A NEW VARIATIONAL INEQUALITY PROBLEMS FOR TWO INVERSE-STRONGLY MONOTONE OPERATORS IN 2-UNIFORMLY SMOOTH AND UNIFORMLY CONVEX BANACH SPACES

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The aim of this paper is to study a viscosity algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions to a new variational inequality problem of two inverse-strongly monotone operators in 2-uniformly smooth and uniformly convex Banach spaces. Under some suitable assumptions imposed on the parameters, we obtain strong convergence theorems. The results obtained in this paper may be an improvement of many recent ones in the literature.

Keywords: Variational inequality, Nonexpansive mapping, Inverse-strongly monotone operators, Fixed point, Banach spaces.

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1. Introduction

Variational inequality theory plays an important role for solving many problems arising in several branches of pure and applied sciences, such as mathematical programming, equilibrium problems and signal recovery problems. See [1-8,21-29] for more details and the references contained therein.

In this paper, we study a generalized viscosity algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions to a new variational inequality problem for two inverse-strongly monotone operators in 2-uniformly smooth and uniformly convex Banach spaces. Strong convergence result of the sequence generated by our algorithm is given under appropriate conditions imposed on the parameters.

2. Definitions and preliminaries

Throughout this paper, let E be a real Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a self-mapping. We always denote by $F(T)$ the set of fixed points of T, that is $F(T) := \{x \in C : x = Tx\}$. Let $J : E \to 2^{E^*}$ be the duality mapping defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2,\ \|x^*\| = \|x\| \right\},\ \forall x \in E.$$

If E is a real Hilbert space, it is easy to see that $J = I$, where I is the identity mapping on E. In addition, when E is smooth, we know from [21] that J is single-valued, which we shall denoted by j. Next we state some basic concepts and facts appeared in this paper.

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A mapping $f : C \to C$ is said to be a strict contraction, if there exists a constant $\delta \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \delta \|x - y\|, \ \forall \ x, y \in C. \tag{1}$$

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall \ x, y \in C. \tag{2}$$

A mapping $A : C \to E$ is said to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0, \forall \ x, y \in C. \tag{3}$$

A mapping $A : C \to E$ is said to be α-inverse strongly accretive if there exists $j(x-y) \in J(x-y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|Ax - Ay\|^2, \forall \ x, y \in C. \tag{4}$$

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) := \sup \left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \ x \in S(E), \ \|y\| \leq t \right\}.$$

A Banach space E is called to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Furthermore, Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well known that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

A Banach space E is called to be strictly convex, if x and y are not colinear, then: $\|x+y\| < \|x\| + \|y\|$. Let $\delta_E(\epsilon)$ be the modulus of convexity of E defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \frac{1}{2}\|x+y\| : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon \right\},$$

for all $\epsilon \in [0,2]$. A Banach space E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that L^p is uniformly smooth and uniformly convex Banach space, where $p > 1$. Precisely, L^p is $\min\{p,2\}$-uniformly smooth and $\max\{p,2\}$-uniformly convex for every $p > 1$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \to H$ be a nonlinear mapping. The classical variational inequality is to find an $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \ \forall \, x \in C. \tag{5}$$

We use $VI(A,C)$ to denote the set of solutions to (5).

Moreover, Ceng *et al.* [4] introduced the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall \, x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall \, x \in C, \end{cases} \tag{6}$$

which is called a general system of variational inequalities, where $A, B : C \to H$ are two nonlinear mappings, $\lambda > 0$ and $\mu > 0$ are two fixed constants. We can see easily that problem (6) contains the classical variational inequality (5) as a special case. At the same time, they introduced a modified Halpern iterative algorithm for finding a common element in the set of solutions to problem (6) and the set of fixed points of a nonexpansive mapping. Strong convergence theorems were obtained under some suitable conditions on the parameters.

