# ALTERNATING $Q$-GROUPS

Ion ARMEANU[1], Didem OZTURK[2]

*In acest articol studiem structura grupurilor alternate finite şi demonstrăm că dintre ele numai cel trivial este un grup avand caracterele ireductibile cu valori rationale. De asemenea, demonstră ca deşi grupurile alternate nu sunt Q-grupuri, totusi concluzia teoremei Brauer-Speiser privitoare la indexul Schur este adevarată.*

*In this paper we shall study the structure of the finite alternating groups and we prove that there are no nontrivial alternating groups whose irreducible characters are rational valued. Also we prove that even the alternating groups are not Q-groups the conclusion of Brauer-Speiser theorem about the Schur index is still true for this class of groups.*

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## 1. Introduction

The notations and terminology are standard ( see for example [2] [3] and [4]). All groups will be finite.

The representation theory of finite groups emerged around the turn of the 19 to 20 century with the work of Frobenius, Schur, and Burnside. While it applied in principle to any finite group, the symmetric group $S_n$ was a simple but important special case. Simple because its characters and irreducible representations are rational (could already be found in the rational field), important because every finite group could be embedded in some symmetric group.

The alternating group $A_n$ is the normal subgroup of the even permutations in $S_n$ and has group order $n!/2$ . For $n > 1$, the group $A_n$ is the commutator subgroup of the symmetric group $S_n$ with index 2. Even more, each element of An is itself a commutator. It is the kernel of the signature group homomorphism sgn : $S_n \to \{1, -1\}$.

[1] Prof., University of Bucharest, Faculty of Physics, Department of Theoretical Physics, and Mathematics, 077125, Magurele, P.O.Box MG-11, Bucharest Romania, E-mail address: ion.armeanu@gmail.com

[2] Assist., Department of Mathematics , Mimar Sinan Fine Arts University, Besiktas-Istanbul-Turkey, E-mail address: dislekel@yahoo.com

The alternating groups represent a very important class of groups. The group $A_n$ is abelian if and only if $n \leq 3$ and simple if and only if $n = 3$ or $n \geq 5$ ([6], p. 295). $A_5$ is the smallest non-abelian simple group (having order 60), and the smallest non-solvable group. $A_4$ is the smallest group demonstrating that the converse of Lagrange's theorem is not true in general. Given a finite group $G$ and a divisor $d$ of $|G|$, there does not necessarily exist a subgroup of $G$ with order $d$: the group $G = A_4$, of order 12, has no subgroup of order 6.

The pure rotational subgroup of the icosahedral group is isomorphic to $A_5$.

The full icosahedral group is isomorphic to the direct product $A_5 \times Z_2$.

$A_4$ is isomorphic to $PSL_2(3)$. $A_5$ is isomorphic to $PSL_2(4)$.

**Definition 1.** *(see [4]) A $Q$-group is a group all whose irreducible characters are rational valued.*

**Proposition 1.** *A group $G$ is a $Q$-group if and only if for every $x, y \in G$ with $\langle x \rangle = \langle y \rangle$ there is a $g \in G$ such that $gxg^{-1} = y$.*

*Proof.* Let ε be a primitive n-th root of 1 in C and let ***E*** = Q(ε). Let Gal be the Galois group of E over Q. Given $(m, n) = 1$, there exists σ∈Gal with $\lambda(x^m) = \lambda(x)^\sigma$ for all x∈ G and all irreducible characters λ∈Irr(G) . Conversely, for every σ∈Gal**,** there is a m such that this formula holds.

**Proposition 2.** *A group $G$ is a $Q$-group if and only if for all $x \in G$, $N_G(\langle x \rangle)/C_G(\langle x \rangle); Aut(\langle x \rangle)$.*

*Proof.* Let $f: N_G(\langle x \rangle) \to Aut(\langle x \rangle)$ defined by $f(g) = gxg^{-1}$. Then f is a homomorphism, $Kerf = C_G(\langle x \rangle)$ and $f$ is in. It is clear that $f$ is onto iff $G$ is a $Q$-group.

