DYNAMICAL SYSTEM TECHNIQUE FOR SOLVING QUASI VARIATIONAL INEQUALITIES

Muhammad Aslam Noor[1], Khalida Inayat Noor[2]

In this paper, we introduce and consider second order dynamical system associated with quasi variational inequalities. Using the forward finite difference schemes, we suggest some iterative methods for solving the quasi variational inequalities. These new methods can be viewed as refinement of the extragradient methods of Korpelevich and Noor. Convergence analysis is investigated under certain mild conditions. Since the quasi variational inequalities include variational inequalities and complementarity problems as special cases, our results continue to hold for these problems. It is an interesting problem to compare these methods with other technique for solving quasi variational inequalities for further research activities.

Keywords: Exponentially convex functions, optimality conditions, variational inequalities

MSC2020: Primary 49J40; Secondary 90C33.

1. Introduction

Variational inequality theory, which was introduced by Stampacchia [45] in potential theory, provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. It is worth mentioning that the variational inequalities can be viewed as a significant and novel generalization of the variational principles. It is very simple fact that the minimum of a differentiable convex functions on the convex sets can be characterized by an inequality, which is called the variational inequality. It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence the recent research, see [5, 6, 7, 8, 13, 17, 18, 19, 23, 24, 25, 26, 27, 28, 29, 30, 33, 34, 35, 36, 37, 39, 40, 41, 42, 43, 45, 46]. If the convex set in the variational inequalities depends upon the solution explicitly or implicity, then the variational inequality is called quasi variational inequality. Benssousan and Lions [5] studied the quasi variational inequalities in the impulse control theory. Quasi-variational inequalities are being used as a mathematical programming tool in modeling various equilibria in economics, operations research, optimization, and regional and transportation science, see [5, 7, 14, 15, 16, 18, 27, 28, 32, 33, 36, 44].

One of the most difficult and important problems in variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inequalities and their variant forms. These methods have been extended and modified in numerous ways. Noor [25] proved that the quasi variational inequalities are equivalent to the fixed point problem. This alternative formulation has been

[1] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan, Corresponding author, E-mail: **noormaslam@gmail.com**

[2] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.

used to consider the existence of a solution, iterative schemes, sensitivity analysis, merit functions and other aspects of the quasi variational inequalities. Noor [12] used this equivalent form to suggest an iterative projection method for solving a class of quasi variational inequalities. Antipin et al. [3] proposed gradient projection and extra gradient methods for finding the solution of quasi variational inequality when the involved operator is strongly monotone and Lipschitz continuous. Mijajlovic et al. [22] introduced a more general gradient projection method with strong convergence for solving the quasi variational inequality in a real Hilbert spaces. It is very important to develop some efficient iterative methods for solving the quasi variational inequalities. Alvarez [1] and Alvarez et al. [2] used the inertial type projection methods for solving variational inequalities. Noor [33] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods have been modified in various directions for solving variational inequalities and related optimization problems. Jabeen et al. [14, 15, 16] and Noor et al [39] analyzed some inertial projection methods for some classes of general quasi variational inequalities. Convergence analysis of these inertial type methods has been considered under some mild conditions.

Dupuis and Nagurney [10] introduced and studied the projected dynamical systems associated with variational inequalities using the equivalent fixed point formulation. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. Thus the equilibrium and nonlinear programming problems, which can be formulated in the setting of the variational inequalities, can now be studied in the more general framework of the dynamical systems. It has been shown [4, 10, 11, 12, 14, 18, 30, 31, 32, 33, 38, 39, 47, 48] that these dynamical systems are useful in developing efficient and powerful numerical techniques for solving variational inequalities.

Motivated and inspired by ongoing research in these fascinations areas, we consider a second order dynamical system associated with quasi variational inequalities. Using the finite difference schemes, we suggest and analyzed some new iterative methods for solving quasi variational inequalities. Some special cases are also pointed as potential applications of the obtained results. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be extended for other classes of quasi variational inequalities and related optimization problems.

2. Basic definitions and results

Let $\mathtt{K}$ be a set in a real Hilbert space $\mathtt{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathtt{T}:\mathtt{H}\longrightarrow\mathtt{H}$ be nonlinear operator. Let $\mathtt{K}:\mathtt{H}\longrightarrow\mathtt{H}$ be a set-valued mapping which, for any element $\mu\in\mathtt{H}$, associates a convex-valued and closed set $\mathtt{K}(\mu)\subset\mathtt{H}$.

