APPROXIMATION ON THE MIXED TYPE ADDITIVE-QUADRATIC-SEXTIC FUNCTIONAL EQUATION

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In this paper, a class of the mixed type additive-quadratic-sextic functional equations is introduced and the common general solutions of elements of this class are obtained. An alternative method of the fixed point theory to study the stability of these new functional equations in the quasi-β-normed spaces has been applied. Furthermore, a hyperstability result and a counterexample for the odd case are indicated.

Keywords: Additive-quadratic-sextic functional equation; Hyers-Ulam-Rassias stability; Quasi-β-normed space.

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1. Introduction

The theory of stability of functional equations is an emerging field in modern mathematics. The investigation of stability of functional equations is initiated by the renowned problem of S. M. Ulam [20] in 1940. D. H. Hyers [13] was the foremost mathematician who provided an answer to the question of Ulam. Later on, various generalizations and extensions of Hyers' result were ascertained by Bourgin [8], Th. M. Rassias [18], T. Aoki [1], J. M. Rassias [17] and P. Găvruţa [12] in different versions. After that, this problem became known as *Hyers-Ulam stability problem* for functional equations. During the last three decades, several stability problems of a large variety of functional equations in miscellaneous spaces have been extensively studied and generalized by a number of mathematicians. Some results regarding the stability of various forms of the mixed type additive-quadratic ([5], [6]), additive-cubic ([16], [23]), additive-quartic ([2], [3]), cubic-quartic ([4], [9]), quadratic-quartic ([7], [21]), additive-quadratic-cubic [15], additive-quartic-cubic [10] and additive-quartic-cubic-quartic [19] functional equations were investigated in normed spaces and algebras.

Motivated by the sextic functional equation given in [14], in this paper, we consider the following mixed type additive-quadratic-sextic functional equations as follows:

$$\begin{aligned} &f(rx+sy)+f(rx-sy)+f(sx+ry)+f(sx-ry) \\ &\quad = r^2s^2(r^2+s^2)[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)] \\ &\quad + 2[f(rx)+f(sx)+f(ry)+f(sy)]-(r+s)(f(y)-f(-y)) \end{aligned} \tag{1}$$

for the fixed integer r and any integer s such that $r, s \neq 0, \pm 1$ and $r+s \neq 0$. It is easily verified that the function $f(x) = ax^6 + bx^2 + cx$ is a common solution of the functional equations given in (1). We obtain the general solution and study the Hyers-Ulam stability of the equation (1) in the quasi-β-normed spaces for the fixed integer r and any integer

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s such that $r, s \neq 0, \pm 1$ and $r + s \neq 0$. In the case that f is an odd mapping satisfying (1), we show that under some mild conditions (1) can be hyperstable. We also present a counterexample for a single case.

2. **Solution of (1)**

In this section, we obtain some results on the general solution of functional equation (1). Given $f : X \longrightarrow Y$, for simplicity, we define the difference operators $\Gamma_{r,s} f : X \times X \longrightarrow Y$ by

$$\begin{aligned}\Gamma_{r,s} f(x,y) &= f(rx+sy) + f(rx-sy) + f(sx+ry) + f(sx-ry) \\ &\quad - r^2 s^2 (r^2+s^2)[f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)] \\ &\quad - 2[f(rx) + f(sx) + f(ry) + f(sy)] + (r+s)(f(y) - f(-y))\end{aligned}$$

for all $x, y \in X$, for the fixed integer r and any integer s such that $r, s \neq 0, \pm 1$ and $r + s \neq 0$. In the sequel, by $\Gamma_{r,s} f(x,y) = 0$, we mean that f satisfies (1) for the fixed integer r and any integer s such that $r, s \neq 0, \pm 1$ and $r + s \neq 0$. Moreover, for the set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n-times}$ by X^n. Recall that a mapping $f : X^n \longrightarrow Y$ is called n-additive if it is additive in each variable. Here, we find out the general solution of (1).

Proposition 2.1. *Let X and Y be real vector spaces. Then, a mapping $f : X \longrightarrow Y$ satisfies the functional equation* (1) *if and only if there exist a unique additive mapping $A : X \longrightarrow Y$, a unique symmetric biadditive mapping $Q : X \times X \longrightarrow Y$ and a unique symmetric* 6*-additive mapping $S : X^6 \longrightarrow Y$ such that $f(x) = A(x) + Q(x,x) + S(x,x,x,x,x,x)$ for all $x \in X$.*

Proof. Suppose that there exist a unique additive mapping $A : X \longrightarrow Y$, a unique symmetric biadditive mapping $Q : X \times X \longrightarrow Y$ and a unique symmetric 6-additive mapping $S : X^6 \longrightarrow Y$ such that $f(x) = A(x) + Q(x,x) + S(x,x,x,x,x,x)$ for all $x \in X$. It is easily verified that f satisfies the functional equation (1) for all $x, y \in X$.

