p is a permutation of x, y, z and $\alpha \in (0,1]$,
5. for any $\alpha \in (0,1]$, there exists a number $\mu \in (0, \alpha]$ such that, $G_\alpha(x, y, z) \leq G_\mu(x, a, a) + G_\mu(a, y, z)$ for all $x, y, z, a \in X$ and $\alpha \in (0,1]$,
6. for any $x, y, z \in X$, $G_\alpha(x, y, z)$ is non-increasing in α and left continuous in α.

A generating space of quasi G-metric is said to be symmetric if $G_\alpha(x, y, y) = G_\alpha(y, x, x)$ for all $x, y \in X$ and $\alpha \in (0,1]$.

Example 3.2 Let X be a metric space with metric G. If we put $G_\alpha(x, y, z) = G(x, y, z)$ for all $\alpha \in (0,1]$ and $x, y, z \in X$, then $(X, G_\alpha: \alpha \in (0,1])$ is a generating space of quasi G-metric family.

Example 3.3 Let $(X, d_\alpha: \alpha \in (0,1])$ be a generating space of quasi metric family. If we put $G_\alpha(x,y,z) = \max\{d_\alpha(x,y), d_\alpha(y,z), d_\alpha(x,z)\}$for all $\alpha \in (0,1]$ and $x,y,z \in X$, then $(X, G_\alpha: \alpha \in (0,1])$ is a generating space of quasi G-metric family.

Example 3.4 Let $(X, \|.\|_\alpha : \alpha \in (0,1])$ be a generating space of quasi-norm metric family [10]. If we put $G_\alpha(x,y,z) = \|x-y\|_\alpha + \|y-z\|_\alpha + \|x-z\|_\alpha$ for all $\alpha \in (0,1]$ and $x,y,z \in X$, then $(X, G_\alpha: \alpha \in (0,1])$ be a generating space of quasi G-metric family.

Generating space of quasi G-metric family has properties of G-metric space.

Definition 3.5 Let $(X, G_\alpha: \alpha \in (0,1])$ be a complete generating space of quasi G-metric family. For $\varepsilon > 0$ and $0 < \alpha \leq 1$ the open ball $U_x(\varepsilon, \alpha)$ is defined by

$$U_x(\varepsilon, \alpha) = \{y \in X : G_\alpha(x,y,y) < \varepsilon\}.$$

Definition 3.6 A subset B of $(X, G_\alpha: \alpha \in (0,1])$ is called open set if for each $x \in B$ there exist $\varepsilon > 0$ and $0 < \alpha \leq 1$ such that $U_x(\varepsilon, \alpha) \subset B$.

Lemma 3.7 In a generating space of quasi G-metric family $(X, G_\alpha: \alpha \in (0,1])$ every open ball is an open set.

Proof: Let $U_x(\varepsilon, \alpha)$ be an open ball and $y \in U_x(\varepsilon, \alpha)$. Then $G_\alpha(x,y,y) < \varepsilon$ and there exists $0 < \beta < \alpha$ that $\lambda = G_\beta(x,y,y) < \varepsilon$. It is enough to prove that there exist $\varepsilon_0 > 0$ and $0 < \alpha_0 \leq 1$ such that $U_y(\varepsilon_0, \alpha_0) \subseteq U_x(\varepsilon, \alpha)$. Setting $\varepsilon_0 = \varepsilon - \lambda$ and $\alpha_0 = \beta$. Then if $z \in U_y(\varepsilon_0, \alpha_0) = U_y(\varepsilon - \lambda, \beta)$ therefor $G_\beta(y,z,z) < \varepsilon - \lambda$. Now by,

$$G_\alpha(x,z,z) \leq G_\beta(x,y,y) + G_\beta(y,z,z) < \lambda + \varepsilon - \lambda = \varepsilon,$$

we obtain $z \in U_x(\varepsilon, \alpha)$ and the proof is completed.

The following lemma can be e asily proved.