We consider the problem of finding $\mu\in\mathtt{K}(\mu)$, such that

$$\left\langle \mathcal{T}\mu,\nu-\mu\right\rangle \geq 0,\quad \forall\,\nu\in\mathcal{K}(\mu), \tag{1}$$

which is called the quasi variational inequality, introduced by Bensoussan and Lions [5]. For more details, see [5, 7, 14, 15, 16, 18, 27, 28, 32, 33, 36, 44] and the references therein.

2.1. Applications

To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem, which have been discussed in Noor et al. [37]. For the sake of completeness and to convey the main ideas, we include all the details.

We consider the problem of finding μ such that

$$\left.\begin{array}{ll} -\mu''(x) \geq f(x) & \text{on } \Omega = [a,b] \\ \mu(x) \geq M(\mu) & \text{on } \Omega = [a,b] \\ [-u''(x) - f(x)][\mu - M(\mu)] = 0 & \text{on } \Omega = [a,b] \\ \mu(a) = 0, \quad \mu^(b) = 0. & \end{array}\right\} \tag{2}$$

where $f(x)$ is a continuous function and $M(\mu)$ is the cost (obstacle) function. The prototype encountered is

$$M(\mu) = k + \inf_i \{\mu^i\}. \tag{3}$$

In (3), k represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $\mu = (\mu^1, \mu^2, \ldots, \mu^i)$. We study the problem (2) in the framework of general quasi variational inequality approach. To do so, we first define the set $\mathtt{K}$ as

$$\mathtt{K}(\mu) = \{\nu : \nu \in H_0^1(\Omega) : \nu \geq \mathtt{M}(\mu), \quad \text{on } \Omega\},$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space, see [5, 13, 19]. One can easily show that the energy functional associated with the problem (2) is

$$\begin{aligned} I[\nu] &= -\int_a^b \left(\frac{d\nu}{dx}\right) \nu dx - 2\int_a^b f(x)\nu dx, \quad \forall \nu \in \mathtt{K}(u) \\ &= \int_a^b \left(\frac{d\nu}{dx}\right)^2 dx - 2\int_a^b f(x)\nu dx \\ &= \langle T\nu, \nu\rangle - 2\langle f, \nu\rangle \end{aligned} \tag{4}$$

where

$$\begin{aligned} \langle T\mu, \nu\rangle &= -\int_a^b (\frac{d^2\mu}{dx^2})\nu dx = \int_a^b \frac{d\mu}{dx}\frac{d\nu}{dx} dx \\ \langle f, \nu\rangle &= \int_a^b f(x)\nu dx. \end{aligned} \tag{5}$$

It is clear that the operator T defined by (5) is linear, symmetric and positive. Using the technique of Noor [29, 33] and Noor et al.[39, 40], one can show that the minimum of the functional $I[v]$ defined by (4) associated with the problem (2) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type

$$\langle \mathtt{T}\mu, \nu - \mu\rangle \geq \langle f, \nu - \mu\rangle, \quad \forall \nu \in \mathtt{K}(u), \tag{6}$$

which is exactly the quasi variational inequality (1).

Special cases

We now discuss some special cases of general quasi variational inequalities (1)

(1) If $\mathtt{K}(\mu) = \mathtt{K}$, then problem (1) is equivalent to finding $\mu \in \mathtt{H} : \mathtt{g}(\mu) \in \mathtt{K}$ such that

$$\left\langle \mathtt{T}\mu, \nu - \mu \right\rangle \geq 0, \qquad \forall\, \nu \in \mathtt{K}, \tag{7}$$

which is called the variational inequality, introduced and studied by Stampacchia [45]. It has been shown a wide class of obstacle boundary value and initial value problems can be studied in the general framework of variational inequalities (6). For the applications, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 25, 26, 36, 37, 38, 44, 45] and the references therein.

(2) If $\mathtt{K}^*(\mu) = \{\mu \in \mathtt{H} : \langle \mu, \nu \rangle \geq 0, \forall \nu \in \mathtt{K}(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\mathtt{K}(\mu)$ in $\mathtt{H}$, then problem (7) is equivalent to finding $\mu \in \mathtt{H}$ such that

$$\mu \in K(\mu), \quad T\mu \in K^*(\mu) \quad \text{and} \quad \langle T\mu, \mu \rangle = 0, \tag{8}$$

which is known as the quasi complementarity problem. For $\mathtt{K}(\mu) = m(\mu) + \mathtt{K}$, where m is a point-to-point mapping, the problem(8) is called the implicit complementarity problem, see Noor [28]. The complementarity problems and their variant forms have been studied extensively in recent years, see [8, 17, 21, 23, 28, 33, 36, 39, 40] and the references therein

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem (1) considered in this paper is more general and unifying one.