**Remark 1.** *$G$ is a $Q$-group iff $Aut(K); N_G(K)/C_G(K)$ for all cyclic subgroups K of G.*

**Remark 2.** The symmetric groups $S_n$ are the prototype for the Q-groups for two resons. First $S_n$ are Q-groups, and even more, the irreducible representations of $S_n$ have Q as splitting field. Secondly, by [1] a group G is a Q-group if and only if it can be embedded without fusion in a $S_n$ (or the conjugacy classes of $S_n$ who are in $A_n$ do not split in $A_n$). The fusion of elements in a finite group is the source of many deep theorems in finite group theory (see [2]).

**Proposition 3.** *Let $G$ be a Q-group group. Then:*

*1. If $N$ is a normal subgroup of $G$, then $G/N$ is a Q-group.*

*2. If $G$ and $H$ are Q-groups then $G \times H$ is a Q-group*
*3. Every non-identity 2-central element of $G$ is an involution.*
*4. The center of $G$, $Z(G)$ is an elementary abelian 2-group.*
*5. If $G$ is abelian then it is an elementary abelian 2-group.*

*Proof.* 1. The irreducible characters of $G/H$ are in fact irreducible characters for $G$.

2. The irreducible characters of $G \times H$ are products of irreducible characters of $G$ and $H$.

3. A 2-central element is a nonidentity element in the center of some Sylow 2-subgroup of G. Let $x \in G$ be a 2-central element and set the order $o(x) = 2^k m$, where $(2, m) = 1$. Then there is $S$ a Sylow 2-subgroup of $G$ such that $S \leq C_G(x) \leq N_G(\langle x \rangle)$, hence $(2: |N_G(\langle x \rangle): C_G(x)|) = 1$. Therefore $m = k = 1$.

4. The nontrivial elements of $Z(G)$ are central.

5. Follows from 4.

**Proposition 4.** *Let $G$ be a Q-group and $G'$ the derived subgroup. Then:*
*1. $G/G'$ is an elementary abelian 2-group.*
*2. $O^2(G) = O^2(G')$.*
*3. If $p$ is an odd prime and $P$ a Sylow p-subgroup of $G$, then $P \leq [P, G]$.*
*4. $G$ is generated by its 2-elements.*
*5. For all $S \in Syl_2(G)$ we have that $C_G(S) = Z(S)$.*
*6. If $G$ is also solvable and $S \in Syl_2(G)$ then $N_G(S) = S$ and $N_G(H) = H$ for every subgroup H with $S \leq H \leq G$.*

*Proof.* 1. Follows from prop.3.5.

2. Observe that $|G: G'| = 2^n$.

3. Let $Foc_G(P)$ be the focal subgroup of $P$ in $G$. $Foc_G(P)$ is generated by the commutators $[x; y]$, $x, y \in G$ which lie in $P$. Let $\lambda: G \mapsto P/Foc_G(P)$ stand for the transfer map. Then $\lambda$ is onto and therefore $P/Foc_G(P)$ is an abelian Q-group. Since $p$ is odd, it follows that $P = Foc_G(P) \leq P \cap [P, G] \leq P$ and therefore $P \leq [P, G]$.

4. Let $H$ be the subgroup of $G$ generated by the 2-elements of $G$. Then $H$ is normal in $G$ and $G/H$ is an ambivalent group of odd order.

5. The elements of $C_G(S)$ are 2-central and by prop 3.2. the 2-central elements are involutions. Because $Z(S)$ is the set of 2-elements of $C_G(S)$, it follows that $Z(S)$ is a Sylow 2-subgroup of $C_G(S)$. By Burnside Transfer Theorem ( see [2]) $Z(S)$ has a normal complement $N$ in $N_G(S)$ and therefore $C_G(S) = Z(S)N$. Since $Z(S)$ is a Sylow 2-subgroup of $C_G(S)$ it follows that $(2: |N|) = 1$. Hence $N$ must be trivial.

6. We prove by induction on the order of $G$ that if $S \in Syl_2(G)$ and $x \in N_G(\langle S \rangle)$ is an odd order element, then $x$ is non-real in $G$.