On the other hand, let C be a nonempty, closed and convex subset of a real Banach space E and $A, B : C \to E$ be two operators. In Banach spaces, Yao *et al.* [7] studied the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \forall\, x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \forall\, x \in C, \end{cases} \tag{7}$$

where $A, B : C \to E$ be two nonlinear operators. Precisely, Yao *et al.* [7] studied the following iterative algorithm:

$$\begin{cases} u, x_0 \in C, \\ y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), n \geq 0, \end{cases} \tag{8}$$

and obtained strong convergence results under some suitable conditions on the parameters.

In this paper, we introduce the new problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda A)(ax^* + (1-a)y^*), j(x - x^*)\rangle \geq 0, \ \forall\, x \in C, \\ \langle y^* - (I - \mu B)x^*, j(x - y^*)\rangle \geq 0, \ \forall\, x \in C, \end{cases} \tag{9}$$

which is called the system of more general variational inequalities in a real Banach space. If $\lambda = \mu = 1$ and $a = 0$, the problem (9) becomes problem (8). If $a = 0$, then (9) becomes

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \forall\, x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \forall\, x \in C. \end{cases} \tag{10}$$

Therefore (9) contains (8) or (10) as a special case.

Let C and D be nonempty subsets of a Banach space E such that C is nonempty, closed and convex and $D \subset C$. A mapping $P : C \to D$ is called to be sunny (see [9, 10]) if $P(x + t(x - P(x))) = P(x)$, $\forall\ x \in C$ and $t \geq 0$, whenever $x + t(x - P(x)) \in C$. A mapping $P : C \to D$ is called a retraction if $Px = x, \forall\ x \in D$. Moreover, P is said to be a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D, which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction P from C onto D (see [11] for more details).

Proposition 2.1 ([9])**.** *Let C be a closed and convex subset of a smooth Banach space E. Let D be a nonempty subset of C. Let $P : C \to D$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(a) P is sunny and nonexpansive;
(b) $\|Px - Py\|^2 \leq \langle x - y, J(Px - Py)\rangle$, $\forall\, x, y \in C$;
(c) $\langle x - Px, J(y - Px)\rangle \leq 0$, $\forall\, x \in C, y \in D$.

Proposition 2.2 (Theorem 4.1,[12])**.** *Let D be a closed and convex subset of a reflexive Banach space E with a uniformly Gâteaux differentiable norm. If C is a nonexpansive retract of D, then it is a sunny nonexpansive retract of D.*

For proving our main results, we need the following lemmas.

Lemma 2.1 ([13])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \delta_n, \ n \geq 0,$$

where

(i) $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([14]). *Let E be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \le \|x\|^2 - 2\langle x, jy\rangle + \|y\|^2$, for all $x, y \in B_r$,where $B_r = \{z \in E : \|z\| \le r\}$.*

Lemma 2.3 ([15],Lemma 2.1). *Let C be a closed convex subset of a strictly convex Banach space X. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1 - \lambda)T_2 x, \forall\, x \in C,$$

where λ is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.4 ([16]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be a α-inverse-strongly accretive. Then the following inequality holds:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \le \|x - y\|^2 - 2\lambda(\alpha - K^2\lambda)\|Ax - Ay\|^2 .$$

In particular, if $0 < \lambda \le \frac{\alpha}{K^2}$, then $I - \lambda A$ is nonexpansive, where K is the 2-uniformly smoothness constant of E (i.e., K is a positive constant (see [20]*) satisfying:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2 , \; x, y \in E.$$

Lemma 2.5 ([17]). *Let C be a nonempty, bounded and closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

Lemma 2.6 ([18]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$, $n \ge 0$ and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.7 ([13]). *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \Pi_C$. Then the sequence $\{x_t\}$ define by*

$$x_t = tf(x_t) + (1 - t)Tx_t$$

converges strongly to a point in $F(T)$. If we define a mapping $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Pi_C.$$

Then $Q(f)$ solves the following variational inequality:

$$\langle (I - f)Q(f), j(Q(f) - p)\rangle \le 0, \quad \forall f \in \Pi_C,\, p \in F(T).$$