We need the following well-known definitions and results in obtaining our results.

Definition 2.1. *Let* $\mathtt{T} : \mathtt{H} \longrightarrow \mathtt{H}$ *be a given mapping.*

i. *The mapping* $\mathtt{T}$ *is called strongly monotone, if there exists a constant* $\alpha \geq 0$ *such that*

$$\left\langle \mathtt{T}\mu - \mathtt{T}\nu\,,\, \mu - \nu \right\rangle \geq \alpha\, \|\mu - \nu\|^2, \qquad \forall\, \mu,\, \nu\, \in\, \mathtt{H}.$$

ii. *The mapping* $\mathtt{T}$ *is called monotone, if*

$$\left\langle \mathtt{T}\mu - \mathtt{T}\nu\,,\, \mu - \nu \right\rangle \geq 0, \quad \mu,\, \nu\, \in\, \mathtt{H}.$$

iii. *The mapping* $\mathtt{T}$ *is called* $\eta-$*Lipschitz continuous, if there exists a constant* $\eta > 0$ *such that*

$$\|\, \mathtt{T}\mu - \mathtt{T}\nu\, \| \leq \eta\, \|\mu - \nu\|, \qquad \forall\, \mu,\, \nu\, \in\, \mathtt{H}.$$

The following projection result plays an indispensable role in achieving our results.

Lemma 2.1. [12, 14] *For a given* $\omega \in \mathtt{H}$, *find* $\mu \in \mathtt{K}(\mu)$, *such that*

$$\left\langle \mu - \omega, \nu - \mu \right\rangle \geq 0, \;\; \forall\, \nu \in \mathtt{K}(\mu)\,,$$

if and only if

$$\mu = \Pi_{\mathtt{K}(\mu)}\left[\omega\right],$$

where $\Pi_{\mathtt{K}(\mu)}$ *is the implicit projection of* $\mathtt{H}$ *onto the closed convex-valued set* $\mathtt{K}(\mu)$ *in* $\mathtt{H}$.

The implicit projection operator $\Pi_{\mathtt{K}(\mu)}$ is nonexpansive and has the following characterization.

Assumption 2.1. [14] *The implicit projection operator* $\Pi_{\mathtt{K}(\mu)}$, *satisfies the condition*

$$\| \Pi_{\mathtt{K}(\mu)}[\omega] - \Pi_{\mathtt{K}(\nu)}[\omega] \| \leq \upsilon \, \| \mu - \nu \| \qquad \forall \, \mu, \, \nu \, , \, \omega \, \in \, \mathtt{H}, \tag{9}$$

where $\upsilon > 0$, *is a constant.*

In many important applications, the convex-valued set $\mathtt{K}(\mu)$ is of the form

$$\mathtt{K}(\mu) = m(\mu) + \mathtt{K}, \tag{10}$$

where m is a point-to-point mapping and $\mathtt{K}$ is a closed convex set. The convex-valued set $\mathtt{K}(\mu) = m(\mu) + \mathtt{K}$ defined by (10) is called the moving set-valued convex set.

In this case,

$$P_{K(\mu)}w = P_{m(\mu)+K}(w) = m(\mu) + P_K[w - m(\mu)], \quad \forall \mu, w \in \mathtt{H}.$$

If m is a Lipschitz continuous with constant β, then

$$\begin{aligned} \|P_{K(\mu)}w - P_{K(\nu)}w\| &= \|m(\mu) - m(\nu) + P_K[w - m(\mu)] - P_K[w - m(\nu)]\| \\ &\leq \|m(\mu) - m(\nu)\| + \|P_K[w - m(\mu)] - P_K[w - m(\nu)]\| \\ &\leq 2\|m(\mu) - m(\nu)\| \leq 2\beta. \end{aligned}$$

This show that the Assumption 2.1 holds.

3. **Main Results**

In this section, we suggest some new inertial-type approximation schemes for solving the quasi variational inequality (1) using the dynamical systems techniques. One can easily show that the quasi variational inequality (1) is equivalent to fixed point problem by using Lemma 2.1.

Lemma 3.1. *The function* $\mu \in \mathtt{H} : \mu \in \mathtt{K}(\mu)$ *is solution of quasi variational inequality (1), if and only if,* $\mu \in \mathtt{H} : \mathtt{g}(\mu) \in \mathtt{K}(\mu)$ *satisfies the relation*

$$\mu = \Pi_{\mathtt{K}(\mu)}[\mu - \rho \mathtt{T}\mu], \tag{11}$$

where $\rho > 0$ *is a constant and* $\Pi_{\mathtt{K}(\mu)}$ *is the projection of* $\mathtt{H}$ *onto the convex-valued set* $\mathtt{K}(\mu)$.