Lemma 3.8 Let $(X, G_\alpha: \alpha \in (0,1])$ be a generating space of quasi G-metric family. Define,

$$\tau_{\{G_\alpha\}} = \{B \subset X : \forall x \in B, \exists \varepsilon > 0, 0 < \alpha \leq 1 \ \textit{such that}\ U_x(\varepsilon, \alpha) \subset B\}.$$

Then $\tau_{\{G_\alpha\}}$ is a topology on X.

Lemma 3.9 Every generating space of quasi G-metric family $(X, G_\alpha: \alpha \in (0,1])$ is Hausdorff.

Definition 3.10 Let $(X, G_\alpha: \alpha \in (0,1])$ be a generating space of quasi G-metric family and $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ is said to be G-convergent to a point $x \in X$ if, for each $\varepsilon > 0$ and $\alpha \in (0,1]$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G_\alpha(x_m, x_n, x) < \varepsilon$.
2. $\{x_n\}$ is said to be G-Cauchy sequence if for each $\varepsilon > 0$ and $\alpha \in (0,1]$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G_\alpha(x_m, x_n, x_l) < \varepsilon$.
3. A generating space of quasi G-metric family $(X, G_\alpha: \alpha \in (0,1])$ that every G-Cauchy sequence is G-convergent is said complete.

Definition 3.11 Let $(X, G_\alpha: \alpha \in (0,1])$ be a generating space of quasi G-metric family. Then a family $\{\Omega_\alpha: \alpha \in (0,1]\}$ of mappings from $\Omega_\alpha: X \times X \times X \to [0,\infty)$ is called a family of Ω_α-quasi distances if the following conditions are satisfied :

1. for any $\alpha \in (0\ 1]$, there exists a number $\mu \in (0,\alpha]$ such that, $\Omega_\alpha(x,y,z) \leq \Omega_\mu(x,a,a) + \Omega_\mu(a,y,z)$ for all $x,y,z,a \in X$.
2. for any $x,y \in X$ and $\alpha \in (0,1]$, $\Omega_\alpha(x,y,.)$, $\Omega_\alpha(x,.,y): X \to [0,\infty)$ are lower semi-continuous.
3. for each $\varepsilon > 0$ and $\alpha \in (0,1]$, there exists $\delta > 0$ and a number $\mu \in (0,\alpha]$ such that $\Omega_\mu(x,a,a) \leq \delta$ and $\Omega_\mu(a,y,z) \leq \delta$ imply $G_\alpha(x,y,z) \leq \varepsilon$.

Example 3.12 Let (X, G) be a G-metric space. Put,

$$\Omega_\alpha(x,y,z) = G_\alpha(x,y,z) = G(x,y,z) \quad \text{for all } \alpha \in (0,1],$$

then $\{\Omega_\alpha: \alpha \in (0,1]\}$ is a family of Ω_α-quasi distances.

Example 3.13 Let $(X, \|.\|_\alpha: \alpha \in (0,1])$ be a generating space of quasi-norm metric family [10]. If we put $G_\alpha(x,y,z) = \|x-y\|_\alpha + \|y-z\|_\alpha + \|x-z\|_\alpha$ and $\Omega_\alpha(x,y,z) = \|x-y\|_\alpha + \|x-z\|_\alpha$ for all $\alpha \in (0,1]$ and $x,y,z \in X$, then $\{\Omega_\alpha: \alpha \in (0,1]\}$ a family of Ω_α-quasi distances.

Lemma 3.14 Let $(X, G_\alpha: \alpha \in (0,1])$ be a generating space of quasi G-metric family and $\{\Omega_\alpha: \alpha \in (0,1]\}$ be a family of Ω_α-quasi distances on X. Let $\{x_n\}, \{y_n\}$ be sequences in X, $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero and let $x,y,z,a \in X$. Then we have the following :