Lemma 2.8 ([19]). *Let C be a nonempty closed convex subset of a real Banach space E which has uniformly Gâteaux differentiable norm, and $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. Assume that $\{z_t\}$ strongly converges to a fixed point z of T as $t \to 0$, where $\{z_t\}$ is defined by Lemma 2.9. Suppose $\{x_n\} \subset C$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then*

$$\limsup_{n\to\infty} \langle f(z) - z, J(x_{n+1} - z)\rangle \le 0.$$

Similar to the proof of Lemma 2.9 and Lemma 2.10 of [16], we can obtain the following lemmas.

Lemma 2.9. *Let C be a nonempty, closed and convex subset of a real 2-uniformly smooth Banach space E. Assume that C is a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E onto C. Let the mapping $A : C \to E$ be α-inverse-strongly accretive and let $B : C \to E$ be β-inverse-strongly accretive. Let $G : C \to C$ be a mapping defined by*

$$G(x) = P_C(I - \lambda A)[aI + (1 - a)P_C(I - \mu B)]x, \ \forall\, x \in C.$$

If $0 < \lambda \leq \frac{\alpha}{K^2}$ and $0 < \mu \leq \frac{\beta}{K^2}$, then $G : C \to C$ is nonexpansive, where K is the 2-uniformly smoothness constant of E.

Lemma 2.10. *Let C be a nonempty, closed and convex subset of a real 2-uniformly smooth Banach space E. Assume that C is a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (9) if and only if $x^* = P_C(I - \lambda A)(ax^* + (1 - a)y^*)$, where $y^* = P_C(x^* - \mu Bx^*)$, that is $x^* = Gx^*$, where G is defined by Lemma 2.9.*

3. Main results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth and uniformly convex Banach space E. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.9. Let $f : C \to C$ be a strict contraction with coefficient $\delta \in [0, 1)$. Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \mu Bx_n), \\ y_n = P_C(I - \lambda A)(ax_n + (1 - a)z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Ty_n, \end{cases} \tag{11}$$

where $0 \leq a < 1$, $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$, where K is the 2-uniformly smooth constant appeared in [20]. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - q, j(p - q)\rangle \leq 0, \ \forall\, p \in F(T) \cap F(G).$$

Proof. Firstly, we can show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ by using similar methods used in [7]. We omit the details.

Next, we show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$. By Lemma 2.10, we have

$$\begin{aligned} \|z_n - y^*\|^2 &= \|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\leq \|x_n - x^* - \mu(Bx_n - Bx^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\mu(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 . \end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|P_C(I-\lambda A)(ax_n + (1-a)z_n) - P_C(I-\lambda A)(ax^* + (1-a)y^*)\|^2 \\
&\le \|(ax_n + (1-a)z_n) - (ax^* + (1-a)y^*) \\
&\quad - \lambda(A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*))\|^2 \\
&\le \|a(x_n - x^*) + (1-a)(z_n - y^*)\|^2 \\
&\quad - 2\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*)\|^2 \\
&\le a\|x_n - x^*\|^2 + (1-a)\|z_n - y^*\|^2 \\
&\quad - 2\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*)\|^2. \qquad (13)
\end{aligned}$$

Substituting (12) into (13), we get

$$\begin{aligned}
\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 &- 2\mu(1-a)(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 \\
&- 2\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*\|^2. \qquad (14)
\end{aligned}$$