Conversely, assume that f satisfies (1). We decompose f into the even part and odd part by setting

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)), \qquad f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad (x \in X).$$

By a simple computation, we see that $\Gamma_{r,s} f_o(x,y) = 0$. Since f_o is an odd mapping, the last equation can be rewritten as follows:

$$\begin{aligned}&f_o(rx+sy) + f_o(rx-sy) + f_o(sx+ry) + f_o(sx-ry) \\ &\quad = r^2 s^2 (r^2+s^2)[f_o(x+y) + f_o(x-y) - 2f_o(x)] \\ &\quad + 2[f_o(rx) + f_o(sx) + f_o(ry) + f_o(sy)] - 2(r+s) f_o(y)\end{aligned} \tag{2}$$

for all $x, y \in X$. Replacing (x, y, s) by $(0, x, r)$ in (2), we have

$$f_o(rx) = r f_o(x) \tag{3}$$

for all $x \in X$. Putting $s = r$ in (2) and using (3), we find $2(r^6 - r)[f_o(x+y) + f_o(x-y)] = 4(r^6 - r) f_o(x)$ for all $x, y \in X$. Since $r \neq 0, 1$, we have

$$f_o(x+y) + f_o(x-y) = 2f_o(x) \tag{4}$$

for all $x, y \in X$. Replacing (x, y) by (y, x) in (4) and using the oddness of f_o, we get

$$f_o(x+y) - f_o(x-y) = 2f_o(y) \tag{5}$$

for all $x, y \in X$. It concludes by the equalities (4) and (5) that $f_o(x+y) = f_o(x) + f_o(y)$ for all $x, y \in X$. This means that f_o is an additive mapping, say it A. Now, similar the above, one can show that $\Gamma_{r,s} f_e(x, y) = 0$. Hence, the last equation is equivalent to the following:

$$\begin{aligned} &f_e(rx+sy) + f_e(rx-sy) + f_e(sx+ry) + f_e(sx-ry) = r^2 s^2 (r^2+s^2)[f_e(x+y) \\ &+ f_e(x-y) - 2f_e(x) - 2f_e(y)] + 2[f_e(rx) + f_e(sx) + f_e(ry) + f_e(sy)]. \end{aligned} \tag{6}$$

Note that $f_e(0) = 0$. Letting $y = x$ and $s = 2r$ in (6) and using the evenness of f, we get

$$f_e(3rx) + f_e(rx) = 10r^6[f_e(2x) - 4f_e(x)] + 2[f_e(2rx) + f_e(rx)] \tag{7}$$

for all $x \in X$. Putting $y = x$ and $s = 3r$ in (6), we obtain

$$f_e(4rx) + f_e(2rx) = 45r^6[f_e(2x) - 4f_e(x)] + 2[f_e(3rx) + f_e(rx)] \tag{8}$$

for all $x \in X$. It follows from (7) and (8) that

$$f_e(4rx) + f_e(2rx) = 65r^6[f_e(2x) - 4f_e(x)] + 4[f_e(2rx) + f_e(rx)] \tag{9}$$

for all $x \in X$. Letting $s = r$ and $y = 2x$ in (6), we have

$$f_e(3rx) + f_e(rx) = r^6[f_e(3x) - 2f_e(2x) - f_e(x)] + 2[f_e(2rx) + f_e(rx)] \tag{10}$$

for all $x \in X$. Once more, by putting $s = r$ and $y = 3x$ in (6), we find

$$\begin{aligned} f_e(4rx) + f_e(2rx) &= r^6[f_e(4x) + f_e(2x) - 2f_e(x) - 2f_e(3x)] \\ &\quad + 2[f_e(3rx) + f_e(rx)] \end{aligned} \tag{11}$$

for all $x \in X$. Plugging (10) into (11), we see that

$$f_e(4rx) + f_e(2rx) = r^6[f_e(4x) - 3f_e(2x) - 4f_e(x)] + 4[f_e(2rx) + f_e(rx)] \tag{12}$$

for all $x \in X$. Since $r \neq 0$, the equalities (9) and (12) imply that $f_e(4x) - 68f_e(2x) + 256f_e(x) = 0$ for all $x \in X$. The last equality means that the mappings $g, h : X \longrightarrow Y$ defined by $g(x) := f_e(2x) - 64f_e(x)$ and $h(x) := f_e(2x) - 4f_e(x)$ are quadratic and sextic, respectively. Thus, there exists a unique symmetric biadditive mapping $Q : X \times X \longrightarrow Y$ and a unique symmetric 6-additive mapping $S : X^6 \longrightarrow Y$ such that $f_e(x) = Q(x, x) + S(x, x, x, x, x, x)$ for all $x \in X$ (see the proofs of [14, Theorem 2.1] and [21, Theorem 2.2]). This completes the proof. □

Corollary 2.1. *Let X and Y be real vector spaces. Suppose that the mapping $f : X \longrightarrow Y$ satisfies the functional equation* (1).

(i) *If f is an even mapping, then it is quadratic-sextic;*
(ii) *If f is an odd mapping, then it is additive.*

3. **Stability of (1)–Odd Case**

In this section, we prove the generalized Hyers-Ulam stability of the mixed type additive-quadratic-sextic functional equation (1) when f is an odd mapping. We firstly recall some basic facts concerning quasi-β-normed space.