1. If $\Omega_\alpha(y,x_n,x_n) \leq \alpha_n$ and $\Omega_\alpha(x_n,y,z) \leq \beta_n$ for $n \in \boldsymbol{N}$ and $\alpha \in (0,1]$, then $G_\alpha(y,y,z) < \varepsilon$ and hence $y = z$.
2. If $\Omega_\alpha(y_n,x_n,x_n) \leq \alpha_n$ and $\Omega_\alpha(x_n,y_m,z) \leq \beta_n$ for $m > n$ and $\alpha \in (0,1]$, then $G_\alpha(y_n,y_m,z) \to 0$ and hence $y_n \to z$.
3. If $\Omega_\alpha(x_n,x_m,x_l) \leq \alpha_n$ for any $l,m,n \in \boldsymbol{N}$ with $n \leq m \leq l$ and $\alpha \in (0,1]$, then $\{x_n\}$ is a G-Cauchy sequence.
4. If $\Omega_\alpha(x_n,a,a) \leq \alpha_n$ for any $n \in \boldsymbol{N}$ and $\alpha \in (0,1]$, then $\{x_n\}$ is a G-Cauchy sequence.

4. Non-convex minimization theorem

Here we prove the non-convex minimization theorems for generating space of quasi G-metric family, which generalize and improve the recent results.

Notation 4.1 Let Φ be the set of all φ such that $\varphi : [0,\infty) \to [0,\infty)$ is sub-additive,

(i.e. $\varphi(x+y) \leq \varphi(x) + \varphi(y)$, for $x,y \in [0,\infty)$) and nondecreasing continuous such that $\varphi^{-1}(\{0\}) = \{0\}$.

For more details see ([17], [18]).

The following theorem is the Takahashi type nonconvex minimization theorem in a complete generating space of quasi G-metric space.

Theorem 4.2 Let $(X, G_\alpha : \alpha \in (0,1])$ be a complete generating space of quasi G-metric family and let $\{\Omega_\alpha : \alpha \in (0,1]\}$ be a family Ω_α-quasi distances on X such that for all $x,y,z \in [0,\infty)$ and $\alpha \in (0\,,1]$,

$$\max\{\Omega_\alpha(x,z,z), \Omega_\alpha(y,y,z) \leq \Omega_\alpha(x,y,y) + \Omega_\alpha(y,z,z)\}.$$

Suppose that $f : X \to (-\infty,\infty]$ is a proper lower semi-continuous function bounded from below. Assume that for each $u \in X$ with $\inf_{x\in X} f(x) < f(y)$ and $\varphi \in \Phi$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + \varphi(\Omega_\alpha(u,v,v)) \leq f(u),$$

for all $\alpha \in (0,1]$. Then there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

Proof: Suppose that $\inf_{x\in X} f(x) < f(y)$ for every $y \in X$. Since f is a proper, let $u_1 \in X$ and $f(u_1) < \infty$. Then inductively, we define the sequence $\{u_n\}$ in X such that,

$$u_{n+1} \in S_n = \{x \in X : \ \varphi(\Omega_\alpha(u_n,x,x)) \leq f(u_n) - f(x), \forall \alpha \in (0,1]\},$$
$$K_n = \inf_{x \in S_n} f(x)$$

and

$$f(u_{n+1}) < K_n + \frac{1}{n}.$$

Since,

$$\varphi(\Omega_\alpha(u_n,u_{n+1},u_{n+1})) \leq f(u_n) - f(u_{n+1}),$$

then $\{f(u_n)\}$ is non-increasing. Therefore, $k = \lim_{n\to\infty} f(u_n)$ exists.

Now, we claim that $\{u_n\}$ is G-Cauchy,i.e. for any $l > m > n$ with $m = n+k$ and $l = m + t$ $(k,t \in \boldsymbol{N})$,

$$\lim_{n,m,l\to\infty} \Omega_\alpha(u_n,u_m,u_l) = 0.$$

We have for some $\mu \in (0,\alpha]$,

$$\Omega_\alpha(u_n,u_m,u_l) = \Omega_\alpha(u_n,u_{n+k},u_{n+k+t})$$
$$\leq \Omega_\mu(u_n,u_{n+k},u_{n+k}) + \Omega_\mu(u_{n+k},u_{n+k},u_{n+k+t}),$$

and,

$$\Omega_\mu(u_{n+k},u_{n+k},u_{n+k+t}) \leq \max\{\Omega_\mu(u_n,u_{n+k+t},u_{n+k+t}), \Omega_\mu(u_{n+k},u_{n+k},u_{n+k+t})\}$$

$$\leq \Omega_{\mu}(u_n, u_{n+k}, u_{n+k}) + \Omega_{\mu}(u_{n+k}, u_{n+k+t}, u_{n+k+t}).$$