Lemma 3.1 implies that the problem (1) is equivalent to a fixed point problem (11). This alternate form is very useful from both numerical and theoretical point of views.

In this section, we use the fixed point formulation to suggest and consider a new second order projection dynamical system associated with quasi variational inequalities (1). We use this dynamical system to suggest and investigate some inertial proximal methods for solving the quasi variational inequalities (1). These inertial implicit methods are constructed using the central finite difference schemes and its variant forms.

To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = \Pi_{\mathtt{K}(\mu)}[\mu - \rho\mathfrak{T}(\mu)], \quad \mu(t_0) = \alpha, \dot{\mu}(t_0) = \beta, \tag{12}$$

where $\gamma \geq 0$, $\eta \geq 0$ and $\rho > 0$ are constants. Problem (12) is called second order dynamical system.

If $\gamma = 0$, then dynamical system (12) reduces to

$$\dot{\mu} + \mu = \Pi_{\mathtt{K}(\mu)}[\mu - \rho\mathfrak{T}(\mu)], \quad \mu(t_0) = \alpha, \tag{13}$$

where $\rho > 0$ is a constant. Problem (13) is called dynamical system associated with quasi variational inequalities.

We discretize the second-order dynamical systems (12) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\eta\frac{\mu_n-\mu_{n-1}}{h}+\mu_n=\Pi_{\mathtt{K}(\mu_n)}[\mu_n-\rho\mathcal{T}(\mu_{n+1})], \tag{14}$$

where h is the step size.

If $\gamma = 1, h = 1$, then, from equation(14) we have

Algorithm 3.1. *For a given $\mu_0 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_{\mathtt{K}(\mu_n)}[\mu_n-\rho\mathcal{T}(\mu_{n+1})], \quad n=0,1,2,\ldots.$$

which is the the extragradient method of Kopervelich [20] for solving the quasi variational inequalities.

Algorithm 3.1 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the two-step inertial method.

Algorithm 3.2. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta_n\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\mathtt{K}(\mu_n)}[\mu_n-\rho\mathcal{T}(y_n)], \quad n=0,1,2,\ldots \end{aligned}$$

where $\theta_n \in [0.1]$ is a constant.

Similarly, we suggest the following iterative method.

Algorithm 3.3. *For given $\mu_0 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\Pi_{\mathtt{K}(\mu_n)}[\mu_{n+1}-\rho\mathcal{T}(\mu_{n+1})], \quad n=0,1,2,\ldots.$$

which is known as the double projection method, introduced and studied by Noor [29, 33] and can be written as

Algorithm 3.4. *For a given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\mathtt{K}(\mu_n)}[y_n-\rho\mathcal{T}(y_n)], \quad n=0,1,2,\ldots. \end{aligned}$$

which is called the two-step inertial iterative Noor method.

Problem (12)can be rewritten as

$$\begin{aligned} \gamma\ddot{\mu}+\dot{\mu}+\mu &= \Pi_{\mathtt{K}((1-\theta_n)\mu+\theta_n u))}[(1-\theta_n)\mu+\theta_n\mu-\rho\mathcal{T}((1-\theta_n)\mu+\theta_n\mu)], \\ & \mu(t_0)=\alpha, \dot{\mu}(t_0)=\beta, \end{aligned} \tag{15}$$

where $\gamma > 0, \theta_n \geq 0$ and $\rho > 0$ are constants.

Discretising the system (15), we have

$$\begin{aligned} &\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+\mu_n \\ &=\Pi_{\mathtt{K}((1-\theta_n)\mu_n+\theta_n\mu_{n-1})}[(1-\theta_n)\mu_n+\theta_n\mu_{n-1}-\rho\mathcal{T}((1-\theta_n)\mu_n+\theta_n\mu_{n-1})] \end{aligned}$$

from which, for $\gamma = 0, \quad h = 1$, we have

Algorithm 3.5. *For a given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_{\mathtt{K}((1-\theta_n)\mu+\theta_n\mu))}[(1-\theta_n)u_n + \theta_n\mu_{n-1} - \rho\mathfrak{T}((1-\theta_n)\mu_n + \theta_n\mu_{n-1})]$$

or equivalently

Algorithm 3.6. *For a given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\mathtt{K}(y_n)}[y_n - \rho\mathfrak{T}y_n] \end{aligned}$$

which is called the new inertial iterative method for solving the quasi variational inequality.