Definition 3.1. *Let β be a fix real number with $0 < \beta < 1$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm is a real-valued function on X satisfying the following:*

(i) $\|x\| \geq 0$ *for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;*
(ii) $\|tx\| = |t|^\beta \|x\|$ *for all $x \in X$ and $t \in \mathbb{K}$;*
(iii) *There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.*

The pair $(X, \|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space. From now on, let X be a linear space and Y be a quasi-β-Banach space with quasi-β-norm $\|\cdot\|_Y$ and K be the modulus of concavity of $\|\cdot\|_Y$, unless otherwise explicitly stated. In this section, by using an idea of Găvruţa [12] we prove the stability of (1) in the spirit of Hyers, Ulam, and Rassias. We recall the following theorem which is a result in fixed point theory [22]. This result plays a fundamental role to obtain our purpose in this paper.

Lemma 3.1. *Let $j \in \{-1, 1\}$ be fixed, $a, s \in \mathbb{N}$ with $a \geq 2$ and $\psi : X \longrightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with $\psi(a^j x) < La^{js\beta}\psi x$ for all $x \in X$. If $f : X \longrightarrow Y$ is a mapping satisfying $\|f(ax) - a^s f(x)\|_Y \leq \psi(x)$ for all $x \in X$, then there exists a uniquely determined mapping $F : X \longrightarrow Y$ such that $F(ax) = a^s F(x)$, $F(x) = \lim_{n\to\infty} a^{-jns} f(a^{jns}x)$ and $\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}|1-L^j|}\psi(x)$ for all $x \in X$.*

In the upcoming result, we prove the stability for the functional equation (1) in quasi-β-normed spaces.

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\phi : X \times X \longrightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with $\phi(t^j x, t^j y) \leqslant t^{j\beta} L\phi(x, y)$ for all $x \in X$, where $t \in \{2, r\}$. Let $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\|\Gamma_{r,s} f(x, y)\|_Y \leqslant \phi(x, y) \tag{13}$$

for all $x, y \in X$. Then, there exist unique additive mappings $\mathcal{A}_i : X \longrightarrow Y$ $(i \in \{1, 2\})$ such that

$$\|f(x) - \mathcal{A}_1(x)\|_Y \leqslant \frac{1}{r^\beta|1 - L^j|}\widetilde{\phi}(x) \tag{14}$$

and

$$\|f(x) - \mathcal{A}_2(x)\|_Y \leqslant \frac{1}{2^\beta|1 - L^j|}\Phi(x) \tag{15}$$

for all $x \in X$, where

$$\widetilde{\phi}(x) := \frac{K}{4^\beta}\left[\left|\frac{r^6}{r-1}\right|^\beta \phi(0, 0) + \phi(0, x)\right], \tag{16}$$

$$\begin{aligned}\Phi(x) :=& \frac{1}{(2(r^6 - 1))^\beta}[K^3 2^\beta \widetilde{\phi}(2x) + K^3\phi(x, x) + K^2 8^\beta \widetilde{\phi}(x)] \\ &+ \frac{K}{(4|r-1|)^\beta}\phi(0, 0).\end{aligned} \tag{17}$$

Proof. Putting $x = y = 0$ and $s = r$ in (13), we get

$$\|f(0)\|_Y \leq \frac{1}{|4(r-1)|^\beta}\phi(0, 0). \tag{18}$$

The relation (18) implies that

$$\|4r^6 f(0)\|_Y \leq \left|\frac{r^6}{r-1}\right|^\beta \phi(0, 0). \tag{19}$$

Replacing (x, y, s) by $(0, x, r)$ in (13), we get

$$\|4r^6 f(0) - 4f(rx) + 4rf(x)\|_Y \leq \phi(0, x) \tag{20}$$

for all $x \in X$. It follows from (19) and (20) that

$$\|f(rx) - rf(x)\|_Y \leq \widetilde{\phi}(x) \tag{21}$$

for all $x \in X$, where $\widetilde{\phi}(x)$ is defined in (16). By Lemma 3.1, there exists a unique mapping $\mathcal{A}_1 : X \longrightarrow Y$ such that $\mathcal{A}_1(rx) = r\mathcal{A}_1(x)$ and (14) holds. It remains to show that $\mathcal{A}_1$ is an additive map. By (13), we have

$$\left\|\frac{\Gamma_{r,s}f(r^{jn}x, r^{jn}y)}{r^{jn}}\right\|_Y \leq r^{-jn\beta}\phi(r^{jn}x, r^{jn}y) \leqslant r^{-jn\beta}(r^{j\beta}L)^n\phi(x,y) = L^n\phi(x,y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we observe that $\Gamma_{r,s}\mathcal{A}_1(x,y) = 0$ for all $x, y \in X$. Therefore, the mapping $\mathcal{A}_1$ is additive, as required. For the case $t = 2$, by setting $s = r$ and $y = x$ in (13), we obtain

$$\|2f(2rx) - 2r^6(f(2x) - 2f(x)) - 2(r^6-1)f(0) - 8f(rx) + 4rf(x)\|_Y \leq \phi(x,x)$$

for all $x \in X$. The above inequality can be modified as follows:

$$\begin{aligned}\|2(f(2rx) - rf(2x)) &- 2(r^6 - r)(f(2x) - 2f(x)) \\ &- 2(r^6-1)f(0) - 8(f(rx) - rf(x))\|_Y \leq \phi(x,x)\end{aligned} \tag{22}$$

for all $x \in X$. It concludes from the relation (21) that

$$\|2(f(2rx) - rf(2x))\|_Y \leq \widetilde{\phi}(2x) \tag{23}$$

for all $x \in X$. Now, the inequalities (18), (21), (22) and (23) imply that

$$\|f(2x) - 2f(x))\|_Y \leq \Phi(x) \tag{24}$$

for all $x \in X$, where $\Phi(x)$ is defined in (17). Hence, Lemma 3.1 necessitates that there exists a unique additive mapping $\mathcal{A}_2 : X \longrightarrow Y$ such that (15) holds. □

Recall that a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. Under some conditions the functional equation (1) can be hyperstable as follows. In all corollaries of the paper, we assume that X is a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and Y is a β-Banach space with quasi-β-norm $\|\cdot\|_Y$.