Therefore by sub-additivity of φ,

$$\varphi\left(\Omega_{\mu}(u_{n+k}, u_{n+k}, u_{n+k+t})\right) \leq \varphi\left(\Omega_{\mu}(u_n, u_{n+k}, u_{n+k}) + \Omega_{\mu}(u_{n+k}, u_{n+k+t}, u_{n+k+t})\right)$$

$$\leq \varphi(\Omega_{\mu}(u_n, u_{n+k}, u_{n+k})) + \varphi\left(\Omega_{\mu}(u_{n+k}, u_{n+k+t}, u_{n+k+t})\right)$$
$$\leq f(u_n) - f(u_{n+k}) + f(u_{n+k}) - f(u_{n+k+t})$$
$$= f(u_n) - f(u_{n+k+t}).$$

Also $f(u_{n+k+t}) \leq f(u_{n+k})$, because $\{f(u_n)\}$ is non-increasing. Then,

$$\varphi\left(\Omega_{\alpha}(u_n, u_m, u_l)\right) \leq f(u_n) - f(u_{n+k}) + f(u_n) - f(u_{n+k+t})$$
$$\leq f(u_n) - f(u_{n+k+t}) + f(u_n) - f(u_{n+k+t})$$
$$= 2\left(f(u_n) - f(u_{n+k+t})\right). \qquad (2.1)$$

Thus, $\lim_{n,m,l\to\infty} \varphi(\Omega_{\alpha}(u_n, u_m, u_l)) = 0$ and consequently $\{u_n\}$ is a G-Cauchy sequence. Since X is complete, $\{u_n\}$ converges to a point $u_0 \in X$. Therefore, in view of the lower semi-continuity of Ω_{α} and (2.1), we have

$$\varphi\left(\Omega_{\alpha}(u_n, u_0, u_0)\right) \leq f(u_n) - k = f(u_n) - \liminf f(u_n) \leq f(u_n) - f(u_0)$$

for all $\alpha \in (0, 1]$. Therefore, there exists $u_1 \in X$ such that $u_0 \neq u_1$ and $f(u_1) + \varphi\left(\Omega_{\alpha}(u_0, u_1, u_1)\right) \leq f(u_0)$. Hence, for some $\mu \in (0, \alpha]$,

$$f(u_1) + \varphi\left(\Omega_{\alpha}(u_n, u_1, u_1)\right) \leq f(u_1) + \varphi\left(\Omega_{\mu}(u_0, u_1, u_1)\right) + \varphi\left(\Omega_{\mu}(u_n, u_0, u_0)\right)$$
$$\leq f(u_0) + \varphi\left(\Omega_{\mu}(u_n, u_0, u_0)\right)$$
$$\leq f(u_n)$$

and consequently $u_1 \in S_n$. Since for every $n \in \boldsymbol{N}$,

$$f(u_0) \leq f(u_{n+1}) < K_n + \frac{1}{n} \leq f(u_1) + \frac{1}{n},$$

we have, $f(u_0) \leq f(u_1)$. Then, $f(u_0) = f(u_1)$ and $\varphi\left(\Omega_{\alpha}(u_0, u_1, u_1)\right) = 0$. By assumption, there exists $u_2 \in X$such that $u_2 \neq u_1$and$f(u_2) + \varphi\left(\Omega_{\alpha}(u_1, u_2, u_2)\right) \leq f(u_1)$.Then, $\varphi\left(\Omega_{\alpha}(u_1, u_2, u_2)\right) = 0$. Now, by Part (3) of the Definition (3.11) and the properties of φ, we have $u_0 = u_2$. Similarly, since $\Omega_{\alpha}(u_2, u_1, u_1) = \Omega_{\alpha}(u_0, u_1, u_1)$, we obtain $u_0 = u_1$. Thus, $u_1 = u_2$, which is a contradiction and this complete the proof.