It follows from (11) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(Ty_n - x^*)\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|Ty_n - x^*\|^2 \\
&\le \alpha_n M_1 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2, \qquad (15)
\end{aligned}$$

where

$$M_1 = \sup_{n\ge 1}\{\|f(x_n) - x^*\|^2\}.$$

Combining (14) and (15), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n M_1 + \beta_n\|x_n - x^*\|^2 + \gamma_n[\|x_n - x^*\|^2 - 2\mu(1-a)(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 \\
&\quad - 2\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*\|^2] \\
&= \alpha_n M_1 + (1-\alpha_n)\|x_n - x^*\|^2 - 2\gamma_n\mu(1-a)(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 \\
&\quad - 2\gamma_n\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*\|^2 \\
&\le \alpha_n M_1 + \|x_n - x^*\|^2 - 2\gamma_n\mu(1-a)(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 \\
&\quad - 2\gamma_n\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*\|^2
\end{aligned}$$

which implies

$$\begin{aligned}
&2\gamma_n\mu(1-a)(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 \\
&\quad + 2\gamma_n\lambda(\alpha - K^2\lambda)\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1 \\
&\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + \alpha_n M_1. \qquad (16)
\end{aligned}$$

Since $0 < \lambda < \frac{\alpha}{K^2}$, $0 < \mu < \frac{\beta}{K^2}$, $\lim_{n\to\infty}\alpha_n = 0$, condition (iii), it follows that

$$\lim_{n\to\infty}\|Bx_n - Bx^*\| = 0, \lim_{n\to\infty}\|A(ax_n + (1-a)z_n) - A(ax^* + (1-a)y^*\| = 0. \qquad (17)$$

Let $r_1 = \sup_{n\geq 0}\{\|z_n - y^*\|, \|x_n - x^*\|\}$. From Proposition 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\|z_n - y^*\|^2 \\
&= \|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\|^2 \\
&\leq \langle x_n - \mu Bx_n - (x^* - \mu Bx^*), j(z_n - y^*)\rangle \\
&= \langle x_n - x^*, j(z_n - y^*)\rangle + \mu\langle Bx^* - Bx_n, j(z_n - y^*)\rangle \\
&\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) \\
&\quad + \mu\langle Bx^* - Bx_n, j(z_n - y^*)\rangle,
\end{aligned}$$

where $g_1 : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function such that $g_1(0) = 0$. Consequently, we have

$$\begin{aligned}
&\|z_n - y^*\|^2 \\
&\leq \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu\langle Bx^* - Bx_n, j(z_n - y^*)\rangle \\
&\leq \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\|. \qquad (18)
\end{aligned}$$

Let

$$r_2 = \sup_{n\geq 0}\{\|x_n - x^*\|, \|y_n - x^*\|\}, r_3 = \sup_{n\geq 0}\{\|z_n - y^*\|, \|y_n - x^*\|\}.$$

Again by Proposition 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|P_C(I - \lambda A)(ax_n + (1-a)z_n) - P_C(I - \lambda A)(ax^* + (1-a)y^*)\|^2 \\
&\leq \big\langle a(x_n - x^*) + (1-a)(z_n - y^*) + \lambda A(ax^* + (1-a)y^*) \\
&\quad - \lambda A(ax_n + (1-a)z_n), j(y_n - x^*)\big\rangle \\
&= a\langle x_n - x^*, j(y_n - x^*)\rangle + (1-a)\langle z_n - y^*, j(y_n - x^*)\rangle \\
&\quad + \lambda\langle A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n), j(y_n - x^*)\rangle \\
&\leq \frac{a}{2}(\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - g_2(\|x_n - y_n\|) \\
&\quad + \frac{1-a}{2}(\|z_n - y^*\|^2 + \|y_n - x^*\|^2 - g_3(\|z_n - y_n + (x^* - y^*)\|) \\
&\quad + \lambda\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\|,
\end{aligned}$$

where $g_2, g_3 : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function such that $g_2(0) = 0$ and $g_3(0) = 0$. It follows that

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&\leq a\|x_n - x^*\|^2 + (1-a)\|z_n - y^*\|^2 - ag_2(\|x_n - y_n\|) \\
&\quad - (1-a)g_3(\|z_n - y_n + (x^* - y^*)\|) \\
&\quad + 2\lambda\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| \\
&\leq a\|x_n - x^*\|^2 + (1-a)\|z_n - y^*\|^2 - (1-a)g_3(\|z_n - y_n + (x^* - y^*)\|) \\
&\quad + 2\lambda\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\|. \qquad (19)
\end{aligned}$$