Let $N$ be a minimal nontrivial subgroup of $G$. Since $G$ is solvable, N is an elementary abelian p-group, for some prime p. If $x \notin N$ then the image of $x$ in $G/N$ is non-real by the induction hypothesis and $x$ is non-real.

If $x \in N$, then we have that $[x, S] \subseteq S \cap N = 1$. Since $C_G(x)$ contains a Sylow 2-subgroup of $G$, it follows that the order of $N_G(\langle x \rangle)/C_G(x)$ is odd. Hence $x$ is a non-real element.

For the second part let $x \in N_G(H)$. Then $S$ and $xSx^{-1}$ are Sylow 2-subgroups of $H$ and hence $xSx^{-1} = zSz^{-1}$ for some $z \in H$. Therefore $z^{-1}x \in N_G(S) = S \leq H$, hence $x \in H$.

## 2. Alternating groups

**Corollary 1.** *Let $S_n$ and $A_n$=$S_n'$( the derived subgroup). Then:*

*1. $O^2(S_n) = O^2(A_n)$*

*2. If $p$ is an odd prime and $P$ a Sylow p-subgroup of $S_n$, then $P \leq [P, S_n]$.*

*3. $S_n$ is generated by its 2-elements.*

*4. For all $S \in Syl_2(S_n)$ we have that $C_G(S) = Z(S)$.*

**Theorem 1.** *There are no nontrivial alternating $Q$-groups.*

*Proof.* It is well known that the symmetric groups $S_n$ are $Q$ groups. The alternating groups $A_n$ are normal in $S_n$ hence it contains full conjugacy classes of $S_n$. Therefore, for an element $x \in A_n$ to be rational it is necessary that the conjugacy class of $x$ , $cl(x)$ in $S_n$ splits in $A_n$. The lenght of $cl(x)$ is the index of the centralizer of $x$ in the corresponding group. clearly $C_{A_n}(x) = C_{S_n}(x) \cap A_n$ . Because $|S_n : A_n| = 2$ , then either $C_{A_n}(x) = C_{S_n}(x)$ or $|C_{S_n}(x) : C_{A_n}(x)| = 2$ . Hence the $S_n$ conjugacy class of $x$ splits in $A_n$ iff $C_{A_n}(x) = C_{S_n}(x)$.

Let's prove now that $C_{A_n}(x) = C_{S_n}(x)$ if and only if the type of $x$ (see [2]) , $T(x) = (t_1, \ldots, t_s)$ has $t_i$ odd and pairwise different.

Suppose $C_{A_n}(x) = C_{S_n}(x)$ . Because $x$ commutes with its cyclic factors, then $x$ cannot have cyclic factors of even lenght. If $x$ has two cyclic factors $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ of the same lenght, then $(i_1 j_1) \ldots (i_k j_k)$ belong to $C_{S_n}(x)$ but is not in $A_n$.

Suppose now that $T(x) = (t_1, \ldots, t_s)$ with $t_i$ odd and pairwise different. Then $|C_{S_n}| = \prod \ t_i$ is odd, therefore $C_{A_n}(x) = C_{S_n}(x)$.

Let $x = (12\dots k)\dots(j\dots n)$ product of odd lengths disjoint cycles. Then $z = (2k)(3(k-1))\dots(j+1n)\dots$ inverts $x$. Clearly, $z \notin A_n$ if and only if the number of cyclic factors of $x$ having lenght congruent to $3 mod 4$ is odd. Suppose $z \notin A_n$ and that there is a $t \notin A_n$ inverting $x$. Then $(t^{-1}z)x(t^{-1}z)^{-1} = x$ and $t^{-1}z \notin A_n$, hence $C_{A_n}(x) \neq C_{S_n}(x)$, contradiction. We study now the existence of such conjugacy classes.

Let $n = 4m + 1$ , $m \geq 2$ we have the partition $(4m - 3,3,1)$ . For $n = 4m + 2$, $m \geq 4$ the partition $(4(m-1) - 3,5,3,1)$. Therefore no alternating group $A_n$ with such a $n$ can be rational.