Corollary 4. 3 Let $(X, G_\alpha : \alpha \in (0,1])$ be a complete generating space of quasi G-metric family and let $\{\Omega_\alpha : \alpha \in (0,1]\}$ be a family Ω_α-quasi distances on X such that for all $x, y, z \in [0, \infty)$ and $\alpha \in (0,1]$,

$$\max\{\Omega_\alpha(x,z,z), \Omega_\alpha(y,y,z)\} \le \Omega_\alpha(x,y,y) + \Omega_\alpha(y,z,z).$$

Suppose that $f : X \to (-\infty, \infty]$ is a proper lower semi-continuous function bounded from below. Assume that for each $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$\Omega_\alpha(u,v,v) \le f(u) - f(v).$$

Then there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

Proof: It is sufficient that put $\varphi(t) = t$ in the Theorem (4.2).

The following Corollary is a generalization of the Takahashi type nonconvex minimization theorem in a complete generating space of quasi metric space ([10]).

Corollary 4.4 Let $(X, d_\alpha : \alpha \in (0,1])$ be a complete generating space of quasi metric family. Let $f : X \to (-\infty, \infty]$ be a proper lower semi-continuous function bounded from below. Assume that there exists a family $\{p_\alpha : \alpha \in (0,1]\}$ of weak quasi metrics on X such that for each $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + p_\alpha(u,v) \le f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

Proof: It is enough to defin $G_\alpha(x,y,z) = \max\{d_\alpha(x,y), d_\alpha(y,z), d_\alpha(x,z)\}$ and $\Omega_\alpha(x,y,z) = \max\{p_\alpha(x,y), p_\alpha(x,z)\}$ for all $x, y, z \in X$ and $\alpha \in (0,1]$.Then, by the previous corollary application, the proof is completed.

The following theorem is the Caristi type fixed point theorem in a complete generating space of quasi G-metric space.

Theorem 4.5 Let $(X, G_\alpha : \alpha \in (0,1])$ be a complete generating space of quasi G-metric family and $\{\Omega_\alpha : \alpha \in (0,1]\}$ be a family Ω_α-quasi distances on X. Assume that $f : X \to (-\infty, \infty]$ is a proper lower semi-continuous function bounded from below and T be a mapping from X into itself. Suppose that for every $x \in X$ and $\varphi \in \Phi$,

$$f(Tx) + \varphi\big(\Omega_\alpha(x, Tx, Tx)\big) \le f(x).$$

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $\Omega_\alpha(x_0, Tx_0, Tx_0) = 0$.

Proof: Since f is proper, there exists $u \in X$ such that $f(u) < \infty$. Put

$$Y = \{x \in X : f(x) \le f(u)\}.$$

Since f is lower semi-continuous, Y is closed. Hence Y is a complete generating

G-metric space. Let $x \in Y$. Since $f(Tx) + \varphi(\Omega_\alpha(x, Tx, Tx)) \leq f(x) \leq f(u)$ for all $\alpha \in (0, 1]$, we have $Tx \in Y$. Thus Y is invariant under T. Now, assume $Tx \neq x$ for every $x \in Y$. Then, by Theorem (4.2), there exists $v_0 \in Y$ such that $f(v_0) = \inf_{x \in Y} f(x)$. Since $f(Tv_0) + \varphi(\Omega_\alpha(v_0, Tv_0, Tv_0)) \leq f(v_0)$, and $f(v_0) = \inf_{x \in Y} f(x)$, we have $f(Tv_0) = f(v_0) = \inf_{x \in Y} f(x)$ and $\varphi(\Omega_\alpha(v_0, Tv_0, Tv_0)) = 0$. Similarly, $f(T^2v_0) = f(Tv_0) = \inf_{x \in Y} f(x)$ and $\varphi(\Omega_\alpha(Tv_0, T^2v_0, T^2v_0)) = 0$. By definition of φ, we obtain $\Omega_\alpha(v_0, Tv_0, Tv_0) = 0$ and $\Omega_\alpha(Tv_0, T^2v_0, T^2v_0) = 0$ for all $\alpha \in (0, 1]$. Therefor, by Part (3) of Definition (3.11), $v_0 = T^2v_0$.
Similarly, since $\Omega_\alpha(T^2v_0, Tv_0, Tv_0) = \Omega_\alpha(v_0, Tv_0, Tv_0)$,
we obtain $Tv_0 = v_0$, which is a contradiction.