We discretize the second-order dynamical systems (12) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = \Pi_{\mathtt{K}(\mu_n)}[\mu_n - \rho\mathfrak{T}(\mu_{n+1})],$$

where h is the step size.
Using this discrete form, we can suggest the following an iterative method for solving the quasi variational inequalities (1).

Algorithm 3.7. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_{\mathtt{K}(\mu_n)}[\mu_n - \rho\mathfrak{T}(\mu_{n+1}) - \frac{\gamma\mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2}], \quad n = 0, 1, 2, \ldots.$$

Algorithm 3.7 is called the inertial proximal method for solving the quasi variational inequalities and related optimization problems. This is a new proposed method.

We can rewrite the Algorithm 3.7 in the equivalent form as follows:

Algorithm 3.8. *For a given $\mu_0 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\langle\rho\mathfrak{T}\mu_{n+1} + \frac{(\gamma + h^2)\mu_{n+1} - (2\gamma - h + h^2)\mu_n + (\gamma - h)\mu_{n-1}}{h^2}, v - \nu_{n+1}\rangle \geq 0, \forall v \in K(\mu). \quad (16)$$

We note that, for $\gamma = 0$, Algorithm 3.8 reduces to the following iterative method for solving quasi variational inequalities (1).

Algorithm 3.9. *For given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_{\mathtt{K}(\mu_n)}[\mu_n - \rho\mathfrak{T}\mu_{n+1} - \frac{\mu_n - \mu_{n-1}}{h}], \quad n = 0, 1, 2, \ldots.$$

We again discretize the second-order dynamical systems (12) using central difference scheme and forward difference scheme to suggest the following inertial proximal method for solving (1).

Algorithm 3.10. *For a given $\mu_0 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_{\mathtt{K}(\mu_n)}[\mu_{n+1} - \rho\mathfrak{T}(\mu_{n+1}) - \frac{(\gamma + h)\mu_{n+1} - (2\gamma + h)\mu_n + \gamma\mu_{n-1}}{h^2}], \quad n = 0, 1, 2, \ldots.$$

Algorithm 3.10 is quite different from other inertial proximal methods for solving the quasi variational inequalities.
If $\gamma = 0$, then Algorithm 3.10 collapses to:

Algorithm 3.11. *For a given $\mu_0 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \Pi_{\mathtt{K}(\mu_n)}[\mu_{n+1} - \rho\mathfrak{T}(\mu_{n+1}) - \frac{\mu_{n+1} - \mu_n}{h}], \quad n = 0, 1, 2, \ldots.$$

Algorithm 3.10 is an proximal method for solving the quasi variational inequalities. Such type of proximal methods were suggested by Noor[33] using the fixed point problems. In brief, by suitable descritization of the second-order dynamical systems (12), one can construct a wide class of explicit and implicit method for solving inequalities.

We now consider the convergence criteria of the Algorithm 3.8 using the technique of Alvarez and Attouch [2], Noor [33] and Noor et al.[39].

Theorem 3.1. *Let $\mu \in \mathtt{H}$ be the solution of quasi variational inequality* (1). *Let μ_{n+1} be the approximate solution obtained from (16). If $\mathfrak{T}$ is monotone, then*

$$(h+h^2)\|\mu-\mu_{n+1}\|^2 \leq (\gamma+h^2)\|\mu-\mu_n\|^2-(\gamma+h^2)\|\mu_{n+1}-\mu_n\|^2+(\gamma-h)\|\mu_{n-1}-\mu_n\|^2. \quad (17)$$

Proof. Let $\mu \in \mathtt{H}$ be the solution of quasi variational inequality (1). Then

$$\langle \rho\mathfrak{T}(v), v-\mu\rangle \geq 0, \quad \forall v \in K(\mu), \quad (18)$$

since T is a monotone operator.

Setting $v = \mu_{n+1}$ in (18), we have

$$\langle \rho\mathfrak{T}(\mu_{n+1}), \mu_{n+1}-\mu\rangle \geq 0. \quad (19)$$

Taking $v = \mu$ in (16), we have

$$\langle \rho\mathfrak{T}(\mu_{n+1}) + \frac{(\gamma+h^2)\mu_{n+1}-(2\gamma-h+h^2)\mu_n+(\gamma-h)\mu_{n-1}}{h^2}, \mu-\mu_{n+1}\rangle \geq 0. \quad (20)$$