Corollary 3.1. *Let θ, m and n be positive numbers with $m + n \neq \frac{\beta}{\alpha}$. If $f : X \longrightarrow Y$ is a mapping satisfying $\|\Gamma_{r,s}f(x,y)\|_Y \leqslant \theta\|x\|_X^m\|y\|_X^n$ for all $x, y \in X$, then f is additive.*

Proof. Taking $\phi(x,y) = \theta\|x\|_X^m\|y\|_X^n$ in Theorem 3.1 in the case $t = r$, we see that $\widetilde{\phi}(x) = 0$ and so f is additive. □

The idea of the following example is taken from [11]. The method of proof is similar but we include it for the sake of completeness.

Example Let $\theta > 0$. For the fixed and arbitrary integers r, s with $r, s \neq 0, \pm 1$ and $r + s \neq 0$, set $a = \frac{\theta}{4[2r^2s^2(r^2+s^2)+r+s+6]t}$, where $t = \max\{|r|, |s|\}$. Consider the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} ax & |x| < 1 \\ a & x \geq 1 \\ -a & x \leq -1. \end{cases}$$

Suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined through $f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n}$ for $x \in \mathbb{R}$. We have $f(0) = 0$. Let $x \in \mathbb{R}^+$. Assume that $N_0 = \min\{n : x \geq \frac{1}{2^n}\}$. Thus, $f(-x) = -N_0x - x\sum_{n=N_0}^{\infty} \frac{1}{2^n} = -f(x)$, and so f is an odd function. Furthermore, ψ is continuous and bounded by a. Since f is a uniformly convergent series of continuous functions, it is continuous and bounded. Indeed, for each $x \in \mathbb{R}$, we have $|f(x)| \leq 2a$. We wish to show that

$$|\Gamma_{r,s}f(x,y)| \leqslant \theta(|x| + |y|) \tag{25}$$

for all $x, y \in \mathbb{R}$. Obviously, (25) holds for $x = y = 0$. Assume that $|x| + |y| < \frac{1}{t}$. We know that $|r|, |s| \geq 2$. Then, there exist positive integers N_1, N_2 such that $|rx| + |sy| < \frac{1}{2^{N_1-1}}$ and $|sx| + |ry| < \frac{1}{2^{N_2-1}}$. Hence, $|2^{N-1}(rx \pm sy)| < 1, |2^{N-1}(sx \pm ry)| < 1, |2^{N-1}rx| < 1, |2^{N-1}sx| < 1, |2^{N-1}ry| < 1, |2^{N-1}sy| < 1, |2^{N-1}(x \pm y)| < 1$, where $N = \min\{N_1, N_2\}$. So, the above inequalities hold for each $n \in \{0, 1, 2, \ldots, N-1\}$. Since, ψ is linear on $(-1, 1)$, by Corollary 2.1, $|\Gamma_{r,s}\psi(2^n x, 2^n y)| = 0$ for all $n \in \{0, 1, 2, \ldots, N-1\}$. The last equality implies that

$$\frac{|\Gamma_{r,s}f(2^n x, 2^n y)|}{|x| + |y|} \leq \sum_{n=N}^{\infty} \frac{|\Gamma_{r,s}\psi(2^n x, 2^n y)|}{2^n(|x| + |y|)} \leq \sum_{l=0}^{\infty} \frac{[4r^2s^2(r^2+s^2) + 2(r+s) + 12]a}{2^l 2^N(|x| + |y|)}$$

$$\leq \sum_{l=0}^{\infty} \frac{[4r^2s^2(r^2+s^2) + 2(r+s) + 12]a}{2^l} \leq 4[2r^2s^2(r^2+s^2) + r + s + 6]at = \theta$$

for all $x, y \in \mathbb{R}$. If $|x| + |y| \geq \frac{1}{t}$, then $\frac{|\Gamma_{r,s}f(2^n x, 2^n y)|}{|x|+|y|} \leq 4[2r^2s^2(r^2+s^2) + r + s + 6]at = \theta$. Therefore, f satisfies (25) for all $x, y \in \mathbb{R}$. Suppose contrary to our claim, that there exists a number $b \in [0, \infty)$ and an additive function $\mathcal{A} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $|f(x) - \mathcal{A}(x)| < b|x|$ for all $x \in \mathbb{R}$. Hence, there exists a constant $c \in \mathbb{R}$ such that $\mathcal{A}(x) = cx$ for all $x \in \mathbb{R}$. So

$$|f(x)| \leq (|c| + b)|x|, \tag{26}$$

for all $x \in \mathbb{R}$. On the other hand, consider $m \in \mathbb{N}$ such that $(m+1)a > |c| + b$. If x is a real number in $\left(0, \frac{1}{2^{N-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \cdots, N-1$. Thus, for such x, we get $f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \geq \sum_{n=0}^{m} \frac{2^n ax}{2^n} = (m+1)ax > (|c| + b)x$. This relation contradicts (26).