Substituting (18) into (19), we obtain

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&\le a\|x_n - x^*\|^2 + (1-a)[\|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) \\
&\quad + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\|] - (1-a)g_3(\|z_n - y_n + (x^* - y^*)\|) \\
&\quad + 2\lambda\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| \\
&= \|x_n - x^*\|^2 - (1-a)g_1(\|x_n - z_n - (x^* - y^*)\|) \\
&\quad - (1-a)g_3(\|z_n - y_n + (x^* - y^*)\|) + 2\mu(1-a)\|Bx_n - Bx^*\|\|z_n - y^*\| \\
&\quad + 2\lambda\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\|. \qquad (20)
\end{aligned}$$

Substituting (20) into (15), we get

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 - (1-a)\gamma_n g_1(\|x_n - z_n - (x^* - y^*)\|) \\
&\quad - (1-a)\gamma_n g_3(\|z_n - y_n + (x^* - y^*)\|) \\
&\quad + 2\mu(1-a)\gamma_n\|Bx_n - Bx^*\|\|z_n - y^*\| \\
&\quad + 2\lambda\gamma_n\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| + \alpha_n M_1 \\
&= (1-\alpha_n)\|x_n - x^*\|^2 - (1-a)\gamma_n g_1(\|x_n - z_n - (x^* - y^*)\|) \\
&\quad - (1-a)\gamma_n g_3(\|z_n - y_n + (x^* - y^*)\|) + 2\mu(1-a)\gamma_n\|Bx_n - Bx^*\|\|z_n - y^*\| \\
&\quad + 2\lambda\gamma_n\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| + \alpha_n M_1 \\
&\le \|x_n - x^*\|^2 - (1-a)\gamma_n g_1(\|x_n - z_n - (x^* - y^*)\|) \\
&\quad - (1-a)\gamma_n g_3(\|z_n - y_n + (x^* - y^*)\|) + 2\mu(1-a)\gamma_n\|Bx_n - Bx^*\|\|z_n - y^*\| \\
&\quad + 2\lambda\gamma_n\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| + \alpha_n M_1,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-a)\gamma_n g_1(\|x_n - z_n - (x^* - y^*)\|) - (1-a)\gamma_n g_3(\|z_n - y_n + (x^* - y^*)\|) \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu(1-a)\gamma_n\|Bx_n - Bx^*\|\|z_n - y^*\| \\
&\quad + 2\lambda\gamma_n\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| + \alpha_n M_1 \\
&\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad + 2\mu(1-a)\gamma_n\|Bx_n - Bx^*\|\|z_n - y^*\| \\
&\quad + 2\lambda\gamma_n\|A(ax^* + (1-a)y^*) - A(ax_n + (1-a)z_n)\|\|y_n - x^*\| + \alpha_n M_1. \qquad (21)
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $0 \le a < 1$, condition (iii) and (17), we get

$$\lim_{n\to\infty} g_1(\|x_n - z_n - (x^* - y^*)\|) = 0, \lim_{n\to\infty} g_3(\|z_n - y_n + (x^* - y^*)\|) = 0.$$

By virtue of the properties of g_1 and g_3, we obtain

$$\lim_{n\to\infty}\|x_n - z_n - (x^* - y^*)\| = 0, \lim_{n\to\infty}\|z_n - y_n + (x^* - y^*)\| = 0. \qquad (22)$$

This implies that

$$\begin{aligned}
\|x_n - y_n\| &\le \|x_n - z_n - (x^* - y^*)\| + \|z_n - y_n + (x^* - y^*)\| \\
&\to 0 \text{ as } n \to \infty. \qquad (23)
\end{aligned}$$