From (19) and (20), we obtain

$$\langle(\gamma+h^2)\mu_{n+1}-(2\gamma-h+h^2)\mu_n+(\gamma-h)\mu_{n-1}, \mu-\mu_{n+1}\rangle \geq 0.$$

Thus

$$\begin{aligned} 0 &\leq (\gamma+h^2)\langle\mu_{n+1}-\mu_n, \mu-\mu_{n+1}\rangle + (\gamma-h)\langle\mu_{n-1}-\mu_n, \mu-\mu_{n+1}\rangle \\ &\leq (\gamma+h^2)\|\mu-\mu_n\|^2-(\gamma+h^2)\|\mu_{n+1}-\mu_n\|^2-(\gamma+h^2)\|\mu-\mu_{n+1}\|^2 \\ &\quad +(\gamma-h)\|\mu_{n-1}-\mu_n\|^2+(\gamma-h)\|\mu-\mu_{n+1}\|^2 \\ &= (\gamma+h^2)\|\mu-\mu_n\|^2-(\gamma+h^2)\|\mu_{n+1}-\mu_n\|^2+(\gamma-h)\|\mu_{n-1}-\mu_n\|^2 \\ &\quad -h(1+h)\|\mu-\mu_{n+1}\|^2, \end{aligned} \quad (21)$$

where we have used the following inequalities

$$2\langle\mu, v\rangle = \|\mu+v\|^2-\|\mu\|^2-\|v\|^2, \quad \forall v.\mu \in \mathtt{H}$$

and

$$2\langle\mu, v\rangle \leq \|\mu\|^2-\|v\|^2.$$

From (21), we have

$$(h+h^2)\|\mu-\mu_{n+1}\|^2 \leq (\gamma+h^2)\|\mu-\mu_n\|^2-(\gamma+h^2)\|\mu_{n+1}-\mu_n\|^2+(\gamma-h)\|\mu_{n-1}-\mu_n\|^2,$$

the required (18). □

We also need the following assumption.

Assumption 3.1. *(i).for any sequence $\mu_n$with $\mu_n \to \mu$, and for any $v \in K(\mu)$,there exists a sequence$\{v_n\}$ such that $v_n \in K(\mu_n))$and $v_n \to v$ as $n \to \infty$.*
(ii). For all sequences $\{\mu_n\}$ and $\{v_n\}$ with $v_n \in K(\mu_n)$, then $v \in K(\mu)$.

Theorem 3.2. *Let $\mu \in K(\mu)$ be a solution of variational inequality* (1). *Let μ_{n+1} be the approximate solution obtained from (16). If Assumption 3.1 holds and the operator* T *is monotone, then μ_{n+1} converges to $\mu \in K(\mu)$ satisfying* (1).

Proof. Let $\mu \in K(\mu)$ be a solution of (1). From (18), it follows that the sequence $\{\|\mu - \mu_i\|\}$ is non-increasing and consequently,$\{\mu_n\}$ is bounded. Also from(18), we have

$$\sum_{i=1}^{\infty} \|\mu_n - \mu_{n+1}\|^2 \leq \|\mu - \mu_1\|^2 + \frac{\gamma - h}{\gamma + h^2}\|\mu_0 - \mu_1\|^2,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\|^2 = 0. \tag{22}$$

Since sequence $\{\mu_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{\mu}$ to which the subsequence $\{\mu_{i_k}\}_{k=i}^{\infty}$ converges. From Assumption 3.1, replacing μ_n by μ_{n_i} in (3.2) and taking the limit as $n_j \longrightarrow \infty$, we have

$$\langle \mathcal{T}(\hat{\mu}), v - \hat{\mu}\rangle \geq 0, \quad \forall v \in K(\mu),$$

which implies that $\hat{\mu}$ solves (1) and

$$\|\mu_{n+1} - \mu\|^2 \leq \frac{\gamma + h^2}{h + h^2}\|\mu - \mu_n\|^2 + \frac{\gamma - h}{h + h^2}\|\mu_n - \mu_{n-1}\|^2 \leq \|\mu - \mu_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{\mu}$ is unique and

$$\lim_{n\to\infty} \mu_{n+1} = \hat{\mu}.$$

□

4. **Applications**

In this section, we show that the quasi variational inequalities are equivalent to the general variational inequalities, which were introduced and investigated by Noor [26].
In many applications, the convex-valued set $K(u)$ is of the form (10). Let $\mu \in K(\mu)$ be a solution of problem (1). Then, from Lemma 3.1, it follows that $\mu \in K(\mu)$ such that

$$\begin{aligned} \mu &= P_{K(\mu)}\big[\mu - \rho T\mu\big] \\ &= P_{K(m(\mu)+K)}\big[\mu - \rho T\mu\big] \\ &= m(\mu) + P_K\big[\mu - m(\mu) - \rho T\mu\big]. \end{aligned} \tag{23}$$