4. **Stability of (1)–Even Case**

In this section, we prove the stability of the functional equation (1) when f is an even mapping.

Theorem 4.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\phi : X \times X \longrightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with $\phi(2^j x, 2^j y) \leqslant 2^{j\beta} L\phi(x, y)$ for all $x \in X$. Let $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\|\Gamma_{r,s}f(x, y)\|_Y \leqslant \phi(x, y) \tag{27}$$

for all $x, y \in X$. Then, there exists a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique sextic mapping $\mathcal{S} : X \longrightarrow Y$ such that

$$\|f(2x) - 64f(x) - \mathcal{Q}(x)\|_Y \leqslant \frac{1}{2^{2\beta}|1 - L^j|}\Lambda(x) \tag{28}$$

and

$$\|f(2x) - 4f(x) - \mathcal{S}(x)\|_Y \leqslant \frac{1}{2^{6\beta}|1 - L^j|}\Lambda(x) \tag{29}$$

for all $x \in X$, where

$$\Lambda(x) = \frac{1}{2r^{6\beta}}\left[K^3(2^\beta + 1)\phi(x, x) + K^2\left(\frac{65r^6}{2(r^6 - 1)}\right)^\beta \phi(0, 0)\right]$$
$$+ \frac{1}{2r^{6\beta}}K^2\left[2^\beta\phi(x, 2x) + \phi(x, 3x)\right]. \tag{30}$$

Proof. Putting $x = y = 0$ and $s = r$ in (27), we have $\|f(0)\|_Y \leq \frac{1}{|4(r^6-1)|^\beta}\phi(0, 0)$. Thus

$$\|130r^6 f(0)\|_Y \leq \left(\frac{65r^6}{2(r^6 - 1)}\right)^\beta \phi(0, 0) \tag{31}$$

for all $x \in X$. Replacing (x, y, s) by $(x, x, 2r)$ in (27), we get

$$\|4f(3rx) + 4f(rx) - 40r^6[f(2x) - 4f(x) + f(0)] - 8[f(2rx) + f(rx)]\|_Y \leq 2^\beta \phi(x, x) \tag{32}$$

for all $x \in X$. Letting $x = y$ and $s = 3r$ in (27), we find

$$\|2f(4rx) + 2f(2rx) - 90r^6[f(2x) - 4f(x) + f(0)] - 4[f(3rx) + f(rx)]\|_Y \leq \phi(x, x) \tag{33}$$

for all $x \in X$. It follows from (32) and (33) that

$$\|2f(4rx) + 2f(2rx) - 130r^6[f(2x) - 4f(x) + f(0)] - 8[f(2rx) + f(rx)]\|_Y \leq K(2^\beta + 1)\phi(x, x) \tag{34}$$

for all $x \in X$. The relations (31) and (34) necessitate that

$$\|2f(4rx) + 2f(2rx) - 130r^6[f(2x) - 4f(x)] - 8[f(2rx) + f(rx)]\|_Y \leq K^2(2^\beta + 1)\phi(x, x) + K\left(\frac{65r^6}{2(r^6 - 1)}\right)^\beta \phi(0, 0) \tag{35}$$

for all $x \in X$. Interchanging (x, y, s) into $(x, 2x, r)$ in (27), we arrive at

$$\|4f(3rx) + 4f(rx) - 4r^6[f(3x) - f(x) - 2f(2x)] - 8[f(2rx) + f(rx)]\|_Y \leq 2^\beta \phi(x, 2x) \tag{36}$$

for all $x \in X$. Putting $y = 3x$ and $s = r$ in (27), we obtain

$$\|2f(4rx) + 2f(2rx) - 2r^6[f(4x) + f(2x) - 2f(x) - 2f(3x)] - 4[f(3rx) + f(rx)]\|_Y \leq \phi(x, 3x) \tag{37}$$

for all $x \in X$. Plugging (36) into (37), we have

$$\|2f(4rx) + 2f(2rx) - 2r^6[f(4x) - 3f(2x) - 4f(x)] - 8[f(2rx) + f(rx)]\|_Y \leq K[2^\beta \phi(x, 2x) + \phi(x, 3x)] \tag{38}$$

for all $x \in X$. Now, the relations (35) and (38) imply that

$$\|2r^6[f(4x) - 68f(2x) + 256f(x)]\|_Y \leq K^3(2^\beta + 1)\phi(x, x) + K^2\left(\frac{65r^6}{2(r^6 - 1)}\right)^\beta \phi(0, 0) + K^2[2^\beta \phi(x, 2x) + \phi(x, 3x)] \tag{39}$$

for all $x \in X$. Hence, $\|g(2x) - 4g(x)\|_Y \leq \Lambda(x)$ for all $x \in X$, in which $g(x) = f(2x) - 64f(x)$ and $\Lambda(x)$ is defined in (30). It now follows from Lemma 3.1 that there exists a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ such that $\mathcal{Q}(2x) = 4\mathcal{Q}(x)$ and $\|f(x) - \mathcal{Q}(x)\|_Y \leqslant \frac{1}{2^{2\beta}|1 - L^j|}\Lambda(x)$ for all $x \in X$. Furthermore, from (39) we have $\|h(2x) - 64h(x)\|_Y \leq \Lambda(x)$ for all $x \in X$, where $h(x) = f(2x) - 4f(x)$. The rest of the proof can be repeated similarly. This finishes the proof. □

5. **Stability of (1)**

In this section, by using Theorems 3.1 and 4.1, we prove the generalized Hyers-Ulam-Rassias stability of the mixed type additive, quadratic and sextic functional equation (1) when f is an arbitrary mapping.