On the other hand, we observe

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n(f(x_n)-x_n)+\gamma_n(Ty_n-x_n)\|\\ &\geq \gamma_n\|Ty_n-x_n\|-\alpha_n\|f(x_n)-x_n\|,\end{aligned}$$

which implies

$$\|Ty_n-x_n\| \leq \frac{1}{\gamma_n}[\alpha_n\|f(x_n)-x_n\|+\|x_{n+1}-x_n\|]. \tag{24}$$

Since $\lim_{n\to\infty}\alpha_n=0$, condition (iii) and (16), we obtain

$$\lim_{n\to\infty}\|Ty_n-x_n\|=0. \tag{25}$$

From (23) and (25), we have

$$\begin{aligned}\|x_n-Tx_n\| &\leq \|x_n-Ty_n\|+\|Ty_n-Tx_n\|\\ &\leq \|x_n-Ty_n\|+\|y_n-x_n\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{26}$$

Define a mapping $U: C\to C$ as $Ux=\rho Tx+(1-\rho)Gx$, where G is defined by Lemma 2.9 and $\rho\in(0,1)$ is a constant. It follows from Lemma 2.3 that U is nonexpansive and $F(U)=F(T)\cap F(G)$. We define $x_t=tf(x_t)+(1-t)Ux_t$, it follows from Lemma 2.7 that $\{x_t\}$ converges strongly to $q\in F(U)=F(T)\cap F(G)$. From (23) and (26), we have

$$\begin{aligned}\|x_n-Ux_n\| &= \|\rho(x_n-Tx_n)+(1-\rho)(x_n-Gx_n)\|\\ &= \|\rho(x_n-Tx_n)+(1-\rho)(x_n-y_n)\|\\ &\leq \rho\|x_n-Tx_n\|+(1-\rho)\|x_n-y_n\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{27}$$

By Lemma 2.8, we have

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_{n+1}-q)\rangle \leq 0. \tag{28}$$

Finally, we prove that $x_n\to q$ as $n\to\infty$. Indeed, we have

$$\begin{aligned}&\|x_{n+1}-q\|^2\\ &= \langle\alpha_n(f(x_n)-q)+\beta_n(x_n-q)+\gamma_n(Ty_n-q), j(x_{n+1}-q)\rangle\\ &= \alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle+\beta_n\langle x_n-q, j(x_{n+1}-q)\rangle\\ &\quad+\gamma_n\langle Ty_n-q, j(x_{n+1}-q)\rangle+\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\ &\leq \alpha_n\delta\|x_n-q\|\|x_{n+1}-q\|+\beta_n\|x_n-q\|\|x_{n+1}-q\|+\gamma_n\|y_n-q\|\|x_{n+1}-q\|\\ &\quad+\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\ &\leq \alpha_n\delta\|x_n-q\|\|x_{n+1}-q\|+\beta_n\|x_n-q\|\|x_{n+1}-q\|+\gamma_n\|x_n-q\|\|x_{n+1}-q\|\\ &\quad+\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\ &= [1-\alpha_n(1-\delta)]\|x_n-q\|\|x_{n+1}-q\|+\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\ &\leq \frac{1-\alpha_n(1-\delta)}{2}[\|x_n-q\|^2+\|x_{n+1}-q\|^2]+\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\ &\leq \frac{1-\alpha_n(1-\delta)}{2}\|x_n-q\|^2+\frac{1}{2}\|x_{n+1}-q\|^2+\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle,\end{aligned}$$

which implies

$$\|x_{n+1}-q\|^2 \leq [1-\alpha_n(1-\delta)]\|x_n-q\|^2+\alpha_n(1-\delta)\frac{2\langle f(q)-q, j(x_{n+1}-q)\rangle}{1-\delta}. \tag{29}$$

Apply Lemma 2.1 to (29), we have $x_n \to q$ as $n \to \infty$. This completes the proof. □

The following results can be easily deduced from Theorem 3.1. We omit the details.