Hence the only possible $n$ are $5,6,10,14$.

For $A_5$ the partition $(5)$ has odd and different elements. For $x = (12345)$,
$Aut(< x >) \cong Z_4$, and by prop. 2 $A_5$ is not rational.

For $A_6$ we have the partition $(5,1)$ and as before $A_6$ is not rational.

For $A_{14}$ we have the partition $(13,1)$ and as before the element $(1\dots.13)$ is not rational.

For $A_{10}$ the only partitions with odd and different elements are $(7,3)$ and $(9,1)$ and as before the element $(1\dots.9)$ is not rational.

We verified this by computing the character table for these groups using GAP ([5]).

For $A_5$ of order 60 the character table is

X.1 1 1 1 1 1
X.2 3 -1 . A *A
X.3 3 -1 . *A A
X.4 4 . 1 -1 -1
X.5 5 1 -1 . .

where $A = -E(5) - E(5)^4 = (1 - ER(5))/2$ is real but not rational. The other characters are rational valued.

For $A_6$ of order 360 the character table is

X.1 1 1 1 1 1 1 1
X.2 5 1 2 -1 -1 . .
X.3 5 1 -1 2 -1 . .
X.4 8 . -1 -1 . A *A
X.5 8 . -1 -1 . *A A
X.6 9 1 . . 1 -1 -1
X.7 10 -2 1 1 . . .

where $A = -E(5) - E(5)^4$ is real but not rational.

For $A_{10}$ of order 1814400 the character table is

X.1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
X.2 9 5 1 6 2 3 -1 . 3 -1 . 1 4 . 1 -1 1 -1 2 -1
X.3 35 11 3 14 2 2 2 -1 3 3 . -1 5 1 -1 . . . . .
X.4 36 8 -4 15 -1 3 -1 . 2 -2 -1 . 6 -2 . 1 -1 1 1 1
X.5 42 6 2 . . 3 3 -3 . -4 . 2 -3 1 . 2 -1 -1 . .
X.6 75 15 3 15 3 . . 3 1 -3 1 -1 . . . . . . -2 1
X.7 84 . -4 21 -3 3 3 3 -2 2 1 . 4 . 1 -1 -1 -1 . .
X.8 90 14 2 6 2 3 -1 . . 4 . 2 -5 -1 1 . -1 1 -1 -1
X.9 126 -14 6 21 1 6 -2 . -4 . -1 -2 1 1 1 1 . . . .
X.10 160 16 . 34 -2 -2 -2 -2 . . . . 5 1 -1 . . . -1 -1
X.11 210 6 -6 -21 3 . . 3 -4 . -1 2 5 1 -1 . . . . .
X.12 224 -16 . 14 2 2 2 -1 . . . . -1 -1 -1 -1 . . . .
X.13 224 -16 . 14 2 2 2 -1 . . . . -1 -1 -1 -1 . . . .
X.14 225 5 9 15 -1 -6 2 . -1 3 -1 1 . . . . . . 1 1
X.15 252 8 4 -21 -1 3 -1 . -2 2 1 . 2 -2 -1 2 1 -1 . .
X.16 288 16 . -6 -2 6 -2 . . . . . -7 1 -1 -2 . . 1 1
X.17 300 . 4 -15 -3 3 3 3 2 -2 -1 . . . . . 1 1 -1 -1
X.18 315 19 -5 21 1 -3 1 . -1 -1 -1 -1 -5 -1 1 . 1 -1 . .
X.19 350 -10 -2 35 -1 -1 -1 -1 -2 -2 1 2 . . . . 1 1 . .
X.20 384 . . -24 . . . -3 . . . . 4 . 1 -1 . . -1 A
X.21 384 . . -24 . . . -3 . . . . 4 . 1 -1 . . -1 *A
X.22 450 10 2 -15 1 -3 1 . -2 -2 1 -2 . . . . -1 1 2 -1
X.23 525 -15 5 . . -3 -3 3 3 -1 . 1 . . . . -1 -1 . .
X.24 567 -9 -9 . . . . . 3 3 . -1 -3 1 . 2 . . . .

where $A = E(21)^2 + E(21)^8 + E(21)^10 + E(21)^11 + E(21)^13 + E(21)^19 = (1 - ER(21))/2$ is real but not rational. The other characters are rational valued.