Theorem 5.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \longrightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with $\varphi(2^j x, 2^j y) \leqslant 2^{j\beta} L\varphi(x, y)$. Let $f : X \longrightarrow Y$ be a mapping satisfying $\|\Gamma_{r,s} f(x, y)\|_Y \leqslant \varphi(x, y)$ for all $x, y \in X$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$, a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique sextic mapping $\mathcal{S} : X \longrightarrow Y$ such that $\|f(x) - \mathcal{A}(x) - \mathcal{Q}(x) - \mathcal{S}(x)\|_Y \leqslant \Psi_\varphi(x)$ for all $x \in X$, where*

$$\Psi_\varphi(x) := \frac{K}{2^\beta |1 - L^j|} \widetilde{\Phi}(x) + K^2 \left[\frac{1}{2^{2\beta}} + \frac{1}{2^{6\beta}} \right] \frac{\Lambda_\Phi(x)}{|1 - L^j|} \tag{40}$$

for which

$$\widetilde{\Phi}(x) := \frac{1}{(2(r^6 - 1))^\beta} [K^3 2^\beta \widetilde{\varphi}(2x) + K^3 \Phi(x, x) + K^2 8^\beta \widetilde{\varphi}(x)] + \frac{K}{(4|r - 1|)^\beta} \Phi(0, 0), \tag{41}$$

$$\begin{aligned} \Lambda_\Phi(x) =& \frac{1}{2r^{6\beta}} \left[K^3 (2^\beta + 1)\Phi(x, x) + K^2 \left(\frac{65r^6}{2(r^6 - 1)} \right)^\beta \Phi(0, 0) \right] \\ &+ \frac{1}{2r^{6\beta}} K^2 \left[2^\beta \Phi(x, 2x) + \Phi(x, 3x) \right] \end{aligned} \tag{42}$$

whereas $\Phi(x, y) = \frac{1}{2}[\varphi(x, y) + \varphi(-x, -y)]$ *and* $\widetilde{\varphi}(x) := \frac{K}{4^\beta} \left[\left| \frac{r^6}{r-1} \right|^\beta \Phi(0, 0) + \Phi(0, x) \right]$.

Proof. We consider the mappings $f_o(x)$ and $f_e(x)$ introduced in Proposition 2.1. We have $\|\Gamma_{r,s} f_o(x, y)\|_Y \leqslant \Phi(x, y)$ and $\|\Gamma_{r,s} f_e(x, y)\|_Y \leqslant \Phi(x, y)$ for all $x, y \in X$. Also, $\Phi(2^j x, 2^j y) \leqslant 2^{j\beta} L\Phi(x, y)$ for all $x, y \in X$. It follows from Theorem 3.1 that there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ such that

$$\|f_o(x) - \mathcal{A}(x)\|_Y \leqslant \frac{1}{2^\beta |1 - L^j|} \widetilde{\Phi}(x) \tag{43}$$

for all $x \in X$ in which $\widetilde{\Phi}(x)$ is defined in (41). Once more, Theorem 4.1 implies that there exists a unique quadratic mapping $\mathcal{Q}_0 : X \longrightarrow Y$ and a unique sextic mapping $\mathcal{S}_0 : X \longrightarrow Y$ such that

$$\|f_e(2x) - 64f_e(x) - \mathcal{Q}_0(x)\|_Y \leqslant \frac{1}{2^{2\beta} |1 - L^j|} \Lambda_\Phi(x) \tag{44}$$

and

$$\|f_e(2x) - 4f_e(x) - \mathcal{S}_0(x)\|_Y \leqslant \frac{1}{2^{6\beta} |1 - L^j|} \Lambda_\Phi(x) \tag{45}$$

for all $x \in X$, where $\Lambda_\Phi(x)$ is introduced in (42). By the inequalities (44) and (45), we have

$$\|f_e(x) - \mathcal{Q}(x) - \mathcal{S}(x)\|_Y \leqslant K \left[\frac{1}{2^{2\beta}} + \frac{1}{2^{6\beta}} \right] \frac{\Lambda_\Phi(x)}{|1 - L^j|} \tag{46}$$

for all $x \in X$, where $\mathcal{Q}(x) = -\frac{1}{60} \mathcal{Q}_0(x)$ and $\mathcal{S}(x) = \frac{1}{60} \mathcal{S}_0(x)$. Plugging the relation (43) into (46), we obtain the desired result. □

In the oncoming corollaries which are direct consequences of Theorem 5.1, $\Gamma_{r,s} f(x, y)$ is bounded by the sum and product of the powers of norms. We present them without proofs.