Corollary 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth and uniformly convex Banach space E. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.9 when $a = 0$. Let $f : C \to C$ be a strict contraction with coefficient $\delta \in [0, 1)$. Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = P_C(z_n - \lambda A z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T y_n, \end{cases} \tag{30}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$, where K is the 2-uniformly smooth constant appeared in [20]. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - q, j(p - q)\rangle \leq 0, \ \forall p \in F(T) \cap F(G).$$

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth and uniformly convex Banach space E. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.9 when $a = \frac{1}{2}$. Let $f : C \to C$ be a strict contraction with coefficient $\delta \in [0, 1)$. Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ u_n = \frac{1}{2}(x_n + z_n), \\ y_n = P_C(u_n - \lambda A u_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T y_n, \end{cases} \tag{31}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$, where K is the 2-uniformly smooth constant appeared in [20]. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - q, j(p - q)\rangle \leq 0, \ \forall p \in F(T) \cap F(G).$$

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where*

$G : C \to C$ is a mapping defined by Lemma 2.9. Let $f : C \to C$ be a strict contraction with coefficient $\delta \in [0,1)$. Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = P_C(I - \lambda A)(a x_n + (1-a) z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T y_n, \end{cases} \tag{32}$$

where $0 \le a < 1,\ 0 < \lambda < 2\alpha$ and $0 < \mu < 2\beta$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf \beta_n \le \limsup \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - q, p - q \rangle \le 0,\ \forall p \in F(T) \cap F(G).$$

4. **Applications**

Now we give an application to variational inequality problem for strict pseudocontractive mappings.

A mapping $T : C \to C$ is said to be λ-strict pseudocontractive if there exists a fixed constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \le \|x - y\|^2 - \lambda \|(I-T)x - (I-T)y\|^2, \tag{33}$$

for some $j(x-y) \in J(x-y)$ and for every $x, y \in C$. It is easy to see that (33) is equivalent to the following inequality:

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \ge \lambda \|(I-T)x - (I-T)y\|^2 \tag{34}$$

for some $j(x-y) \in J(x-y)$ and for every $x, y \in C$. Therefore $I - T$ is λ-inverse-strongly accretive.

We can obtain the following results easily by Theorem 3.1.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth and uniformly convex Banach space E. Let the mappings $A, B : C \to C$ be α-strict pseudocontractive and β-strict pseudocontractive, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.9. Let $f : C \to C$ be a strict contraction with coefficient $\delta \in [0,1)$. Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\mu) x_n + \mu B x_n, \\ u_n = a x_n + (1-a) z_n, \\ y_n = (1-\lambda) u_n + \lambda A u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T y_n, \end{cases} \tag{35}$$

where $0 \le a < 1,\ 0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$, where K is the 2-uniformly smooth constant appeared in [20]. *Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf \beta_n \le \limsup \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - q, j(p - q) \rangle \le 0,\ \forall p \in F(T) \cap F(G).$$

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth and uniformly convex Banach space E. Let the mappings $A, B : C \to C$ be α-strict pseudocontractive and β-strict pseudocontractive, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.9 when $a = 0$. Let $f : C \to C$ be a strict contraction with coefficient $\delta \in [0, 1)$. Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\mu)x_n + \mu Bx_n, \\ y_n = (1-\lambda)z_n + \lambda Az_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Ty_n, \end{cases} \tag{36}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$, where K is the 2-uniformly smooth constant appeared in [20]. *Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - q, j(p - q)\rangle \leq 0, \ \forall\, p \in F(T) \cap F(G).$$

5. Conclusions

Variational inequality theory has many applications in pure and applied sciences. There are some numerical methods for solving variational inequality problems and related optimization problems in recent years. By using a modified extragradient method, we study a generalized viscosity algorithm for finding a common element for the set of fixed points of one nonexpansive mapping and the set of solutions of new variational inequality problems for two inverse-strongly monotone operators in 2-uniformly smooth and uniformly convex Banach spaces. Strong convergence theorems are obtained under some suitable conditions on the parameters.

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