$A_{14}$ of order 43589145600 has 72 irreducible characters and the non rational numbers

$$A = -2 * E(5) + E(5)^2 + E(5)^3 - 2 * E(5)^4$$

$$B = E(33)^5 + E(33)^7 + E(33)^10 + E(33)^13 + E(33)^14 + E(33)^19 + E(33)^20 + E(33)^23 + E(33)^26 + E(33)^28$$

$$C = -E(13) - E(13)^3 - E(13)^4 - E(13)^9 - E(13)^10 - E(13)^12$$

appear as values.

**Definition 2.** *Let G be a finite group, K is a splitting field for G, and* $\chi$ *a character of an irreducible linear representation R of G over K. Suppose k is the subfield of K generated by the character values* $\chi(g), g \in G$ *. The* ***Schur index*** *of* $\chi$ *is defined as the smallest positive integer m=m(*$\chi$ *) such that there exists a field extension L of k of degree m, [L:k]=m, such that the representation R can be realized over L ( i.e., we can change basis such that all the matrix entries are from L).*

**Theorem 2.** *Let* $S_n$ *and* $S_n'=A_n$*. Then* $m_Q(\mu) = m(\mu) \leq 2$ *for all* $\mu \in Irr(A_n)$ .

*Proof.* Let $\mu \in Irr(A_n)$.

If $\mu$ is real valued, then $m_Q(\mu) = m(\mu) \leq 2$ by Brauer-Speiser theorem (see [3]).

Suppose now that $\mu$ is not a real valued irreducible character. Because $S_n$ is a Q-group, there is a $S_n$-conjugate of $\mu$ which is the complex conjugate of $\mu$, that is $\mu^g = \mu^{-1}$ fore some $g \in S_n$. Since $g^2 \in A_n$, $g^2$ stabilizes $\mu$. Let $H = \langle S_n, g\rangle$. Clearly, $\nu = \mu^H$ is a real valued irreducible character of $H$. Denote by $\nu_1, \nu_2, \ldots, \nu_k$ all the distinct algebraic conjugates of $\nu$ over $Q$. By Brauer-Speiser theorem we have that $\tau = 2(\nu_1 + \nu_2 + \ldots + \nu_k)$ is rational valued and $m_Q(\tau) = 1$, that is $\tau$ is the character of a rational representation of $H$. In $\tau_G$, the character $\mu$ occurs twice, hence $m_Q(\mu) = m(\mu) \leq 2$.

## 3. Conclusions

The symmetric groups are Q-groups, and even more, Q is a splitting field for these groups. The alternating groups are very close to the symmetric groups as normal subgroups of index 2. **Theorem 1** says that even so, the characters and the representations of the alternating groups do not have similar properties with those of the symmetric groups.

This theorem, via [1], says also that the alternating groups can not be embedded without fusion in a symmetric group.

**Theorem 2** recover the conclusions of Brauer-Speiser theorem for the alternating groups. An interesting step will be to prove that for the alternating groups they are irreducible characters with Schur index 2.

## REFERENCES

[1] *V. Alexandru, I.Armeanu***,** Sur les Characteres d'un Groupe Fini, C.R.Acad.Sc.Paris, t.298, Serie I, no. 6, 1984, 99-102.

[2*] B.Huppert*, Endliche Gruppen, Springer Verlang, 1976.

[3*] I. M. Isaacs*, Character Theory of Finite Groups, Academic Press, 1976.

[4] *D.Kletzing*, Structure and Representations of Q-groups, Lecture Notes in Mathematics, Springer-Verlag, 1984.
[5] [GAP2008] The GAP Group, GAP -- Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)
[6] *W. R. Scott*, Group Theory. New York: Dover, 1987.