Corollary 5.1. *Let θ, λ be positive numbers with $\lambda \neq \frac{\beta}{\alpha}, 2\frac{\beta}{\alpha}, 6\frac{\beta}{\alpha}$. If $f : X \longrightarrow Y$ is a mapping satisfying $\|\Gamma_{r,s} f(x, y)\|_Y \leqslant \theta(\|x\|_X^\lambda + \|y\|_X^\lambda)$ for all $x, y \in X$, then there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$, a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique sextic mapping $\mathcal{S} : X \longrightarrow Y$ such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{Q}(x) - \mathcal{S}(x)\|_Y$$

$$\leq \begin{cases} \left[\frac{\theta\Gamma_\lambda}{2^{\beta}-2^{\alpha\lambda}} + \frac{\theta\Lambda_\lambda}{2^{2\beta}-2^{\alpha\lambda}} + \frac{\theta\Lambda_\lambda}{2^{6\beta}-2^{\alpha\lambda}}\right] \|x\|_X^\lambda & 0 < \lambda < \frac{\beta}{\alpha} \\ \left[\frac{2^{\alpha\lambda}\Gamma_\lambda\theta}{2^{\beta}(2^{\alpha\lambda}-2^{\beta})} + \frac{\theta\Lambda_\lambda}{2^{2\beta}-2^{\alpha\lambda}} + \frac{\theta\Lambda_\lambda}{2^{6\beta}-2^{\alpha\lambda}}\right] \|x\|_X^\lambda & \frac{\beta}{\alpha} < \lambda < 2\frac{\beta}{\alpha} \\ \left[\frac{2^{\alpha\lambda}\Gamma_\lambda\theta}{2^{\beta}(2^{\alpha\lambda}-2^{\beta})} + \frac{2^{\alpha\lambda}\theta\Lambda_\lambda}{2^{2\beta}(2^{\alpha\lambda}-2^{2\beta})} + \frac{\theta\Lambda_\lambda}{2^{6\beta}-2^{\alpha\lambda}}\right] \|x\|_X^\lambda & 2\frac{\beta}{\alpha} < \lambda < 6\frac{\beta}{\alpha} \\ \left[\frac{2^{\alpha\lambda}\Gamma_\lambda\theta}{2^{\beta}(2^{\alpha\lambda}-2^{\beta})} + \frac{2^{\alpha\lambda}\theta\Lambda_\lambda}{2^{2\beta}(2^{\alpha\lambda}-2^{2\beta})} + \frac{2^{\alpha\lambda}\theta\Lambda_\lambda}{2^{6\beta}(2^{\alpha\lambda}-2^{6\beta})}\right] \|x\|_X^\lambda & \lambda > 6\frac{\beta}{\alpha} \end{cases}$$

for all $x \in X$*, where* $\Gamma_\lambda = \frac{1}{(4(r^6-1))^\beta}[K^3 2^{\alpha\lambda} + K^3 2^{\beta+1} + K^2 4^\beta]$ *and*

$$\Lambda_\lambda = \frac{1}{2r^{6\beta}}[2K^3(2^\beta+1) + K^2[2^\beta(1+2^{\alpha\lambda}) + 1 + 3^{\alpha\lambda}]].$$

Corollary 5.2. *Let* θ*,* m *and* n *be positive numbers with* $\lambda = m + n \neq \frac{\beta}{\alpha}, 2\frac{\beta}{\alpha}, 6\frac{\beta}{\alpha}$*. If* $f : X \longrightarrow Y$ *is a mapping satisfying* $\|\Gamma_{r,s}f(x,y)\|_Y \leqslant \theta\|x\|_X^m\|y\|_X^n$ *for all* $x, y \in X$*, then there exists a unique additive mapping* $\mathcal{A} : X \longrightarrow Y$*, a unique quadratic mapping* $\mathcal{Q} : X \longrightarrow Y$ *and a unique sextic mapping* $\mathcal{S} : X \longrightarrow Y$ *such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{Q}(x) - \mathcal{S}(x)\|_Y$$

$$\leq \begin{cases} \left[\frac{2K^3\theta}{(2(r^6-1))^\beta(2^\beta-2^{\alpha\lambda})} + \frac{\theta\Lambda_{\lambda,n}}{2^{2\beta}-2^{\alpha\lambda}} + \frac{\theta(\Lambda_{\lambda,n})}{2^{6\beta}-2^{\alpha\lambda}}\right] \|x\|_X^\lambda & 0 < \lambda < \frac{\beta}{\alpha} \\ \left[\frac{2K^3 2^{\alpha\lambda}\theta}{2^\beta(2(r^6-1))^\beta(2^{\alpha\lambda}-2^\beta)} + \frac{\theta\Lambda_{\lambda,n}}{2^{2\beta}-2^{\alpha\lambda}} + \frac{\theta(\Lambda_{\lambda,n})}{2^{6\beta}-2^{\alpha\lambda}}\right] \|x\|_X^\lambda & \frac{\beta}{\alpha} < \lambda < 2\frac{\beta}{\alpha} \\ \left[\frac{2K^3 2^{\alpha\lambda}\theta}{2^\beta(2(r^6-1))^\beta(2^{\alpha\lambda}-2^\beta)} + \frac{2^{\alpha\lambda}\theta\Lambda_{\lambda,n}}{2^{2\beta}(2^{\alpha\lambda}-2^{2\beta})} + \frac{\theta\Lambda_{\lambda,n}}{2^{6\beta}-2^{\alpha\lambda}}\right] \|x\|_X^\lambda & 2\frac{\beta}{\alpha} < \lambda < 6\frac{\beta}{\alpha} \\ \left[\frac{2K^3 2^{\alpha\lambda}\theta}{2^\beta(2(r^6-1))^\beta(2^{\alpha\lambda}-2^\beta)} + \frac{2^{\alpha\lambda}\theta\Lambda_{\lambda,n}}{2^{2\beta}(2^{\alpha\lambda}-2^{2\beta})} + \frac{2^{\alpha\lambda}\theta\Lambda_{\lambda,n}}{2^{6\beta}(2^{\alpha\lambda}-2^{2\beta})}\right] \|x\|_X^\lambda & \lambda > 6\frac{\beta}{\alpha} \end{cases}$$