Corollary 4.6 Let $(X, G_\alpha: \alpha \in (0, 1])$ be a complete generating space of quasi G-metric family and $\{\Omega_\alpha: \alpha \in (0,1]\}$ a family Ω_α-quasi distances on X. Assume that $f : X \to (-\infty, \infty]$ is a proper lower semi-continuous function bounded from below. Let that T be a mapping from X into itself. Suppose that for every $x \in X$,

$$f(Tx) + \Omega_\alpha(x, Tx, Tx) \leq f(x).$$

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $\Omega_\alpha(x_0, Tx_0, Tx_0) = 0$.
The following Corollary is a generalization of the Caristi type fixed point theorem in a complete generating space of quasi metric space ([10]).

Corollary 4.7 Let $(X, d_\alpha: \alpha \in (0,1])$ be a complete generating space of quasi metric family and let $f : X \to (-\infty, \infty]$ be a proper lower semi-continuous function bounded from below and that T be a mapping from X into itself. Assume that there exists a family $\{p_\alpha: \alpha \in (0,1]\}$ of weak quasi metrics on X such that for every $x \in X$,

$$f(Tx) + p_\alpha(x, Tx) \leq f(x).$$

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p_\alpha(x_0, Tx_0) = 0$.
The following theorem is Ekeland's ε-variational principle in a complete generating space of quasi G-metric space.

Theorem 4.8 Let $(X, G_\alpha: \alpha \in (0, 1])$ be a complete generating space of quasi G-metric family and $\{\Omega_\alpha: \alpha \in (0,1]\}$ a family Ω_α-quasi distances on X such that for all $x, y, z \in X$ and $\alpha \in (0, 1]$,

$$\max\{\Omega_\alpha(x, z, z), \Omega_\alpha(y, y, z)\} \leq \Omega_\alpha(x, y, y) + \Omega_\alpha(y, z, z).$$

Assume that $f : X \to (-\infty, \infty]$ is a proper lower semi-continuous function bounded from below. Then we have
(1) for every $\varepsilon > 0$ and $u \in X$ such that $\Omega_\alpha(u, u, u) = 0$ for all $\alpha \in (0, 1]$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon\lambda$, there exists a $v \in X$ such that $f(v) \leq f(u)$,

$\varphi(\Omega_\alpha(u,v,v)) \leq \lambda$ for all $\alpha \in (0,1]$, and for every $w \in X$ with $w \neq v$, there exists a $\beta \in (0,1]$ such that,

$$f(w) > f(v) - \varepsilon\varphi(\Omega_\beta(v,w,w)).$$

(2) for every $u \in X$ with $f(u) < \infty$, there exists a $v \in X$ such that $f(v) \leq f(u)$, and for every $w \in X$ with $w \neq v$, there exists a $\beta \in (0,1]$ such that

$$f(w) > f(v) - \varphi(\Omega_\beta(v,w,w)).$$

Proof: (1) Let $M = \{x \in X : f(x) \leq f(u) - \varepsilon\varphi(\Omega_\alpha(u,x,x)), \alpha \in (0,1]\}$.
Then M is non-empty and complete. Moreover, for every $x \in M$,

$$\varepsilon\varphi(\Omega_\alpha(u,x,x)) \leq f(u) - f(x) \leq f(u) - \inf_{x\in X} f(x) \leq \varepsilon\lambda$$