This implies that

$$\mu - m(\mu) = P_K\big[\mu - m(\mu) - \rho T\mu\big].$$

which is equivalent to finding $\mu \in K$ such that

$$\langle T\mu, g(v) - g(\mu)\rangle \geq 0, \quad \forall v \in K, \tag{24}$$

where $g(\mu) = \mu - m(\mu)$. Conversely, if $g(\mu) = \mu - m(\mu)$, then the general variational inequality (24) is equivalent to the quasi variational inequality (1). It is worth mentioning that general variational inequality (24) was introduced and investigated by Noor [26]. It has been shown by Noor [33] that odd-order and nonsymmetric obstacle boundary value problems can be studied in the general variational inequalities. Thus all the results proved for quasi variational inequalities continue to hold for general variational inequalities of the type (24) with suitable modifications and adjustment. Despite the recent research activates, very few numerical results are available. The development of efficient numerical methods requires further efforts.

Conclusion: In this paper, we have used a second-order resolvent dynamical systems to suggest some inertial proximal methods for solving quasi variational inequalities. The convergence analysis of these methods have beeb considered under some weaker conditions. Our method of convergence criteria is very simple as compared with other techniques. Comparison and implementation of these new methods need further research efforts. We have only discussed the theoretical aspects of the proposed iterative methods. It is an interesting problem to discuss the implementation and performance of these new methods with other methods. Similar methods can be suggested for stochastic variational inequalities, which is another interesting and challenging problem. The ideas and techniques presented in this paper may be starting point for further developments.

Acknowledgments: The authors wish to express their deepest gratitude to our respected teachers, students, colleagues, collaborators and friends, who have directly or indirectly contributed in the preparation of this paper.

Conflict of Interest: The authors declare that they have no competing interests.

REFERENCES

[1] F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, SIAM J. Optim., 14(2003), 773–782.

[2] F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Anal., 9(2001), 3–11.

[3] A. S. Antipin, M. Jacimovic and N. Mijajlovic, Extra gradient method for solving quasi variational inequalities, Optimization, 67(2018), 103–112.

[4] S. Batool, M. A. Noor and K. I. Noor, Absolute value variational inequalities and dynamical systems, Inter. J. Anal. Appl. 18(3)(2020), 337-355.

[5] A. Bensoussan and J. L. Lions, Application des Inéqualities Variationnelles en Contrôl Stochastique, Paris: Bordas(Dunod), 1978.

[6] B. Bin-Mohsin, M. A. Noor, K. I. Noor and R. Latif, Resolvent dynamical systems and mixed variational inequalities, J. Nonl. Sci. Appl. 10(2017), 2925-2933.

[7] D. Chan and J. Pang, The generalized quasi-variational inequality problem, Math. Oper. Res., 7(1982), 211–222.

[8] R. W. Cottle, J. S. Pang and R. E. Stone, The Linear Complementarity Problem, SIAM Publication, 1992.

[9] J. Dong, D. H. Zhang and A. Nagurney, A projected dynamical systems model of general equilibrium with stability analysis, Math. Computer Modelling, 24(1996), 35 44.

[10] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities, Annal Opers. Research 44(1993), 19-42.

[11] T. L. Friesz, D. H. Berstein, N. J. Mehta, R. L. Tobin and S. Ganjliazadeh, Day-to day dynamic network disequilibrium and idealized traveler information systems, Opers. Res. 42(1994), 1120-1136.

[12] T. L. Friesz, D. H. Berstein and R. Stough, Dynamical systems, variational inequalities and control theoretical models for predicting time-varying urban network ows, Trans. Sci. 30(1996), 14-31.

[13] R. Glowinski, J. L. Lions and R. Tremolieres, Numerical Analysis of Variational Inequalities, New York: North-Holland Publishing Company, 1981.

[14] S. Jabeen, M. A. Noor and K. I. Noor, Inertial iterative methods for general quasi variational inequalities and dynamical systems, J. Math. Anal., 11(2020), 14–29.

[15] S. Jabeen, M. A. Noor and K. I. Noor, Some new inertial projection methods for quasi variational Inequalities, Appl. Math. E-Notes, 2021 (2021).

[16] S. Jabeen, B. Bin-Mohsen, M. A. Noor and K. I. Noor, Inertial projection methods for solving general quasi-variational inequalities, AIMS Math., 6(2021), 1075-1086.