for all $x \in X$*, where* $\Lambda_{\lambda,n} = \frac{1}{2r^{6\beta}}[2K^3(2^\beta+1) + K^2(2^{\beta+\alpha n} + 3^{\alpha n})]$.

6. **Conclusions**

In this paper, the authors introduced a class of the mixed type additive-quadratic-sextic functional equations and investigated their stability in the quasi-β-normed spaces.

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REFERENCES

[1] *T. Aoki*, On the stability of the linear transformation in Banach spaces, J. Math . Soc. Japan. **2** (1950), 64–66.

[2] *A. Bodaghi*, Approximate mixed type additive and quartic functional equation, Bol. Soc. Paran. Mat. **35** (1) (2017), 43–56.

[3] *A. Bodaghi*, Stability of a mixed type additive and quartic function equation, Filomat. **28** (8) (2014), 1629–1640.

[4] *A. Bodaghi, D. Kang and J. M. Rassias*, The mixed cubic-quartic functional equation, An. Stiint. Univ. Al. I. Cuza Iasi Mat. (N.S.), **63** (1) (2017), 215-227.

[5] *A. Bodaghi and S. O. Kim*, Ulam's type stability of a functional equation deriving from quadratic and additive functions, J. Math. Ineq. **9**, No. 1 (2015), 73–84.

[6] *A. Bodaghi and S. O. Kim*, Stability of a functional equation deriving from quadratic and additive functions in non-Archimedean normed spaces, Abstr. Appl. Anal. **2013**, Art. ID 198018, 10 pages, doi:10.1155/2013/198018.

[7] *A. Bodaghi and P. Narasimman*, Stability of the general form of quadratic-quartic functional equations in non-Archimedean $\mathcal{L}$-fuzzy normed spaces, Tbilisi Math. J., **11** (1) (2018), 15–29.

[8] *D. G. Bourgin*, Classes of transformations and bordering transformations, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[9] *M. Eshaghi Gordji, M. B. Savadkouhi, J. M. Rassias, S. Zolfaghari*, Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces, Abstr. Appl. Anal. **2009**, Art. ID 417473, 14 pages.

[10] *M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias and S. Zolfaghari*, Solution and stability of a mixed type additive, quadratic, and cubic functional equation, Abstr. Appl. Anal. **2009**, Art. ID 826130, doi:10.1155/2009/826130.

[11] *Z. Gajda*, On stability of additive mappings, Int. J. Math. Math. Sci. **14**, no. 3 (1991), 431–434.

[12] *P. Găvruţa*, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184** (1994), 431–436.

[13] *D. H. Hyers*, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. **27** (1941), 222–224.

[14] *D. Kang and H. Koh*, A fixed pint approach to the stability of sextic Lie $*$-derivations, Filomat. **31** (15) (2017), 4933–4944.

[15] *M. Maghsoudi, A. Bodaghi, A. Niazi Motlagh and M. Karami*, Almost additive-quadratic-cubic mappings on modular spaces, Revista De La UMA, In Press.

[16] *P. Narasimman and A. Bodaghi*, Solution and stability of a mixed type functional equation, Filomat. **31** (5) (2017), 1229–1239.

[17] *J. M. Rassias*, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. **46** (1982), 126–130.

[18] *Th. M. Rassias*, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[19] *K. Ravi, J. M. Rassias, M. Arunkumar and R. Kodandan*, Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation, J. Ineq. Pure & Appl. Math. **10** (4)(2009), 1–29.

[20] *S. M. Ulam*, Problems in Modern Mathematics, Chapter VI, Wiley-Interscience, New York, 1964.

[21] *T. Z. Xu, M. J. Rassias and W. X. Xu*, A generalized mixed quadratic-quartic functional equation, Bull. Malays. Math. Sci. Soc. **35** (3) (2012), 633-649.

[22] *T. Z. Xu, J. M. Rassias, M. J. Rassias and W. X. Xu*, A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces, J. Inequal. Appl. **2010**, Article ID 423231, 23 pages, doi:10.1155/2010/423231.

[23] *T. Z. Xu, M. J. Rassias and W. X. Xu*, Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces, J. Math. Physics. Art. 093508, 19 pages, **51** (2010).