for all $\alpha \in (0,1]$. Then $\varphi(\Omega_\alpha(u,x,x)) \leq \lambda$ and $f(x) \leq f(u)$. Assume that for every $x \in X$ there exists a $w \in X$ such that $w \neq x$ and $f(w) \leq f(x) - \varepsilon\varphi(\Omega_\alpha(x,w,w))$ for all $\alpha \in (0,1]$. Thus, $f(w) \leq f(x) - \varepsilon\varphi(\Omega_\alpha(x,w,w)) \leq f(x)$.
Now, for every $\alpha \in (0,1]$, there exists a $\mu \in (0,\alpha]$ such that,

$$\varepsilon\varphi(\Omega_\alpha(u,w,w)) \leq \varepsilon\varphi(\Omega_\mu(u,x,x)) + \varepsilon\varphi(\Omega_\mu(x,w,w)).$$

Then,

$$\varepsilon\varphi(\Omega_\alpha(u,w,w)) \leq f(u) - f(x) + f(x) - f(w) = f(u) - f(w)$$

for all $\alpha \in (0,1]$. This implies that $w \in M$. Therefore, for every $x \in M$, there exists a $w \in M$ such that $w \neq x$ and $f(w) \leq f(x) - \varepsilon\varphi(\Omega_\alpha(x,w,w))$ for all $\alpha \in (0,1]$. Based on Theorem (4.2), there exists an $x_0 \in X$ such that $f(x_0) = \inf_{x\in M} f(x)$. Now, for such x_0, there exists $x_1 \in M$ such that $x_1 \neq x_0$ and

$$f(x_1) \leq f(x_0) - \varepsilon\varphi(\Omega_\alpha(x_0,x_1,x_1)),$$

for all $\alpha \in (0,1]$. Thus, $f(x_1) = f(x_0) = \inf_{x\in M} f(x)$ and $\varphi(\Omega_\alpha(x_0,x_1,x_1)) = 0$ for all $\alpha \in (0,1]$. Therefor, $\Omega_\alpha(x_0,x_1,x_1) = 0$. Similarly, there exists $x_2 \in M$ such that $x_1 \neq x_2$ and $\Omega_\alpha(x_1,x_2,x_2) = 0$ for all $\alpha \in (0,1]$. According to part (3) of Definition (3.11), we obtain $x_0 = x_2$ and therefor $x_0 = x_1$. This is contradiction.
(2) Let $W = \{x \in X : f(x) \leq f(u)\}$. Then W is non-empty and complete. Moreover, for every $x \in W$ there exists $v \in X$ such that $v \neq x$ and

$$f(v) \leq f(x) - \varphi(\Omega_\alpha(x,v,v))$$

for all $\alpha \in (0,1]$. Since $f(v) \leq f(x) \leq f(u)$, then $v \in W$. Thus, by Theorem (4.2) and above proof, we can prove (2).
The following Corollary is a generalization of Ekeland's ε-variational principle in a complete generating space of quasi metric space ([10]).

Corollary 4.9 Let $(X, d_\alpha : \alpha \in (0,1])$ be a complete generating space of quasi metric family and $\{p_\alpha : \alpha \in (0,1]\}$ a family weak quasi metrics. Assume $f : X \to (-\infty,\infty]$ is a proper lower semi-continuous function bounded from below. Then :

(1) for every $\varepsilon > 0$ and $u \in X$ such that $p_\alpha(u,u) = 0$ for all $\alpha \in (0,1]$ and $f(u) \leq \inf_{x\in X} f(x) + \varepsilon\lambda$, there exists a $v \in X$ such that $f(v) \leq f(u)$, $p_\alpha(u,v) \leq \lambda$ for all $\alpha \in (0,1]$, and for every $w \in X$ with $w \neq v$, there exists a $\beta \in (0,1]$ such that,

$$f(w) > f(v) - \varepsilon\, p_\beta(v,w)).$$

(2) for every $u \in X$ with $f(u) < \infty$, there exists a $v \in X$ such that $f(v) \leq f(u)$, and for every $w \in X$ with $w \neq v$, there exists a $\beta \in (0,1]$ such that

$$f(w) > f(v) - p_\beta(v,w).$$

REFERENCES

[1] *B. Ahmad, M. Ashraf, B. Rhoades*, Fixed Point theorem for expansive mappings in G-metric spaces, J. Pure Appl. Math 32(10)(2001) 1513-1518.