[17] S. Karamardian, Generalized complementarity problem, J. Optim. Theory Appl., 8(1971), 161–168.

[18] A. G. Khan, M. A. Noor and K. I. Noor, Dynamical systems for quasi variational inequalities, Ann. Funct. Anal. 6(1)(2015), 193-209. DOI: 10.15352/afa/06-1-14

[19] D. Kinderlehrer and G. Stampachia, An Introduction to Variational Inequalities and Their Applications, New York: Academic Press, 1980.

[20] G. M. Korpelevich, The extragradient method for finding saddle points and other problems, Matecom, 12(1976), 203-213.

[21] C. F. Lemke, Bimatrix equilibrium points and mathematical programming, Manage. Sci., 11(1965), 681–689.

[22] N. Mijajlovic, J. Milojica and M. A. Noor, Gradient-type projection methods for quasi variational inequalities, Optim. Lett., 13(2019), 1885–1896.

[23] K. G. Murty, Linear Complementarity, Linear and Nonlinear Programming, Berlin, Heldermann Verlag, 1988

[24] M. A. Noor, On Variational Inequalities, Ph.D Thesis, Brunel University, London, UK, 1975.

[25] M. A. Noor, An iterative scheme for a class of quasi variational inequalities, J. Math. Anal. Appl., 110(1985), 463–468.

[26] M. A. Noor, General variational inequalities, Appl. Math. Letters, 1(1988), 119–122.

[27] M. A. Noor, Quasi variational inequalities, Appl. Math. Letters, 1(1988), 367–370.

[28] M. A. Noor, Nonlinear quasi complementarity problems, Appl. Math. Letters, 2(3)(1989), 251-254.

[29] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., 251(1)(2000), 217-229.

[30] M. A. Noor, Stability of the modified projected dynamical systems, Computers Math. Appl. 44(2002), 1-5.

[31] M. A. Noor, A Wiener-Hopf dynamical system for variational inequalities, New Zealand J. Math. 31(2002), 173-182.

[32] M. A. Noor, Implicit dynamical systems and quasi variational inequalities, Appl. Math. Comput., 134(2003), 69–81.

[33] M. A. Noor, Some developments in general variational inequalities, Appl. Math. Comput., 152(2004), 199–277.

[34] M. A. Noor, Merit functions for quasi variational inequalities, J. Math. Inequal., 1(2007), 259–269.

[35] M. A. Noor, Extended general variational inequalities, Appl. Math. Letters, 22(2009), 182–185.

[36] M. A. Noor and W. Oettli, On general nonlinear complementarity problems and quasi equilibria, Le Mathematiche, 49(1994), 313–331.

[37] M. A. Noor and K. I. Noor, Some novel aspects of quasi variational inequalities, Earthline J. Math. Sci., 10(1)(2022), 1-66

[38] M. A. Noor, K. I. Noor and R. Latif, Dynamical systems and variational inequalities, J. Inequal. Special Funct. 8(5)(2017), 22-29.

[39] M. A. Noor, K. I. Noor and M. Th. Rassias, New trends in general variational inequalities, Acta Appl. Mathematica, 170(1) (2020), 981–1064.

[40] M. A. Noor, K. I. Noor and T. M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math., 47(1993), 285–312.

[41] A. Pitea, M. Postolache: Duality theorems for a new class of multitime multiobjective variational problems, J. Glob. Optim. 53(2012), No. 1, 47-58;

[42] A. Pitea, M. Postolache: Minimization of vectors of curvilinear functionals on the second order jet bundle. Necessary conditions, Optim. Lett. 6(2012), No. 3, 459-470;

[43] A. Pitea, M. Postolache: Minimization of vectors of curvilinear functionals on the second order jet bundle. Sufficient efficiency conditions, Optim. Lett. 6(2012), No. 8, 1657-1669.

[44] Y. Shehu, A. Gibali and, S. Sagratella, Inertial projection-type method for solving quasi variational inequalities in real Hilbert space, J. Optim. Theory Appl., 184(2020), 877–894.

[45] G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, C. R. Acad. Sci., 258 (1964), 4413–4416.

[46] B.S. Thakur and M. Postolache, Existence and approximation of solutions for generalized extended nonlinear variational inequalities. J. Inequal. Appl. 2013(2013): Art. No. 590.

[47] Y. S. Xia and J. Wang, A recurrent neural network for solving linear projection equations, Neural Network, 13(2000), 337-350.

[48] Y. S. Xia and J. Wang, On the stability of globally projected dynamical systems, J. Optim. Theory Appl., 106(2000), 129-150.