[2] *M. Abbas, B. Rhoades*, Common fixed point results for non-commuting mappings without continuity in generalized metric spaces , Appl. Math. Comput. 215(2009) 262-269.

[3] *M. Abbas, T. Nazir, B. Rhoades*, Common fixed point results in G-metric spaces, preprint.

[4] *J. Caristi*, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215(1976) 241-251.

[5] *S. Chang, Y.Cho, S. Kang*, Probailistic metric spaces and operator theory, Sichuan University Press, Chendgu, China, (1994).

[6] *S. Chang, Y. Cho, B. Lee, J. Jung, S. Kang*, Coincidence point theorems and minimization theorems in fuzzy metric spaces, Fuzzy Sets and Systems 88(1)(1997) 119-128.

[7] *I. Ekland*, Nonconvex minimization problems, Bull. Amer. Math. Soc. 13(1979) 443-474.

[8] *O. Kaleva, S. Seikkala*, On fuzzy metric spaces, Fuzzy Sets and Systems 12(1984) 215-226.

[9] *O. Kada, T. Suzuki, W. Takahashi*, Nonconvex minimization theorems and fixed point theorems in complete metric space, Math. Japonica 44(1996) 381-391.

[10] *G.Lee, B. Lee, J.Jung, S.Chang*, Minimization theorems and fixed point theorems in generating spaces of quasi- metric family, Fuzzy Sets and Systems 101(1999) 143-152.

[11] *Z. Mustafa, F. Awawded, W. Shantanawi*, Fixed Point theorem for expansive mappings in G-metric spaces, Int. J. Contemp. Math. Sciences. 5, (2010) , no. 50, 2463-2472.

[12] *S. Manro, S.S. Bhatia, S. Kumar*, Expansion mappings theorems in G-metric spaces, J. Contemp. Math. Sciences, Vol. 5, (2010), no. 51, 2529 - 2535.

[13] *Z.Mustafa, T. Obiedat, F. Awawdeh*, Some Fixed Point theorems for mapping on Complete G-metric space, Fixed Point theory Appl.(2008)12. Article ID 189870.

[14] *Z. Mustafa, B. Sims*, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006) 289-297.

[15] *Z. Mustafa, B. Sims*, Fixed Point theorems for contractive mappings in complete G-metric spaces, Fixed Point theory Appl. (2009) 10. Article ID 917175.

[16] *D. O'Regan, R. Saadati*, Nonlinear contraction theorems in probabilistic spaces, Appl. Math. Comput. 195(2008) 86-93.

[17] *J.H.Qiu*, Ekeland's variational principle in Fréchet spaces and the density of extremal points, Studia Math. 168(2005) 81-94.

[18] *J.H.Qiu*, A generalized Ekeland vector variational principle and its applications in

optimization, Nonlinear Anal. 71(2009) 4705-4717.
[19] *W. Shatanawi*, Fixed Point theory for contractive mappings satisfying Φ-maps in G-metric spaces, Fixed Point theory Appl. (2010) 9. Article ID 181650.
[20] *B. Schweizer, A. Sklar*, Statistical metric spaced, Pacific J. Math. 10(1960) 313-334.
[21] *R. Saadati, S.M. Vaezpour, P. Vetro*, *B.E. Rhoades*, Fixed Point theorems in generalized partially ordered G-metric spaces, Math. Comput. 52(2010) 797-801.
[22] *G. Sun, K. Yang*, Generalized on fuzzy metric spaces with properties, Appl. Sciences, Engineering and Technology 2(7)(2010) 673-678.
[23] *W. Takahashi*, Existence theorem generating fixed point theorems for multivalued mapping Fixed Point theory Appl (M.A. Thera and J.-B. Baillon, eds.), Pitman Research Notes in Math. Series 152(1991) 397-406.